Preface

The importance of controlling pedestrian flow especially during emergencies is being understood by researchers to be a very important research area. Currently, the use of static emergency routes is not efficient, since during emergencies, the preferred routes might be congested, or worse yet might not even exist. Hence, it is very important to use sensors to measure the current traffic and conditions on the routes and give real-time guidance to pedestrians using feedback control. This book is the first book that provides feedback control design for pedestrian movement control in one and two-dimensional problems using lumped and distributed parameter model settings. There is much more development that is needed in this important work, but the authors hope that this book provides inspiration for other researchers to continue work in this area.

Evacuation can be from a small area, single floor of a building, a entire building, a parking area, or from a much bigger region such as an entire city. The feedback control design for evacation of pedestrians in small areas falls under the framework presented in this book. Evacuation from bigger regions such as a city requires vehicular traffic control from highways, which can involve modeling of networks using digraphs. Network control for evacuation is not covered in this book.

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of this book as well as Dr. M. P. Singh from Virginia Tech. Some earlier work also involved development of visualization software (not presented in this book) by Thomas A. Merrell.

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Chapter 2

Traffic Flow Theory for 1-D

2.1 Introduction

Interest in modeling traffic flow has been around since the appearance of traffic jams. Ideally, if you can correctly predict the behavior of vehicle flow given an initial set of data, then, in theory, adjusting the flow in crucial areas can maximize the overall throughput of traffic along a stretch of road. This is of particular interest in regions of high traffic density, which may be caused by high volume peak time traffic, accidents or closure of one or more lanes of the road.

The development of the pedestrian evacuation dynamic systems follows from the traffic flow theory in 1-D space [2, 39, 51]. In many ways, the pedestrian evacuation system is similar to the vehicle traffic flow problem [73]. The main conservation equations used in modeling the vehicle traffic flow and the pedestrian evacuation flow are the same, with the exception that vehicle traffic is a 1-D space problem and the evacuation system is a 2-D space problem. Other similarities exist from having escape routes and escape times in both problems. In most of the situations a vehicle or a pedestrian has more than one route to a destination and each route has an associated cost, such as time.

In this chapter we will give the necessary background on traffic flow theory and survey the existing macroscopic mathematical models for single-lane, 1-D space traffic flow. These models will be used for crowd flow in 1-D, and they will be modified in Chap. 3 for 2-D flow. In Sect. 2.2, we start with the concept of macroscopic vs microscopic ways of modeling the traffic flow problem, followed by Sect. 2.3, where a microscopic model is introduced. The derivation of the traffic flow theory based on conservation of mass law, and the relationships between velocity and density are given in Sect. 2.4. In Sect. 2.5, four macroscopic traffic flow models are presented, derived, and analyzed based on their mathematical characteristics. Finally, the exact and weak solutions to the scalar traffic flow PDE, and the concepts of shock wave, rarefaction wave, and the admissibility of a solution are considered.

2.2 Microscopic vs Macroscopic

In the traffic flow problem, there are two classes of models: Macroscopic, which is concerned with average behavior, such as traffic density, average speed and module area, and a second class of models based on individual behavior referred to as microscopic models. The latter is classified into different types. The most famous one is the *Car-Following* models [6, 17, 57], where the driver adjusts his or her acceleration according to the conditions in front. In these models the vehicle position is treated as a continuous function and each vehicle is governed by an ordinary differential equation (ODE) that depends on speed and distance of the car in front. Another type of microscopic models are the *Cellular Automata* or vehicle hopping which differs from *Car-Following* in that it is a fully discrete model. It considers the road as a string of cells which are either empty or occupied by one vehicle. One such model is the *Stochastic Traffic Cellular Automata*, given in [75]. However, microscopic approaches are computationally expensive, as each car has an ODE to be solved at each time step, and as the number of cars increases, so does the size of the system to be solved. On the other hand, the macroscopic models are computationally less expensive because they have fewer design details in terms of interaction among vehicles and between vehicles and their environment. Therefore, it is desirable to use macroscopic models if a good model can be found satisfactorily to describe the traffic flow. In addition, this idea provides flexibility since detailed interactions are overlooked, and the model's characteristics are shifted toward parameters such as flow rate $f(\rho, v)$, concentration ρ (also known as traffic density), and average speed v, all functions of 1-D or 2-D space (x, y), and time (t). This is also true for first-order fluid dynamic models of isothermal flow and gases through pipes.

Two main prototypes set the stage for macroscopic traffic flow: the first is called the LWR model which is a *non-linear*, first-order hyperbolic PDE based on law of conservation of mass. The second one is a *second-order* model known as the PW model, which is based on two coupled PDE's one given by the conservation of mass and a second equation that mimics traffic flow.

2.3 Car-Following Model

We present the well known *car-following* microscopic traffic flow model. In [93], a 2-D version of this model was used for pedestrian flow in 2-D space. To derive the 1-D model, first assume cars can not pass each other. Then the idea is that a car in 1-D can move and accelerate forward based on two parameters; the headway distance between the current car and the one in front, and their speed difference. Hence, it is called *following*, where a car from behind follows the one in front, and this is the anisotropic property. This property is also desirable in macroscopic models, since it reflects the actual observed behavior of traffic flow [23].

Suppose the nth car location is $x_n(t)$, then the nonlinear model is given by

$$\ddot{x}_n(t) = c\,\frac{\dot{x}_n(t) - \dot{x}_{n-1}(t)}{x_n(t) - x_{n-1}(t)}, \tag{2.1}$$

The acceleration of the current car $\ddot{x}_n(t)$ depends on the front car speed and location, c is the sensitivity parameter. Integrating the above yields

$$\dot{x}_n(t) = c \ln\left(x_n(t) - x_{n-1}(t)\right) + d_n. \tag{2.2}$$

Since by the definition of the density (number of cars per unit area)

$$\frac{1}{\rho(x,t)} = x_n(t) - x_{n-1}(t), \tag{2.3}$$

and the integration constant d_n is chosen such that at jam density ρ_m, the velocity is zero. Then for steady-state we get

$$v = -c \ln \frac{\rho}{\rho_m}. \tag{2.4}$$

We see that for $\rho \to 0$ we get in trouble, but from observations in low traffic densities, car speed is the maximum allowed speed, hence we can assume $v = v_{\max}$, which is the maximum allowed speed.

2.4 Traffic Flow Theory

In this section we will cover the vehicle traffic flow fundamentals for the macroscopic modeling approach. The relation between density, velocity and flow is presented for traffic flow. Then we derive the conservation of vehicles, which is the main governing equation for scalar macroscopic traffic models. Finally, the velocity–density functions that makes the conservation equation a function of only one variable (density) are given.

