Introduction

The first purpose of an introduction is to explain what distinguishes the newly written book from other books that might as well have the same title. This book deals with quadratic mappings between modules over an arbitrary ring K (commutative, associative, with unit element); therefore it requires an effective mastery of some little part of commutative algebra. It is especially interested in quadratic forms and in their Clifford algebras. The most common object under consideration is a *quadratic module* (M, q), that is any module M provided with a quadratic form $q : M \to K$, and the deepest results are obtained when (M, q) is a *quadratic space*, in other words, when M is a finitely generated projective module and q induces a bijection from M onto the dual module M^*. In particular the study of Clifford algebras of quadratic spaces shall (very progressively) lead to sophisticated theories involving noncommutative algebras over the ring K (Azumaya algebras, Morita theory, separability).

This book is almost never interested in results that would follow from some special properties of the basic ring K; therefore much more emphasis has been put on a serious study of Clifford algebras than on sophisticated properties of quadratic forms which always depend on subtle hypotheses on the ring K. Here, when K is not an arbitrary ring, it is a local ring, or even a field; the consideration of such particular rings is justified by the importance of localization and globalization in many chapters, and the important role of residue fields at some critical moments. Besides, many useful applications of Clifford algebras outside mathematics involve quadratic spaces over fields.

Another essential feature of this book is the narrowness of the set of prerequisites, and its constancy from the beginning to the end. These prerequisites are made precise below, and although they are not elementary, they are much less difficult and fewer than would be required for a pioneering or scholarly work. All essential properties of Clifford algebras have been reached by elementary means in the first five chapters before more difficult theories are presented in Chapter 6. The concern of the authors about teaching has led them to limit the amount of prerequisites, and to prove all results in the core of the book (almost the whole book) on the basis of these prerequisites; for all these results the complete path leading to their proof (sometimes by new simpler means) is explained.

Of course it has not been possible to impose the above-mentioned features on the whole book. We thought it sensible to present interesting examples involving theories outside the scope of the book, and to give information about related topics which do not appear in the core of the book. Thus for the proof of several statements it has been necessary to refer to other publications. For instance, quadratic forms over the ring of integers often afford illuminating applications of general theories; but since this book does not deal with arithmetic, it just mentions which arithmetical knowledge is indispensable.

Readers are assumed already to know elementary algebra (rings, fields, groups, quotients,...), and also linear and multilinear algebra over fields, especially tensor products and exterior algebras. They are assumed to know the usual properties of quadratic forms over the usual fields $\mathbb{R}$ and $\mathbb{C}$, which should enable a rapid understanding of the properties of more general quadratic mappings. Even some knowledge of linear algebra over rings (over commutative, associative rings with unit) is required: exact sequences, projective and flat modules,.... Most of these prerequisites are briefly recalled, especially in Chapter **1**. A self-contained yet concise exposition of commutative algebra is provided; it only covers the small part that is needed, essentially localization and globalization, and finitely generated modules. Homological algebra is never involved, except in isolated allusions.

Many pages are devoted to "exercises"; their purposes are varied. Some of them are training exercises, in other words, direct applications. Others present still more results, which have seemed less important to the authors, but which nevertheless deserve to be stated with indications about how to prove them. Others present examples enlightening the reader on some particular features or some unexpected difficulties. There are also developments showing applications in other domains, and some few extracts from the existing literature. The levels of difficulty are varied; when an exercise has seemed to be very difficult, or to require some knowledge that is not treated in the book, an asterisk has been put on its number, and often a hint has been supplied.

In the opinion of the authors, many applications of Clifford algebras outside algebra, and even outside mathematics, raise problems that are universally interesting, even for algebraists. It is the duty of algebraists to find clear concepts and effective treatments, especially in places that are usually obscured by a lot of cumbersome calculations. In many applications of Clifford algebras there are interior multiplications; here (in Chapter **4**) it is explained that they can be derived from the comultiplication that makes every Clifford algebra become a comodule over the exterior algebra (treated as a coalgebra). In many applications of Clifford algebras the calculations need two multiplications, a Clifford multiplication and an exterior one; here (in Chapter **4**) this practice is related to the concept of "deformation of Clifford algebra", which allows an elaborate presentation of a well-known result stated for instance in [Chevalley 1954], §2.1, and with more generality in [Bourbaki 1959, *Algèbre*, Chap. 9] (see Proposition 3 in §9, $n^o 3$). But the true meaning of this essential result only appears when it is stated that it gives isomorphisms of comodules over exterior algebras, and not merely isomorphisms of K-modules.

Spinor spaces in quantum mechanics raise problems for which insightful algebraic interpretation and smart proof eschewing tedious calculations are still objects of discussion. Spinor spaces are often said to be Clifford modules although they are actually *graded* Clifford modules (see Example (6.2.2) in this book); the word "graded" refers to a parity grading which distinguishes even and odd elements. Whereas the theory of Clifford modules is a long sequence of particular cases, graded Clifford modules come under a unified and effective theory. The last exercises of Chapter **6** propose a smart and effective path to the essential algebraic properties that are needed in quantum mechanics.

This comment about spinor spaces is just one example of the constant emphasis put on parity gradings (from Chapter **3** to the end), in full agreement with C.T.C. Wall and H. Bass. In many cases the reversion of two odd factors must be compensated by a multiplication by -1, and here this rule is systematically enforced in all contexts in which it is relevant; indeed only a systematic treatment of parity gradings can avoid repeated hesitations about such multiplications by -1. For instance if f and g are linear forms on M, their exterior product can be defined as the linear form on $\bigwedge^2(M)$ that takes this value on the exterior product of two elements x and y of M :

$$(f \wedge g)(x \wedge y) \;=\; -f(x)\; g(y) \;+ f(y)\; g(x)\;;$$

the sign $-$ before $f(x)g(y)$ comes from the reversion of the odd factors g and x; but in $f(y)g(x)$ the odd factor y has jumped over two odd factors x and g, whence the sign $+$.

