Preface

This book presents fundamental material that should be part of the education of every practicing mathematician. This material will also be of interest to computer scientists, physicists, and engineers.

Complex analysis is also known as function theory. In this text we address the theory of complex-valued functions of a single complex variable. This is a prerequisite for the study of many current and rapidly developing areas of mathematics, including the theory of several and infinitely many complex variables, the theory of groups, hyperbolic geometry and three-manifolds, and number theory. Complex analysis has connections and applications to many other many other subjects in mathematics, and also to other sciences as an area where the classic and the modern techniques meet and benefit from each other. We will try to illustrate this in the applications we give.

Because function theory has been used by generations of practicing mathematicians working in a number of different fields, the basic results have been developed and redeveloped from a number of different perspectives. We are not wedded to any one viewpoint. Rather we will try to exploit the richness of the subject and explain and interpret standard definitions and results using the most convenient tools from analysis, geometry, and algebra.

The key first step in the theory is to extend the concept of differentiability from real-valued functions of a real variable to complex-valued functions of a complex variable. Although the definition of complex differentiability resembles the definition of real differentiability, its consequences are profoundly different. A complex-valued function of a complex variable that is differentiable is called *holomorphic* or *analytic*, and the first part of this book is a study of the many equivalent ways of understanding the concept of analyticity. Many of the equivalent ways of formulating the concept of an analytic function are summarized in what we term the Fundamental Theorem for functions of a complex variable. Chapter 1 begins with two motivating examples, followed by the statement of the Fundamental Theorem, an outline of the plan for proving it, and a description of the text contents: the plan for the rest of the book.

In devoting the first part of this book to the precise goal of stating and proving the Fundamental Theorem, we follow a path charted for us by Lipman Bers, from whom we learned the subject. In his teaching, expository, and research writing he often started by introducing a main, often technical, result and then proceeded to derive its important and seemingly surprising consequences. Some of the grace and elegance of this subject will not emerge until a more technical framework has been established. In the second part of the text, we proceed to the leisurely exploration of interesting ramifications and applications of the Fundamental Theorem.

We are grateful to Lipman Bers for introducing us to the beauty of the subject. The book is an outgrowth of notes from Bers's original lectures. Versions of these notes have been used by us at our respective home institutions, some for more than 20 years, as well as by others at various universities. We are grateful to many colleagues and students who read and commented on these notes. Our interaction with them helped shape this book. We tried to follow all useful advice and correct, of course, any mistakes or shortcomings they identified. Those that remain are entirely our responsibility.

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CHAPTER 2

Foundations

The first section of this chapter introduces the complex plane, fixes notation, and discusses some useful concepts from real analysis. Some readers may initially choose to skim this section. The second section contains the definition and elementary properties of the class of holomorphic functions - the basic object of our study.

2.1. Introduction and preliminaries

This section is a summary of basic notation, a description of some of the basic properties of the complex number system, and a disjoint collection of needed facts from real analysis (advanced calculus). We remind the reader of some of the formalities behind the standard notation, which we usually approach informally.

We start with some **Notation**: $\mathbb{Z}_{>0} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \widehat{\mathbb{C}}$. Here $\mathbb{Z}$ represents the integers, $\mathbb{Z}_{>0}$ the positive integers,[1] $\mathbb{Q}$ the rationals (the integer n is included in the rationals as the equivalence class of $\frac{n}{1}$), and $\mathbb{R}$ the reals. Whether one views the reals as the completion of the rationals or identifies them with Dedekind cuts, the most important property from the perspective of complex variables is the least upper bound property: Every nonempty set of real numbers that has an upper bound has a least upper bound.