2.4.1 Flow

In this section, we will illustrate the close relationship between the three variables: density, velocity and traffic flow. Suppose there is a road with cars moving with constant velocity v_0, and constant density ρ_0 such that the distance between the cars is also constant as shown in the Fig. 2.1a. Now let an observer measure the number of cars per unit time τ that pass him (i.e. traffic flow f). In τ time, each car has moved $v_0\tau$ distance, and hence the number of cars that pass the observer in τ time is the number of cars in $v_0\tau$ distance, see Fig. 2.1b.

Since the density ρ_0 is the number of cars per unit area and there is $v_0\tau$ distance, then the traffic flow is given by

$$f = \rho_0 v_0 \tag{2.5}$$

This is the same equation as in the time varying case, i.e.,

$$f(\rho, v) = \rho(x,t)v(x,t). \tag{2.6}$$

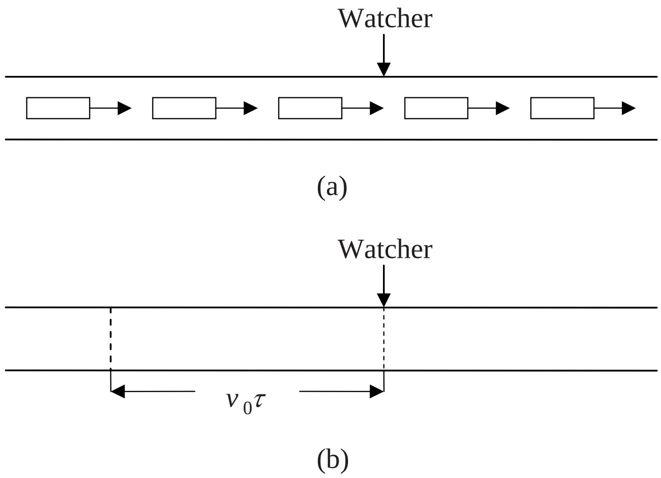

Fig. 2.1. (**a**) Constant flow of cars; (**b**) Distance traveled in τ hours for a single car

To show this, consider the number of cars that pass point $x = x_0$ in a very small time Δt. In this period of time the cars have not moved far and hence $v(x,t)$, and $\rho(x,t)$ can be approximated by their constant values at $x = x_0$ and $t = t_0$. Then, the number of cars passing the observer occupy a short distance, and they are approximately equal to $\rho(x,t)v(x,t)\Delta t$, where the traffic flow is given by (2.6).

2.4.2 Conservation Law

The models for traffic, whether they are one-equation or system of equations, are based on the physical principle of *conservation*. When physical quantities remain the same during some process, these quantities are said to be conserved. Putting this principle into a mathematical representation will make it possible to predict the densities and velocities patterns at future time. In our case, the number of cars in a segment of a highway $[x_1, x_2]$ are our physical quantities, and the process is to keep them fixed (i.e., the number of cars coming in equals the number of cars going out of the segment). The derivation of the conservation law is given in [26, 37], and it is presented here for completion. Consider a stretch of highway on which cars are

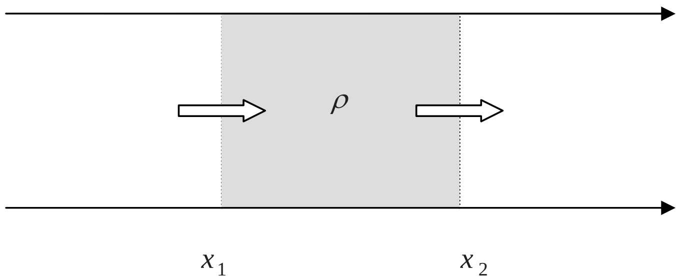

Fig. 2.2. One-dimension flow

moving from left to right as show in Fig. 2.2. It is assumed here that there are no exit or entrance ramps. The number of cars within $[x_1, x_2]$ at a given time t is the integral of the traffic density given by

$$N = \int_{x_1}^{x_2} \rho(x,t)\,\mathrm{d}x. \tag{2.7}$$

In the above equation, it is implied that the number of people within $[x_1, x_2]$ is at maximum when traffic density is equal to jam density ρ_m which is associated with the maximum number of cars that could possibly fit in a unit area.

The number of cars can still change (increase or decrease) in time due to cars crossing both ends of the segment. Assuming no cars are crated or destroyed, then the change of the number of cars is due to the change at the boundaries only. Therefore, the rate of change of the number of cars is given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = f_{\mathrm{in}}(\rho, v) - f_{\mathrm{out}}(\rho, v), \tag{2.8}$$

since the number of cars per unit time is the flow $f(\rho, v)$. Combining (2.7), and (2.8), yields the **integral conservation law**

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} \rho(x,t)\,\mathrm{d}x = f_{\mathrm{in}}(\rho, v) - f_{\mathrm{out}}(\rho, v). \tag{2.9}$$

This equation represents the fact that change in number of cars is due to the flows at the boundaries. Now let the end points be independent variables (not fixed with time), then the full derivative is replaced by partial derivative to get

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(x,t)\,\mathrm{d}x = f_{\mathrm{in}}(\rho,v) - f_{\mathrm{out}}(\rho,v). \tag{2.10}$$

The change in the number of cars with respect to distance is given by

$$f_{\mathrm{in}}(\rho,v) - f_{\mathrm{out}}(\rho,v) = -\int_{x_1}^{x_2} \frac{\partial f}{\partial x}(\rho,v)\,\mathrm{d}x, \tag{2.11}$$

and by setting the last two equations equal to each other, we get

$$\int_{x_1}^{x_2} \left[\frac{\partial \rho}{\partial t}(x,t) + \frac{\partial f}{\partial x}(\rho,v)\right]\mathrm{d}x = 0. \tag{2.12}$$

This equation states that the definite integral of some quantity is always zero for all values of the independent varying limits of the integral. The only function with this feature is the zero function. Therefore, assuming $\rho(x,t)$, and $q(x,t)$ are both smooth, the **1-D conservation law** is found to be

$$\rho_t + f_x(\rho,v) = 0. \tag{2.13}$$

We need to mention that this equation is valid for traffic and many more physical quantities. The idea here is conservation, and for vehicle traffic flow, the flow is given by (2.6).

