Preface

Discrete differential geometry (DDG) is a new and active mathematical terrain where differential geometry (providing the classical theory of smooth manifolds) interacts with discrete geometry (concerned with polytopes, simplicial complexes, etc.), using tools and ideas from all parts of mathematics. DDG aims to develop discrete equivalents of the geometric notions and methods of classical differential geometry. Current interest in this field derives not only from its importance in pure mathematics but also from its relevance for other fields such as computer graphics.

Discrete differential geometry initially arose from the observation that when a notion from smooth geometry (such as that of a minimal surface) is discretized "properly", the discrete objects are not merely approximations of the smooth ones, but have special properties of their own, which make them form a coherent entity by themselves. One might suggest many different reasonable discretizations with the same smooth limit. Among these, which one is the best? From the theoretical point of view, the best discretization is the one which preserves the fundamental properties of the smooth theory. Often such a discretization clarifies the structures of the smooth theory and possesses important connections to other fields of mathematics, for instance to projective geometry, integrable systems, algebraic geometry, or complex analysis. The discrete theory is in a sense the more fundamental one: the smooth theory can always be recovered as a limit, while it is a nontrivial problem to find which discretization has the desired properties.

The problems considered in discrete differential geometry are numerous and include in particular: discrete notions of curvature, special classes of discrete surfaces (such as those with constant curvature), cubical complexes (including quad-meshes), discrete analogs of special parametrization of surfaces (such as conformal and curvature-line parametrizations), the existence and rigidity of polyhedral surfaces (for example, of a given combinatorial type), discrete analogs of various functionals (such as bending energy), and approximation theory. Since computers work with discrete representations of data, it is no surprise that many of the applications of DDG are found within computer science, particularly in the areas of computational geometry, graphics and geometry processing.

Despite much effort by various individuals with exceptional scientific breadth, large gaps remain between the various mathematical subcommunities working in discrete differential geometry. The scientific opportunities and potential applications here are very substantial. The goal of the Oberwolfach Seminar "Discrete Differential Geometry" held in May–June 2004 was to bring together mathematicians from various subcommunities working in different aspects of DDG to give lecture courses addressed to a general mathematical audience. The seminar was primarily addressed to students and postdocs, but some more senior specialists working in the field also participated.

There were four main lecture courses given by the editors of this volume, corresponding to the four parts of this book:

I: Discretization of Surfaces: Special Classes and Parametrizations,
II: Curvatures of Discrete Curves and Surfaces,
III: Geometric Realizations of Combinatorial Surfaces,
IV: Geometry Processing and Modeling with Discrete Differential Geometry.

These courses were complemented by related lectures by other participants. The topics were chosen to cover (as much as possible) the whole spectrum of DDG—from differential geometry and discrete geometry to applications in geometry processing.

Part I of this book focuses on special discretizations of surfaces, including those related to integrable systems. Bobenko's "Surfaces from Circles" discusses several ways to discretize surfaces in terms of circles and spheres, in particular a Möbius-invariant discretization of Willmore energy and S-isothermic discrete minimal surfaces. The latter are explored in more detail, with many examples, in Bücking's article. Pinkall constructs discrete surfaces of constant negative curvature, documenting an interactive computer tool that works in real time. The final three articles focus on connections between quad-surfaces and integrable systems: Schief, Bobenko and Hoffmann consider the rigidity of quad-surfaces; Hoffmann constructs discrete versions of the smoke-ring flow and Hashimoto surfaces; and Suris considers discrete holomorphic and harmonic functions on quad-graphs.

Part II considers discretizations of the usual notions of curvature for curves and surfaces in space. Sullivan's "Curves of Finite Total Curvature" gives a unified treatment of curvatures for smooth and polygonal curves in the framework of such FTC curves. The article by Denne and Sullivan considers isotopy and convergence results for FTC graphs, with applications to geometric knot theory. Sullivan's "Curvatures of Smooth and Discrete Surfaces" introduces different discretizations of Gauss and mean curvature for polyhedral surfaces from the point of view of preserving integral curvature relations.

Part III considers the question of realizability: which polyhedral surfaces can be embedded in space with flat faces. Ziegler's "Polyhedral Surfaces of High Genus" describes constructions of triangulated surfaces with n vertices having genus $O(n^2)$ (not known to be realizable) or genus $O(n \log n)$ (realizable). Timmreck gives some new criteria which could be used to show surfaces are not realizable. Lutz discusses automated methods to enumerate triangulated surfaces and to search for realizations. Bokowski discusses heuristic methods for finding realizations, which he has used by hand.

Part IV focuses on applications of discrete differential geometry. Schröder's "What Can We Measure?" gives an overview of intrinsic volumes, Steiner's formula and Hadwiger's theorem. Wardetzky shows that normal convergence of polyhedral surfaces to a smooth limit suffices to get convergence of area and of mean curvature as defined by the cotangent formula. Desbrun, Kanso and Tong discuss the use of a discrete exterior calculus for computational modeling. Grinspun considers a discrete model, based on bending energy, for thin shells.

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Our hope is that this book will stimulate the interest of other mathematicians to work in the field of discrete differential geometry, which we find so fascinating.

Alexander I. Bobenko
Peter Schröder
John M. Sullivan
Günter M. Ziegler

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