The inclusion of $\mathbb{R}$ into the complex numbers $\mathbb{C}$ needs a bit more explanation. It is specified as follows: For $z \in \mathbb{C}$, we write $z = x + \imath\, y$ with x and y in $\mathbb{R}$ where the symbol $\imath$ represents a square root of -1 so that $\imath^2 = -1$. With these conventions we can define addition and multiplication of complex numbers[2] using the usual rules for these operation on the reals: For all $x, y, \xi, \eta \in \mathbb{R}$,

$$(x + \imath y) + (\xi + \imath\eta) = (x + \xi) + \imath(y + \eta)$$

and

$$(x + \imath y)(\xi + \imath\eta) = (x\xi - y\eta) + \imath(x\eta + y\xi).$$

[1] In general $X_{\text{condition}}$ and $\{x \in X : \text{condition}\}$ will describe the set $x \in X$ that satisfy the condition indicated.

[2] With these operations $(\mathbb{C}, +, \cdot)$ is a field.

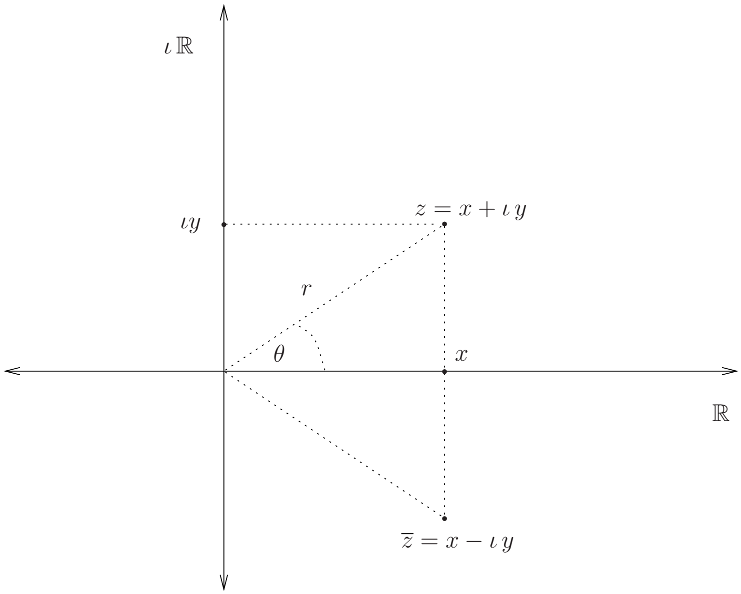

FIGURE 2.1. The complex plane

The *reals* $\mathbb{R}$ are identified with the subset of $\mathbb{C}$ consisting of those numbers with $y = 0$; the *imaginary* numbers $\imath\mathbb{R}$ are those with $x = 0$. For $z = x + \imath y$ in $\mathbb{C}$ with x and y in $\mathbb{R}$, we write $x = \Re z$, the *real part* of z, and $y = \Im z$, the *imaginary part* of z. Geometrically, $\mathbb{R}$ and $\imath\mathbb{R}$ represent the *real* and *imaginary axes* of $\mathbb{C}$, viewed as the complex plane and identified with the cartesian product $\mathbb{R}^2$ or, equivalently, $\mathbb{R} \times \mathbb{R}$.

The complex plane can be viewed as a subset of the complex sphere $\widehat{\mathbb{C}}$, which is $\mathbb{C}$ compactified by adjoining a point, known as *the point at infinity*, so that $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. $\widehat{\mathbb{C}}$ is also called *the extended complex plane* or the *Riemann sphere*. See Exercise 3.18 for a justification of the name.

The complex number $\overline{z} = x - \imath y$ is the *complex conjugate* of the complex number $z = x + \imath y$. Note that $\Re z = \dfrac{z + \overline{z}}{2}$ and $\Im z = \dfrac{z - \overline{z}}{2\imath}$. It is easy to verify the following properties:

Properties of conjugation. For z and $w \in \mathbb{C}$,

(a) $\overline{z + w} = \overline{z} + \overline{w}$,
(b) $\overline{zw} = \overline{z}\,\overline{w}$, and
(c) $\overline{\overline{z}} = z$.