2.4.3 Velocity–Density Relationship(s)

Traffic density and vehicle velocity are related by one equation, conservation of vehicles,

$$\rho_t(x,t) + (\rho(x,t)v(x,t))_x = 0, \tag{2.14}$$

where the notation $(\cdot)_\phi = \frac{\partial(\cdot)}{\partial\phi}$ will be used from here on. If the initial density and the velocity field are known, the above equation can be used to predict future traffic density. This leads us to choose the velocity function for the traffic flow model to be dependent on density and call it $V(\rho)$. The choice of such function depends on the behavior the model is trying to mimic. The following is a brief description of models that have been recognized and used by researchers [57], with emphasis on Greenshield model that will be used in several traffic (crowd) models throughout this book.

Greenshield's Model [34]

This model is simple and widely used. It is assumed here that the velocity is a linearly decreasing function of the traffic flow density, and it is given by

$$V(\rho) = v_f(1 - \frac{\rho}{\rho_m}) \tag{2.15}$$

where v_f is the free flow speed and ρ_m is the maximum density. Figure 2.3 shows the speed $V(\rho(x,t))$ as a monotonically decreasing function. For zero density the model allows free flow speed v_f, while for maximum density ρ_m no car can move in or out.

The flux–density relationship for Greenshield's model (2.15) is given in Fig. 2.4, where it shows the flux increases to a maximum which occurs at some density $\hat{\rho}$ and then it goes back to zero. This kind of behavior is due to the fact that $f''(\rho) < 0$ (note that $f(\rho) = \rho V(\rho)$ is the flux flow).

Greenberg Model [33]

In this model the speed–density function is given by

$$V(\rho) = v_f \ln(\frac{\rho}{\rho_m}) \tag{2.16}$$

Underwood Model [33]

In the Underwood model the velocity–density function is represented by

$$V(\rho) = v_f \, \exp(\frac{-\rho}{\rho_m}) \tag{2.17}$$

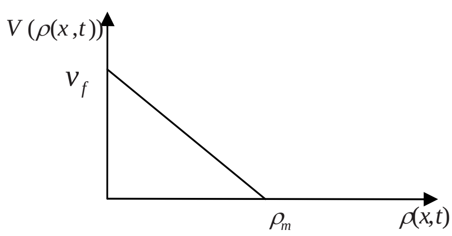

Fig. 2.3. Greenshield's model for traffic flow speed

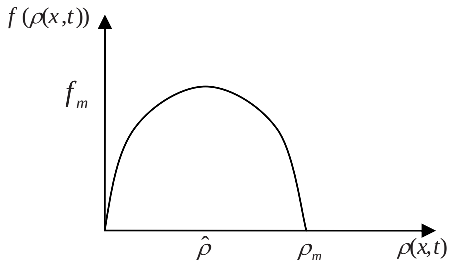

Fig. 2.4. Traffic flow flux as a function of density

Diffusion Model [14, 74]

Diffusion is a good extension to the model given by (2.15), where the effect of gradual rather than instantaneous reduction of speed by the driver takes place in response to shock waves. This kind of reaction can be accomplished by adding an extra term such that the modified Greenshield model will become

$$V(\rho) = v_f(1 - \frac{\rho}{\rho_m}) - \frac{D}{\rho}(\frac{\partial \rho}{\partial x}) \tag{2.18}$$

where D is a diffusion coefficient given by

$$D = \tau \; v_r^2$$

and v_r is a random velocity, τ is a relaxation parameter.

2.5 Traffic Flow Model 1-D

In this section, we will present four different models for traffic flow in 1-D space. The first model is one equation model, and the rest are systems of two-equation models. All the models are described by partial differential equations and based on conservation of mass and a second equation that is intended to capture the complex interactions observed in traffic flow motion. In addition, this second equation provides another way to couple velocity and density. Hence the flow is not in equilibrium like the one equation model. This is one of the main differences between the scalar and the system models.

2.5.1 LWR Model

The first model used in describing the traffic flow problem known as the LWR model, named after the authors in [68] and [87]. The LWR model is a scalar, time-varying, non-linear, hyperbolic partial differential equation. The model governing equation is (2.14). In this equation, traffic density is the conserved quantity, and we rewrite the model as

$$\rho_t(x,t) + (\rho(x,t)V(\rho(x,t)))_x = 0 \tag{2.19}$$

with the flux being replaced by the velocity–density relationship

$$f(x,t) = \rho(x,t)V(\rho(x,t)) \tag{2.20}$$

and $V(\rho(x,t))$ is the velocity function given by (2.15).

One of the basic assumptions in the LWR model regarding the velocity is its dependence on density alone. Any changes to density will be reflected in the velocity. The drawback for this assumption, as pointed out in [23], is that traffic is in equilibrium when such velocity–density functions are used, i.e., given a particular density, especially for light traffic, the velocity will be fixed and the model does not recognize that there is a distribution of desired velocities across vehicles. Therefore the model is not able to describe observed behavior in light traffic, although one can argue that v_f is an average speed which might take care of this issue. On the plus side, the model is anisotropic as the nature of the observed traffic flow, i.e. vehicle behavior is affected by mostly the car in front. This can be found from the model eigenvalue given by

$$f'(\rho(x,t)) = \frac{\partial f}{\partial \rho}(\rho(x,t)) = V(\rho(x,t)) + \rho V'(\rho(x,t)), \tag{2.21}$$

which means that the model allows information to travel as fast as the flow of traffic, and not more, since it satisfies $0 < f'(\rho) < V(\rho)$, because

$$V'(\rho) = -\frac{v_f}{\rho_m}. \tag{2.22}$$

The LWR model given by (2.19) and (2.15) is a simple model and it is unable to capture all of the complex interactions for a realistic traffic flow model. For this reason, modifications to the LWR model have been suggested. One way is by the various velocity–density functions we gave in Sect. 2.4.3. The second way is by coupling the conservation of mass with a second equation that tries to mimic traffic motion instead of the velocity–density models as given next.