Lipschitz, the forgotten pioneer

Rudolf O.S. Lipschitz (1832–1903) discovered Clifford algebras in 1880, two years after William K. Clifford (1845–1879) and independently of him, and he was the first to use them in the study of orthogonal transformations. Up to 1950 people mentioned "Clifford-Lipschitz numbers" when they referred to this discovery of Lipschitz. Yet Lipschitz's name suddenly disappeared from the publications involving Clifford algebras; for instance Claude Chevalley (1909–1984) gave the name "Clifford group" to an object that is never mentioned in Clifford's works, but stems from Lipschitz's. The oblivion of Lipschitz's role is corroborated by [Weil], a letter that A. Weil first published anonymously, probably to protest against authors who discovered again some of Lipschitz's results in complete ignorance of his priority. Pertti Lounesto (1945–2002) contributed greatly to recalling the importance of Lipschitz's role: see his historical comment in [Riesz, 1993].

This extraordinary oblivion has generated two different controversies, a historical one and a mathematical one. On one side, some people claimed that the name "Clifford group" was historically incorrect and should be replaced with "Lipschitz group"; their action at least convinced other mathematicians to make correct references to Lipschitz when they had to invent *new* terms for objects that still had no name, even when they were reluctant to forsake the name "Clifford group". On the other side, some people were not satisfied with Chevalley's presentation of the so-called Clifford group, and completed it with additional developments that meant a return to Lipschitz's ideas; this is especially flagrant in [Sato, Miwa, Jimbo 1978], where the authors discovered again some of Lipschitz's results and gave them much more generality and effectiveness; the same might be said about [Helmstetter 1977, 1982]; but since the Japanese team showed applications of his cliffordian ideas to difficult problems involving differential operators (the "holonomic quantum fields"), the necessity of going beyond Chevalley's ideas became obvious for external reasons too. The fact that all these authors at that time completely ignored Lipschitz's contribution proves that the mathematical controversy is independent of the historical one.

The part of this book devoted to orthogonal transformations can be understood as a modernization of Lipschitz's theory. Whereas Lipschitz only considered real positive definite quadratic forms for which Clifford–Lipschitz groups may look quite satisfying, with more general quadratic forms it becomes necessary to attach importance to "Lipschitz monoids" from which "Lipschitz groups" are derived. Here the historically incorrect "Clifford groups" are still accepted (with the usual improved definition that pays due attention to the parity grading), but they only play an incidental role. They coincide with the Lipschitz groups in the classical case of quadratic spaces; but when beyond this classical case Clifford groups and Lipschitz groups no longer coincide, the latter prove to be much more interesting. Thus the mathematical controversy happens to prevail over the historical one.

Contraction and expansion are opposite and equally indispensable stages in all scientific research. At Chevalley's time it was opportune to contract the arguments and to exclude developments that no longer looked useful; but in Chevalley's works there is at least one part (in [Chevalley 1954], Chapter 3) that should have led him to reinstate Lipschitz if he had continued developing it. Our Chapter **7** is an expansion of this part of Chevalley's work, which for a long time has remained as he left it. This expansion involves the contributions of both Lipschitz and Chevalley, and should give evidence that it is much better to accept the *whole* heritage from *all* pioneers without prolonging inopportune exclusions. Besides, Lipschitz's ideas also proved to be very helpful in the cliffordian treatment of Weyl algebras.

Weyl algebras

Weyl algebras represent for alternate bilinear forms the same structure as Clifford algebras for quadratic forms, and in some publications they are even called "symplectic Clifford algebras". In [Dixmier 1968] you can find a concise exposition of what was known about them before cliffordian mathematicians became interested in them. Revoy was probably the first to propose a cliffordian treatment of Weyl algebras; see [Nouazé, Revoy 1972] and [Revoy 1978]. Later and independently, Crumeyrolle in France and the Japanese team Sato–Miwa–Jimbo produced some publications developing the cliffordian treatment of Weyl algebras, although they ignored (at least in their first publications) that these algebras had been already studied, and had received H. Weyl's name. Revoy's isolated work was hardly noticed, the cliffordian ideas of the Japanese team (which the renewal of Lipschitz's ideas mentioned above) were inserted in a very long and difficult work devoted to differential operators, which discouraged many people, and Crumeyrolle's statements bumped up against severe and serious objections. That is why the cliffordian treatment of Weyl algebras has not yet won complete acknowledgement.

There is no systematic presentation of Weyl algebras in this book, which already deals with a large number of other subjects. But at the end of Chapters **4**, **5** and **7**, many exercises about them have been proposed; Weyl algebras are defined in (4.ex.18). These exercises explain the cliffordian treatment of Weyl algebras as long as it is an imitation, or at least an adaptation, of the analogous treatment of Clifford algebras. For the most difficult results that require Fourier analysis and related theories, a short summary has been supplied; it should help readers to understand the purposes and the achievements of this new theory.

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During the writing of our text, we took advantage of the services of the Mathematical Department (Fourier Institute) of the University of Grenoble; our text was prepared in this Institute and its Library was very often visited. Therefore we are grateful to the Director and to the Librarian for their help. Several colleagues in this Institute suggested some good ideas, or tried to answer embarrassing questions; they also deserve our gratitude.

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