There is a simple and useful **geometric interpretation of conjugation**: It is represented by mirror reflection in real axis. Since $\overline{\overline{z}} = z$, the map $z \mapsto \overline{z}$ defines an involution of $\mathbb{C}$

$$\bar{\ } : \mathbb{C} \to \mathbb{C}\,.$$

Another important map, $z \mapsto |z|$ or

$$|\ | : \mathbb{C} \to \mathbb{R}_{\geq 0}$$

is defined by $r = |z| = (z\overline{z})^{\frac{1}{2}} = (x^2 + y^2)^{\frac{1}{2}}$. Here $z = x + \imath y$ and we use the usual convention that unless otherwise specified the square root of a nonnegative number is chosen to be nonnegative. The nonnegative real number r is called the *absolute value* or *norm* or *modulus* of the complex number z.

Properties of absolute value. For z and $w \in \mathbb{C}$,

(a) $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$.
(b) $|zw| = |z|\,|w|$.
(c) $|z + w| \leq |z| + |w|$. Equality holds whenever either z or w is equal to 0. If $z \neq 0$ and $w \neq 0$, then equality holds if and only if $w = az$ with $a \in \mathbb{R}_{>0}$.
(d) $|z| = |\overline{z}|$.

Linear representation of $\mathbb{C}$. As a vector space over $\mathbb{R}$, we can identify $\mathbb{C}$ with $\mathbb{R}^2$. Vector addition agrees with complex addition. Scalar multiplication $\mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is the restriction of complex multiplication $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

Polar coordinates. A nonzero vector can be described by polar coordinates (r, θ) as well as by the rectangular coordinates (x, y) we have been using. If $z \in \mathbb{C}$ and $z \neq 0$, then we can write

$$z = x + \imath\, y = r\,(\cos\theta + \imath\,\sin\theta)\,,$$

where $r = |z|$ and $\theta = \arg z$ (an *argument* of z) $= \arcsin\dfrac{y}{r} = \arccos\dfrac{x}{r}$.

Note that the last two identities are needed to define the argument and that[3] $\arg z$ is defined up to the addition of an integral multiple of 2π.

If $w = \rho[\cos\varphi + \imath\,\sin\varphi] \neq 0$, then using the addition formulas for the sine and cosine functions, one has

$$zw = (r\rho)[\cos(\theta + \varphi) + \imath\,\sin(\theta + \varphi)]\,.$$

[3]The number π will be defined rigorously in Definition 3.29. Trigonometric functions will be introduced in the next chapter. Hence, for the moment, polar coordinates should not be used in proofs.

This polar form of the multiplication formula shows that complex multiplication involves real multiplication of the moduli and addition of the arguments, giving a geometric interpretation of how the operation of multiplication acts on vectors given in polar coordinates.

In particular, it follows that if $n \in \mathbb{Z}$ and $z = r\,(\cos\theta + \imath\,\sin\theta)$ is a nonzero complex number, then

$$z^n = r^n[\cos n\theta + \imath\,\sin n\theta]\ .$$

Therefore, for n in $\mathbb{Z}_{>0}$, each nonzero complex number z has (precisely) n n-th roots given by

$$r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + \imath\,\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]\ ,$$

with $k = 0, 1, \ldots, n-1$.

The formula $d(z, w) = |z - w|$, for z and $w \in \mathbb{C}$, defines a **metric for** $\mathbb{C}$ that agrees with the Euclidean metric on $\mathbb{R}^2$ (under the linear representation of the complex plane).

DEFINITION 2.1. We say that a sequence (indexed by $n \in \mathbb{Z}_{>0}$) $\{z_n\}$ of complex numbers *converges* to $\alpha \in \mathbb{C}$ if given $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{>0}$ such that $|z_n - \alpha| < \epsilon$ for all $n > N$; in this case we write

$$\lim_{n\to\infty} z_n = \alpha\ .$$

A sequence $\{z_n\}$ of complex numbers is called *Cauchy* if given $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{>0}$ such that $|z_n - z_m| < \epsilon$ for all $n, m > N$.