2.5.2 PW Model

The first system model to be presented is a two-equation model proposed in the 1970s independently in [82] and [102]. Their model was the first model to couple velocity dynamics as a second equation, and it is referred to as the PW model. The first equation is the conservation of mass as discussed in the previous sections

$$\rho_t + (\rho v)_x = 0, \tag{2.23}$$

where the flux function $f(\rho, v) = \rho v$. In the LWR scalar equation model, a particular form of v was assumed where velocity is a function of density, but in high order models, v and ρ are assumed to be independent and a second equation is formed to link them, as in fluid and gas models. The second equation is derived from the Navier-Stokes equation of motion for a 1-D compressible flow, but with the pressure term replaced by $P = C_0^2 \rho$, where C_0 is the anticipation term that describes the response of macroscopic driver to traffic density, i.e. space concentration, and the pressure now is not "pressure" as such. The model also includes a traffic relaxation term that keeps speed concentration in equilibrium

$$\frac{V(\rho) - v}{\tau},$$

where τ is a relaxation time, and the velocity $V(\rho)$ is the maximum out-of-danger velocity meant to mimic driver's behavior given earlier by (2.15) to (2.18). The second equation of the PW model in nonconservative form is then given by

$$v_t + v\, v_x = \frac{V(\rho) - v}{\tau} - \frac{C_0^2}{\rho}\, \rho_x. \tag{2.24}$$

To study this model, we have to find its eigenvalues by first rewriting the model in conservation form. The first step is to use the product rule

$$(\rho v)_t = \rho v_t + v \rho_t, \tag{2.25}$$

and by multiplying (2.23) by v, we get

$$v\rho_t + v(\rho v)_x = 0. \tag{2.26}$$

Then by substituting in for $v\rho_t$ from the product rule (2.25), we get

$$v(\rho v)_x + (\rho v)_t - \rho v_t = 0, \tag{2.27}$$

and by substituting (2.27) into (2.24), and multiply the result by ρ we get

$$\rho v_t + \rho v \; v_x = \rho \left(\frac{V(\rho) - v}{\tau} \right) - C_0^2 \rho_x.$$

Then by substituting for ρv_t from (2.27) we get

$$v(\rho v)_x + (\rho v)_t + \rho v \; v_x = \rho \left(\frac{V(\rho) - v}{\tau} \right) - C_0^2 \rho_x. \tag{2.28}$$

Again using the product rule on $(\rho v v)_x$, i.e.,

$$(\rho v v)_x = (\rho v)_x v + (\rho v) v_x \tag{2.29}$$

and substituting in (2.28) we obtain

$$(\rho v^2)(\rho v)_t + (\rho v^2)_x = \rho \left(\frac{V(\rho) - v}{\tau} \right) - C_0^2 \rho_x. \tag{2.30}$$

Hence we obtain (2.24) with the lefthand side in conservation form

$$(\rho v)_t + (\rho v^2 + C_0^2 \rho)_x = \rho \left(\frac{V(\rho) - v}{\tau} \right) \tag{2.31}$$

where now ρ and ρv are the conserved variables. Equations (2.23) and (2.31) can be written in vector form as

$$Q_t + F(Q)_x = S \tag{2.32}$$

where,

$$Q = \begin{bmatrix} \rho \\ \rho v \end{bmatrix}, \; F(Q) = \begin{bmatrix} \rho v \\ \rho v^2 + C_0^2 \rho \end{bmatrix}, \; S = \begin{bmatrix} 0 \\ \rho \left(\frac{V(\rho) - v}{\tau} \right) \end{bmatrix}. \tag{2.33}$$

Setting the source term $S = 0$, we can rewrite the system in quasi-linear form as

$$Q_t + A(Q)\,Q_x = 0, \tag{2.34}$$

where

$$A(Q) = \frac{\partial F}{\partial Q} = \begin{bmatrix} 0 & 1 \\ C_0^2 - \rho v^2 & 2\,v \end{bmatrix}. \tag{2.35}$$

Finally by solving for the eigenvalues from

$$|A(Q) - \lambda I| = 0 \tag{2.36}$$

we get two distinct and real eigenvalues

$$\lambda_{1,2} = v \pm C_0, \tag{2.37}$$

therefore, the system is strictly hyperbolic.

The model has a major drawback that researchers (see for example [23]) are concerned about, mainly, that the model strongly follows the fluid flow theory. In fluids, the behavior of a particle is affected by its surrounding particles. Thus the anisotropic nature of traffic is not preserved since the vehicles are allowed to move with negative velocity, i.e. against the flow. This is clear from the eigenvalue, where one of $\lambda_{1,2} = v \pm C_0$ is always greater than the vehicle speed v. So, information from behind affects the behavior of the driver, and this is not true for observed traffic flow. This is called the isotropic property.

2.5.3 AR Model

A new model in [4] and improved in [84] is argued to be an improvement on the PW model. The authors of this model say that other researchers have stuck too closely to fluid flow models and have not allowed for a significant difference between traffic and fluids, e.g., traffic is more concerned with the flow in front, rather than behind. Therefore in order to move away from fluids and toward the anisotropic property of traffic, they argue that replacing the "pressure" term with an anticipation term describing how the average driver behaves is not a sufficient fix for the differences between the two types of flow. They claim that the drawback in the PW model (letting information travel faster than the flow) is due to an incorrect anticipation factor involving the derivative of the pressure w.r.t. x. Therefore they suggest the correct dependence must involve the convective derivative (full derivative) of the pressure term. The convective derivative in its general form is given by

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + (\vec{v}\cdot\nabla)\phi, \tag{2.38}$$

for $\phi(\vec{x},t)$, and $\vec{x} \in \Re^n$. They support their claim by the following example: "Assuming that in front of a driver traveling with speed v the density is increasing with respect to x, but decreasing with respect to $(x-vt)$. Then the PW type models predict that this driver would slow down, since the density ahead is increasing with respect to x! On the contrary, any reasonable driver would accelerate, since this denser traffic travels faster than him."

We call the model AR for short, and the first PDE equation is the same conservation of cars given by (2.23). However, in the AR model the next lagrangian equation replaces the second PDE equation given in the PW model. This second equation is found by applying the full derivative (2.38) to describe traffic motion dynamics, and it is given by

$$(v + P(\rho))_t + v\,(v + P(\rho))_x = 0, \tag{2.39}$$

where $P(\rho)$ is an increasing function of density. This choice is to ensure that this model caries the anisotropic property, and it is given by

$$P(\rho) = C_0^2\,\rho^\gamma, \tag{2.40}$$

where $\gamma > 0$, and $C_0 = 1$. Next we will put the second equation in conservation form, and find the system eigenvalues. First multiply (2.39) by ρ, then by using the product rule

$$(\rho\,(v + P(\rho)))_t = \rho_t(v + P(\rho)) + \rho\,(v + P(\rho))_t, \tag{2.41}$$

$$(\rho v\,(v + P(\rho)))_x = (\rho v)_x(v + P(\rho)) + (\rho v)(v + P(\rho))_x, \tag{2.42}$$

we obtain

$$(\rho\,(v+P(\rho))_t - \rho_t(v+P(\rho)) + (\rho v\,(v+P(\rho)))_x - (\rho v)_x(v+P(\rho)) = 0. \tag{2.43}$$

