Preface

This is the first comprehensive introduction to the authors' recent attempts toward a better understanding of the global concepts behind spinor representations of surfaces in 3-space. The important new aspect is a quaternionic-valued function theory, whose "meromorphic functions" are conformal maps into $\mathbb{H}$, which extends the classical complex function theory on Riemann surfaces. The first results along these lines were presented at the ICM 98 in Berlin [10], and a detailed exposition will appear in [4]. Basic constructions of complex Riemann surface theory, such as holomorphic line bundles, holomorphic curves in projective space, Kodaira embedding, and Riemann-Roch, carry over to the quaternionic setting. Additionally, an important new invariant of the quaternionic holomorphic theory is the Willmore energy. For quaternionic holomorphic curves in $\mathbb{H}P^1$ this energy is the classical Willmore energy of conformal surfaces.

The present lecture note is based on a course given by Dirk Ferus at the Summer School on Differential Geometry at Coimbra in September, 1999, [3]. It centers on Willmore surfaces in the conformal 4-sphere $\mathbb{H}P^1$. The first three sections introduce linear algebra over the quaternions and the quaternionic projective line as a model for the conformal 4-sphere. Conformal surfaces $f : M \to \mathbb{H}P^1$ are identified with the pull-back of the tautological bundle. They are treated as quaternionic line subbundles of the trivial bundle $M \times \mathbb{H}^2$. A central object, explained in section 5, is the mean curvature sphere (or conformal Gauss map) of such a surface, which is a complex structure on $M \times \mathbb{H}^2$. It leads to the definition of the Willmore energy, the critical points of which are called Willmore surfaces. In section 7 we identify the new notions of our quaternionic theory with notions in classical submanifold theory. The rest of the paper is devoted to applications: We classify super-conformal immersions as twistor projections from $\mathbb{C}P^3$ in the sense of Penrose, we construct Bäcklund transformations for Willmore surfaces in $\mathbb{H}P^1$, we set up a duality between Willmore surfaces in S^3 and certain minimal surfaces in hyperbolic 3-space, and we give a new proof of the classification of Willmore 2-spheres in the 4-sphere, see Ejiri [2], Musso [9] and Montiel [8]. Finally we explain a close similarity between the theory of constant mean curvature spheres in $\mathbb{R}^3$ and that of Willmore surfaces in $\mathbb{H}P^1$, and use it to construct Darboux transforms for the latter.

Bath/Berlin,
August 2001

Francis Burstall, Dirk Ferus, Katrin Leschke,
Franz Pedit, Ulrich Pinkall