THEOREM 2.2. *If $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences of complex numbers, then*

(a) $\{z_n + \alpha\, w_n\}$ is Cauchy for all $\alpha \in \mathbb{C}$.
(b) $\{\overline{z}_n\}$ is Cauchy.
(c) $\{|z_n|\} \subset \mathbb{R}_{\geq 0}$ is Cauchy.

PROOF. (a) It suffices to assume that $\alpha \neq 0$. Given $\epsilon > 0$, choose N_1 such that $|z_n - z_m| < \frac{\epsilon}{2}$ for all $n, m > N_1$ and choose N_2 such that $|w_n - w_m| < \frac{\epsilon}{2|\alpha|}$ for all $n, m > N_2$. Choose $N = \max\{N_1, N_2\}$. Then for n and $m > N$, we have

$$|(z_n + \alpha\, w_n) - (z_m + \alpha\, w_m)| \leq |z_n - z_m| + |\alpha|\,|w_n - w_m| < \epsilon.$$

(b) $|\overline{z}_n - \overline{z}_m| = |z_n - z_m|$.

(c) Note that for all z and ζ in $\mathbb{C}$, we have

$$|z| = |z - \zeta + \zeta| \leq |z - \zeta| + |\zeta|,$$

and hence, we conclude that

$$||z| - |\zeta|| \leq |z - \zeta|.$$

Thus we have

$$||z_n| - |z_m|| \leq |z_n - z_m|.$$

□

REMARK 2.3. The above arguments mimic arguments in real analysis needed to establish the corresponding results for real sequences. We will, in the sequel, leave such routine arguments as exercises for the reader.

COROLLARY 2.4. *$\{z_n\}$ is a Cauchy sequence of complex numbers if and only if $\{\Re z_n\}$ and $\{\Im z_n\}$ are Cauchy sequences of real numbers.*

COROLLARY 2.5. *$(\mathbb{C}, d)$ is a complete metric space; that is, every Cauchy sequence of complex numbers converges to a complex number.*

PROOF. The metric on $\mathbb{C}$ restricts to the Euclidean metric on $\mathbb{R}$, which is complete. □

DEFINITION 2.6. Let $A \subseteq \mathbb{C}$. Define

$$\|A\| = \{|z|\,;\ z \in A\} \subset \mathbb{R}_{\geq 0}.$$

We say that A is *bounded* if and only if $\|A\|$ is; that is, if there exists a positive real number M such that $|z| < M$ for all z in A.

DEFINITION 2.7. Let $\zeta \in \mathbb{C}$ and $\epsilon > 0$. The *ϵ-ball about ζ* is the set

$$U_\zeta(\epsilon) = U(\zeta, \epsilon) = \{z \in \mathbb{C}; |z - \zeta| < \epsilon\}.$$

PROPOSITION 2.8. *A subset A of $\mathbb{C}$ is bounded if and only if there exists a $\zeta \in \mathbb{C}$ and an $R > 0$ such that $A \subset U(\zeta, R)$.*

REMARK 2.9. A proof is omitted for one of three reasons (in addition to the reason described in Remark 2.3): Either it is trivial, it follows directly from results in real analysis, or it appears as an exercise at the end of the corresponding chapter.[4] The third possibility is always labeled; when standard results in real analysis are needed, there is some indication of what they are and where to find them. It should

[4]Exercises can be found at the end of each chapter and are numbered by chapter, so that Exercise 2.7 is to be found at the end of Chapter 2.

be clear from the context when the first possibility occurs. It is recommended that the reader check that he/she can supply an appropriate proof when none is given.