Now, using the conservation law (2.23) for ρ_t, we can simplify the above equation to

$$(\rho\,(v + P(\rho)))_t + (\rho v\,(v + P(\rho)))_x = 0, \tag{2.44}$$

which is the conservation form of (2.39). Then our conserved variables are ρ, and $\rho\,(v + P(\rho))$. We proceed now to find the system eigenvalues, let $X = \rho\,(v + P(\rho))$ for simplification, then the AR model given by (2.23), and (2.44) can be rewritten as

$$\begin{cases} \rho_t + (X - \rho P(\rho))_x & = \quad 0 \\ X_t + \left(\dfrac{X^2}{\rho} - XP(\rho)\right)_x & = \quad 0 \end{cases} \tag{2.45}$$

and in vector form (2.32), the stats and the flux are given by

$$Q = \begin{bmatrix} \rho \\ X \end{bmatrix}, \; F(Q) = \begin{bmatrix} X - \rho P(\rho) \\ \dfrac{X^2}{\rho} - XP(\rho) \end{bmatrix}. \tag{2.46}$$

For the quasi-linear form (2.34), the Jacobian is given by

$$A(Q) = \frac{\partial F}{\partial Q} = \begin{bmatrix} -(\gamma+1) & 1 \\ -\left(\dfrac{X^2}{\rho^2} + \dfrac{\gamma X P(\rho)}{\rho}\right) & \left(\dfrac{2X}{\rho} - P(\rho)\right) \end{bmatrix}. \tag{2.47}$$

Finally, solving for the eigenvalues from $|A(Q) - \lambda I| = 0$, we find two distinct and real eigenvalues

$$\lambda_1 = v - \gamma P(\rho) \quad \& \quad \lambda_2 = v. \tag{2.48}$$

Therefore, the system is strictly hyperbolic and since the "pressure" is an increasing function, then it is guaranteed that $\lambda_1 < \lambda_2$ due to the fact that the maximum wave speed is equal to the velocity of the flow v. Hence, the anisotropic property of traffic is preserved.

2.5.4 Zhang Model

We present here a model that was proposed in [104, 105], and it claims not to be of fluid, or gas-like behavior. The model caries the anisotropic property, because the second equation is derived from the microscopic *car-following* model. Hence, a micro-to-macro link is established for this model. Again, as in the PW and AR models, the Zhang model is also a system consisting of the conservation of cars (2.23), and coupled with a second PDE that describe car motion given by

$$v_t + v v_x + \rho V'(\rho) v_x = 0 \tag{2.49}$$

We start the micro-to-macro derivation from the *homogeneous* microscopic *car-following* model (the relaxation term can be added for 2-D crowd flow given in the next chapter) given by

$$\tau(s_n(t))\,\ddot{x}_n(t) = \dot{x}_{n-1}(t) - \dot{x}_n(t), \tag{2.50}$$

where

$$s_n(t) = x_{n-1}(t) - x_n(t), \tag{2.51}$$

and $s_n(t)$ is a function of the local spacing between cars, $x_n(t)$ is the position of the nth car, $\ddot{x}_n(t)$ is the acceleration, $\dot{x}_n(t)$ is the velocity, and $\tau(s_n(t))$ is the average response time to the headway distance. Using the above notations, we rewrite (2.50), and define the velocity as $v(x,t) = \dot{x}(t)$ to obtain the following

$$\tau(s(x(t),t))\,\frac{\mathrm{d}v(x,t)}{\mathrm{d}t} = \frac{\mathrm{d}(s(x(t),t))}{\mathrm{d}t}, \tag{2.52}$$

and by using convective derivative $\partial_t + v\partial_x$ on the velocity component, we get

$$\tau(s)\,(v_t + vv_x) = (s_t + vs_x). \tag{2.53}$$

From the conservation law (2.23), let $\rho = 1/s$, and by using the following derivative form

$$D_x(\frac{a}{b}) = \frac{bD_xa - aD_xb}{b^2}, \tag{2.54}$$

for any a and $b \neq 0$, we get

$$s_t + vs_x + us_y = sv_x + su_y, \tag{2.55}$$

and by direct substituting in the right hand side of (2.53), we obtain our desired equation in the following form

$$(v_t + vv_x) = \frac{s}{\tau(s)}v_x, \tag{2.56}$$

where

$$\frac{s}{\tau(s)} = -C(\rho) = -\rho V'(\rho) \geq 0 \tag{2.57}$$

is the sound wave speed. This completes the derivation of the macroscopic model (2.49) from its microscopic counterpart.

The conservative form of this model is derived next. First collect terms and rewrite (2.49) to get

$$v_t + (v + \rho V'(\rho))\, v_x = 0, \tag{2.58}$$

then expand the conservation of mass equation

$$\rho_t + \rho v_x + v\rho_x = 0. \tag{2.59}$$

We substitute for ρv_x from (2.59) into (2.58) to obtain

$$v_t + v v_x + V'(\rho)\,(-\rho_t - v\rho_x) = 0, \tag{2.60}$$

which can be rewritten as

$$v_t + v v_x - (V(\rho))_t - v\,(V(\rho))_x = 0, \tag{2.61}$$

or in lagrangian form as

$$(v - V(\rho))_t + v\,(v - V(\rho))_x = 0. \tag{2.62}$$

We now proceed to find the conservation form by multiplying (2.62) by ρ and using the product rules

$$(\rho(v - V(\rho)))_t = \rho_t(v - V(\rho)) + \rho(v - V(\rho))_t, \tag{2.63}$$
$$(\rho\, v\,(v - V(\rho)))_x = (\rho v)_x(v - V(\rho)) + \rho v\,(v - V(\rho))_x, \tag{2.64}$$

to get

$$(\rho(v-V(\rho)))_t - \rho_t(v-V(\rho)) + (\rho v\,(v-V(\rho)))_x - (v\rho)_x(v-V(\rho)) = 0. \tag{2.65}$$

From (2.59), we substitute for ρ_t in the above equation to obtain our final conservation form given by

$$(\rho\,(v - V(\rho)))_t + (\rho v\,(v - V(\rho)))_x = 0, \tag{2.66}$$

where our states are given by ρ and $\rho(v-V(\rho))$. For the quasi-linear form (2.34), let $X = \rho(v - V(\rho))$, and write the system in vector form (2.32), such that

$$Q = \begin{bmatrix} \rho \\ X \end{bmatrix}, \; F(Q) = \begin{bmatrix} X + \rho V(\rho) \\ \dfrac{X^2}{\rho} - XV(\rho) \end{bmatrix}. \tag{2.67}$$

The Jacobian is then can be found to be

$$A(Q) = \frac{\partial F}{\partial Q} = \begin{bmatrix} V(\rho) + \rho V'(\rho) & 1 \\ -\left(\frac{X^2}{\rho^2} - XV'(\rho)\right) & \left(\frac{2X}{\rho} + V(\rho)\right) \end{bmatrix}. \quad (2.68)$$

Finally by solving for the eigenvalues from $|A(Q) - \lambda I| = 0$, we find two distinct and real eigenvalues

$$\lambda_1 = v + \rho V'(\rho) \quad \& \quad \lambda_2 = v \quad (2.69)$$

therefore the system is strictly hyperbolic. Since $V'(\rho)$ is negative and given by (2.22), then the maximum the information can travel is equal to the vehicle speed v.

2.5.5 Models Summary

The one-equation LWR model consists of a single wave whose velocity is given by the derivative of the flux function, and information travels forward at a maximum not faster than the speed of traffic. Therefore the model behavior is anisotropic, i.e. only reacts to conditions ahead. For the two-equation models, the PW has two waves traveling at speeds given by $v \pm C_0$, one of them will always be traveling faster than the current speed v. This is a major cause of criticism of this model. The AR model has wave speeds given by v and $v - \gamma P(\rho)$. This seems reasonable since, as in the LWR model, the faster wave will move at the same speed as the traffic v. This is also true for the Zhang model, whose wave speeds are given by v and $v + \rho V'(\rho)$ with $V'(\rho) \leq 0$. This demonstrates the desirable anisotropic nature of the LWR, AR and Zhang models, and the isotropic nature of the PW model, for which it is severely criticized. In addition, Zhang model has a microscopic counterpart, which is not true for the PW and AR models. Although, the AR model has an indirect micro-to-macro relationship [4], where a numerical discretization of the AR model and a microscopic *car-following* model gave the same numerical formula. This suggest that the macroscopic AR model can be considered as an upper limit to the microscopic model.

2.6 Method of Characteristics

A typical problem in partial differential equation consists of finding the solution of a PDE subject to boundary conditions (BVP), initial conditions (IVP), or both (IBVP). In most cases it is difficult to find the exact (classical) solution of a hyperbolic PDE, but due to the simplicity of the LWR model and the fact it is a scalar 1-D space model we are able to find the exact solution by method of characteristics. The method of characteristics is a widely used technique to solve hyperbolic PDE's [24, 58, 77, 86].

2.6.1 LWR Model Classification

The partial derivative scalar conservation law in (2.19) is classified as first-order quasi-linear partial differential equation. This is due to the fact that the derivative of the highest partial occurs linearly. We can rewrite (2.19) as

$$\rho_t(x,t) + f'(\rho(x,t))\ \rho_x = 0 \tag{2.70}$$

where $f'(\rho)$ is the vehicle speed, and it is called the characteristic slop or the eigenvalue of the PDE. By using Greenshield's model (2.15), we get

$$f'(\rho(x,t)) = \frac{\mathrm{d}f(\rho(x,t))}{\mathrm{d}x} = v_f - \frac{2\ v_f\ \rho(x,t)}{\rho_m}, \tag{2.71}$$

where we see that this eigenvalue is real. Therefore, the LWR model is classified as strictly hyperbolic PDE. Figure 2.5 below shows the changes in speed $f'(\rho)$ with respect to the changes in density ρ. This relationship is important in finding the solution to the traffic flow model (2.70) by using method of characteristics discussed in the following section.

2.6.2 Exact Solution

Here we will use the method of characteristics to solve the initial value problem (IVP) and also called the Cauchy problem given by

$$\begin{cases} \rho_t(x,t) + f(\rho(x,t))_x \ = \ 0 \\ \qquad\qquad \rho(x,0) = \rho_0(x) \end{cases} \tag{2.72}$$

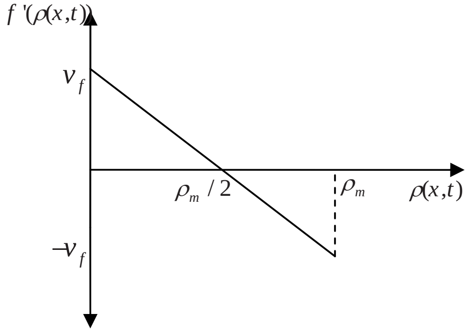

Fig. 2.5. Characteristic slops vs density

where $x \in \Re$ and time $t \in \Re^+$. Flow rate $f : \Re \mapsto \Re$ is assumed to be a smooth function, at least C^2 (i.e., twice differentiable) and the initial condition $\rho_0 : \Re \mapsto \Re$ is continuous. For the single conservation law, the eigenvalue of the PDE in (2.72) is given by the slope of the characteristic curve found from the quasi-linear form (2.70) as

$$\lambda(\rho) = f'(\rho) \tag{2.73}$$

Theorem 2.6.1 *Any C^1 solution of the single conservation law in (2.72) is constant along its characteristics. Accordingly, characteristic curves for the partial derivative conservation law in (2.72) are straight lines.*

Proof See [86].

The above theorem implies that any curve of the form

$$x(t) = k\,t + x(0) \tag{2.74}$$

is a characteristic curve where $x(t)$ is the solution, $k = f'(\rho(x(t), t))$ is the constant slope of the characteristic rays and $x(0)$ is the initial position of the characteristics rays. To show that (2.74) is indeed a solution to our Cauchy problem, let us first define what is meant by a solution.

Definition 2.6.1 Let $f : \Re \mapsto \Re$ be smooth, and let $\rho_0 : \Re \mapsto \Re$ be continuous. We say that $\rho(x,t) : (\Re \times \Re^+) \mapsto \Re$ is a classical solution of the Cauchy problem if $\rho(x,t) \in C^1\left(\Re \times \Re^+\right) \cap C^0\left(\Re \times \Re^+\right)$ and (2.70) is satisfied.

We can verify that it is indeed a solution by substituting for $x(0) = x - f't$ from (2.74) to get

$$\rho(x,t) = \rho_0(x - f'\,t), \tag{2.75}$$

then, by taking partial derivatives with respect to t and x, respectively, we obtain

$$\rho_t = \rho_0'(x - f'\,t)(-f'), \quad \text{and} \quad \rho_x = \rho_0'(x - f'\,t).$$

Substituting the above in (2.70), we get

$$\rho_0'(x - f'\,t)(-f') + f'\rho_0'(x - f'\,t) = 0,$$

which shows (2.72) is satisfied. So, the exact solution is basically the initial data shifted by the slope of the characteristic as shown in Fig. 2.6.

From the initial data we are able to generate the slopes of the characteristic rays originating from the x-axis. For some certain profiles of initial data, this gives us a method for solving the Cauchy problem. To illustrate this method, let us consider the initial data in Fig. 2.7, where we have the density profile of heavy traffic density at one end and light at the other end. Substituting for the initial density values in (2.71) will give the slopes of each of the characteristics. Then the solution follows from (2.74) along the rays of the characteristics as shown on the same figure.

2.6.3 Blowup of Smooth Solutions

Unfortunately, other examples with different initial data show how easily the procedure above fails. In Fig. 2.8, we see a density profile

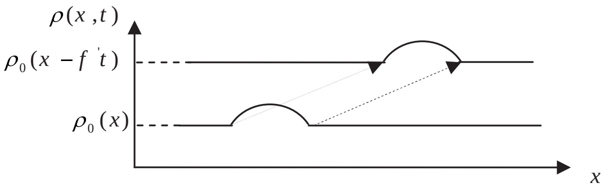

Fig. 2.6. Exact solution is achieved by shifting the initial density profile

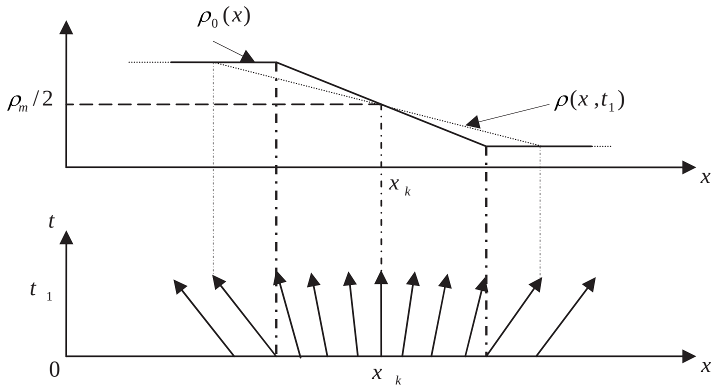

Fig. 2.7. Density distribution in the upper part, and the corresponding characteristic rays in the lower part

that describes road condition when cars are approaching red traffic light. Although initial density is continuous, the characteristics overlap at some later point in time. Since our solution cannot be multi-valued, we must conclude (in light of Theorem 2.6.1) that the solution shown cannot be smooth. For this type of initial data, a theory of discontinuous solutions, or *shock wave* solution is used.

Moreover, in Fig. 2.9, we face a different kind of problem. This time we have initial profile corresponding to heavy traffic at the beginning, then at some point x_k it is lighter. This example can

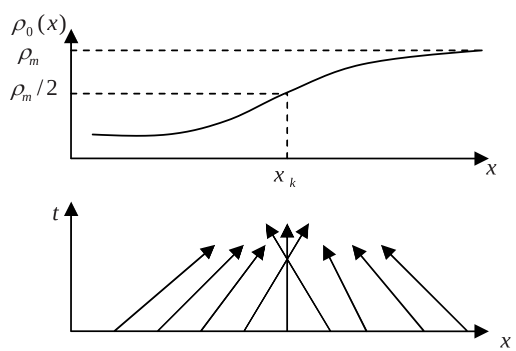

Fig. 2.8. Overlapping characteristics from continuous initial data

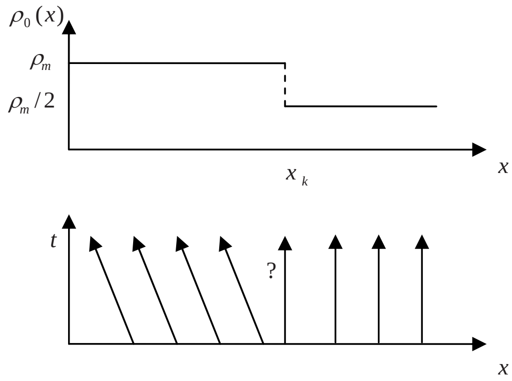

Fig. 2.9. Characteristics do not specify solution in the wedge

be related to conditions of a red light turns to green. As we all observe in real situation, cars start to accelerate from high density (low speed) to low density (higher speed). The exact solution for this data shows that there is a region untouched by any characteristics from the given initial data. Thus, the method of characteristics did not identify a solution in this region. As we shall see next, for this case we will be able to identify a continuous solution called a *rarefaction or fan wave* solution to fill the wedge.

2.6.4 Weak Solution

From the discussion above, smooth solutions of a single conservation laws can blow up (develop discontinuities or singularities) in finite time which fails to make any sense. Therefore, one cannot follow the practice of accepting solutions to the hyperbolic partial differential equations as given directly by method of characteristics. In order to understand discontinuous solutions, one needs to extend the notion of solution itself. One of the main features of the quasi-linear theory for hyperbolic PDE's is the notion of *weak solutions.* For a given initial data,

Definition 2.6.2 Let $\rho_0 \in L^\infty$. Then ρ is a weak solution or a solution in the distributional sense of (2.72) if and only if $\rho \in L^\infty(\Re \times \Re^+)$ and,

$$\int_0^\infty \int_{-\infty}^\infty [\rho(x,t)\,\phi_t(x,t) \quad + \quad f(\rho(x,t))\,\phi_x(x,t)]\;\mathrm{d}x\,\mathrm{d}t \\ + \quad \int_{-\infty}^\infty \rho_0(x)\,\phi(x,0)\;\mathrm{d}x = 0 \quad (2.76)$$

is satisfied for every $\phi \in C_0^\infty(\Re \times [0,\infty[)$

and $C_0^\infty(\Re \times [0,\infty[) := \{C_0^\infty(\Re \times [0,\infty[) \mid \exists\, r > 0$ s.t. support of $\phi \subset B_r(0,0) \cap (\Re \times [0,\infty[)\}$.

Here $\phi(x,t)$ is a test function with compact support on the boundary (i.e., $\phi(x,t)$ is zero outside the boundary). In the weak solution (2.76), the partial derivative is moved to the test function that is guaranteed to be smooth. In addition, the definition above is an extension of the classical solution according to

Theorem 2.6.2 *Suppose* $\rho \in C^1(\Re \times [0,\infty[)$ *is a classical solution of* (2.72)*. Then* ρ *is also a weak solution.*

Proof is given in [86].

We have to keep in mind that a *weak solution* might not be a classical solution. So we need necessary and sufficient conditions for the weak solutions to be the correct solution. We start by the necessary condition for a piecewise-smooth weak solution known as the Rankine-Hugoniot condition given by

$$s = \frac{[f(\rho)]}{[\rho]} = \frac{f(\rho_R) - f(\rho_L)}{\rho_R, -\rho_L} \tag{2.77}$$

where s is the shock speed. Let's look at the earlier example in Fig. 2.8, where the initial density was given by

$$\rho(x,0) = \begin{cases} \frac{\rho_m}{2} & x < 0, \\ \rho_m & x \geq 0, \end{cases} \tag{2.78}$$

and we seek a solution to our Cauchy problem (2.72), using the method of characteristics. Since the characteristics do overlap at some point in time, the shock speed is calculated from (2.77)

$$\begin{aligned} s &= \frac{(v_f \rho_R - v_f \rho_R^2/\rho_m) - (v_f \rho_L - v_f \rho_L^2/\rho_m)}{\rho_R - \rho_L} \\ &= v_f - \frac{v_f}{\rho_m}(\rho_R + \rho_L) \\ &= v_f - \frac{v_f}{\rho_m}(\rho_m + \frac{\rho_m}{2}) \\ &= -0.5\, v_f \end{aligned}$$

and the solution is given by

$$\rho(x,t) = \begin{cases} \frac{\rho_m}{2} & x < s\,t, \\ \rho_m & x \geq s\,t. \end{cases} \tag{2.79}$$

This solution is shown in Fig. 2.10.

Let's look now at the example of Fig. 2.9, and try to solve the traffic flow problem there. We will give two methods to find the solution and discuss which one must be used to get the correct solution. Using the initial density values

$$\rho(x,0) = \begin{cases} \rho_m & x < 0, \\ \rho_m/2 & x \geq 0, \end{cases} \tag{2.80}$$

we get the first solution as a shock wave given by

$$s \;=\; v_f - \frac{v_f}{\rho_m}(\rho_R + \rho_L) \;=\; v_f - \frac{v_f}{\rho_m}(\rho_m/2 + \rho_m) \;=\; -0.5\,v_f$$

As we can see "$-0.5\,v_f$" is the same shock speed as in the previous example and it is plotted in the Fig. 2.11b. The second solution is continuous and it provides another way to fill the wedge. It is called the *rarefaction wave* solution. The general form of the solution for traffic flow is given by

$$\rho(x,t) = \begin{cases} \rho_L & & x & < f'(\rho_L)\,t, \\ f'(\frac{x}{t})^{-1} & f'(\rho_L)\,t \leq & x & < f'(\rho_R)\,t, \\ \rho_R & & x & \geq f'(\rho_R)\,t, \end{cases} \tag{2.81}$$

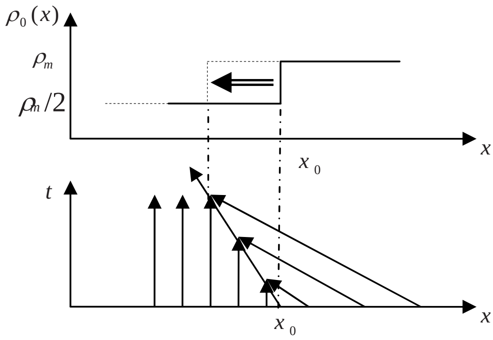

Fig. 2.10. Shock solution

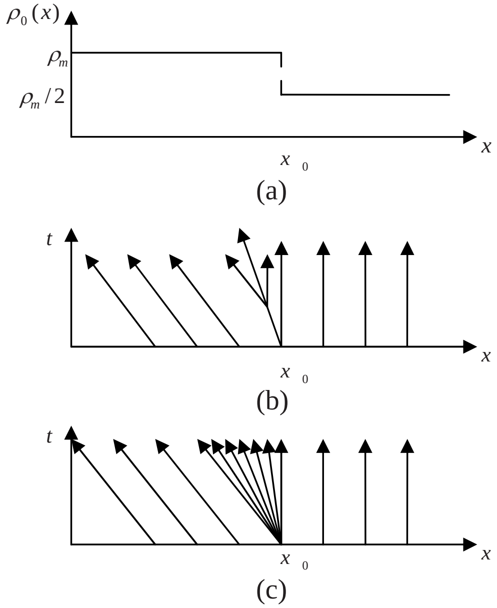

Fig. 2.11. Initial density profile followed by two weak solutions, shock and fan respectively

where the "−1" is an inverse mapping. For the traffic flow problem, $f'(\frac{x}{t})^{-1}$ can be found by letting

$$f'(\frac{x}{t})^{-1} = v_f - \frac{v_f \rho}{\rho_m} = \frac{x}{t}, \tag{2.82}$$

and solving for the density solution to get

$$\rho(x,t) = f'(\frac{x}{t})^{-1} = \frac{\rho_m}{2} - \frac{\rho_m x}{2 v_f t}. \tag{2.83}$$

Then, the continuous solution for the PDE is given by

$$\rho(x,t) = \begin{cases} \rho_L & x < -v_f t, \\ f'(\frac{x}{t})^{-1} \quad -v_f t \le & x < 0, \\ \rho_R & x \ge 0. \end{cases} \tag{2.84}$$

For the initial data given in Fig. 2.11a, the *rarefaction wave* solution is plotted in Fig. 2.11c.

Fig. 2.12. Lax shock condition for traffic flow problem: (**a**) not a shock, (**b**) shock

Such multiplicity of solutions is unacceptable. Thus we need a selection criterion that picks out the physically reasonable solution from among the possible weak solutions (2.81), and (2.84). Lax shock condition or *Lax entropy* condition [66] is a sufficient condition for scalar conservation laws. The condition for the traffic flow PDE states that the discontinuous solution for the traffic flow problem is admissible (i.e., a shock solution is selected with the direction of increased entropy) if

$$\rho_L < s < \rho_R, \tag{2.85}$$

otherwise, the rarefaction wave solution is the admissible one (see Fig. 2.12 for easy interpretation). In the example of Fig. 2.11a, and according to the Lax condition, the rarefaction solution is the admissible one and the shock solution is not admissible. Finally, we summarize the solution for the LWR model by mentioning the following two points:

- The solution is piece-wise smooth as $t \mapsto \infty$ with jumps in density (shocks) separating the pieces.
- This means traffic is predicted to be stable with transition between stable regions approximated by discontinuous shocks.