THEOREM 2.10 (**Bolzano–Weierstrass Theorem**). *Every infinite bounded set S in $\mathbb{C}$ has at least one limit point; that is, there exists at least one $a \in \mathbb{C}$ such that for all $\epsilon > 0$, $U(a, \epsilon)$ contains a point $z \neq a, z \in S$. Such an a is called a limit point of S.*

THEOREM 2.11. *A set $K \subset \mathbb{C}$ is compact if and only if it is closed and bounded.*

DEFINITION 2.12. Let f be a function defined on a set S in $\mathbb{C}$. We always (unless otherwise stated) assume that f is complex valued. Thus, f may be viewed as a map from S into $\mathbb{R}^2$ or $\mathbb{C}$ and as two real-valued functions defined on the set S. Let ζ be a limit point of S and $\alpha \in \mathbb{C}$. Then

$$\lim_{z \to \zeta} f(z) = \alpha$$

if and only if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - \alpha| < \epsilon \text{ whenever } z \in S \text{ and } 0 < |z - \zeta| < \delta \,.$$

REMARK 2.13. In addition to the usual algebraic operations on pairs of functions $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ familiar from real analysis, such as $f + cg$ with $c \in \mathbb{C}$, fg, and $\frac{f}{g}$ (provided g does not vanish on S), we will consider other new functions constructed from a single function f. Among these

$$(\Re f)(z) = \Re f(z), \ (\Im f)(z) = \Im f(z), \ \overline{f}(z) = \overline{f(z)}, \ |f|\,(z) = |f(z)|\,.$$

THEOREM 2.14. *Let S be a set in $\mathbb{C}$, and let f and g be functions on S. Let ζ be a limit point of S. Then*

(a) $\lim_{z \to \zeta}(f + c\,g)(z) = \lim_{z \to \zeta} f(z) + c \lim_{z \to \zeta} g(z)$ *for all* $c \in \mathbb{C}$,

(b) $\lim_{z \to \zeta}(fg)(z) = \lim_{z \to \zeta} f(z) \lim_{z \to \zeta} g(z)$,

(c) $\lim_{z \to \zeta} |f|\,(z) = \left|\lim_{z \to \zeta} f(z)\right|$, *and*

(d) $\lim_{z \to \zeta} \overline{f}(z) = \overline{\lim_{z \to \zeta} f(z)}$.

REMARK 2.15. The usual interpretation of the above formulas is used here and in the rest of the book: The LHS[5] exists whenever the RHS exists, and then we have the stated equality.

[5]LHS (RHS) are standard abbreviations for left-(right)-hand side and will be used throughout this book.

COROLLARY 2.16. *Let* $u = \Re f$, $v = \Im f$ *(so that* $f(z) = u(z) + \imath\, v(z)$*) and* $\alpha \in \mathbb{C}$. *Then*

$$\lim_{z \to \zeta} f(z) = \alpha$$

if and only if

$$\lim_{z \to \zeta} u(z) = \Re\alpha \text{ and } \lim_{z \to \zeta} v(z) = \Im\alpha \ .$$

DEFINITION 2.17. Let S be a subset of $\mathbb{C}$, $f : S \to \mathbb{C}$ and ζ a limit point of S, which is also in S. We say that f is *continuous at* ζ if $\lim_{z \to \zeta} f(z) = f(\zeta)$, that f is *continuous on* S if it is continuous at each ζ in S, and that f is *uniformly continuous on* S if and only if for all $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(w)| < \epsilon \text{ for all } z \text{ and } w \text{ in } S \text{ with } |z - w| < \delta \ .$$

REMARK 2.18. Uniform continuity implies continuity.

THEOREM 2.19. *Let* f *and* g *be functions defined in appropriate sets; that is, sets where composition of these functions makes sense.*

(a) *If* f *is continuous at* ζ *and* $f(\zeta) \neq 0$, *then* $\frac{1}{f}$ *is defined in a neighborhood of* ζ *and is continuous at* ζ.

(b) *If* f *is continuous at* ζ *and* g *is continuous at* $f(\zeta)$, *then* $g \circ f$ *is continuous at* ζ.

THEOREM 2.20. *Let* $K \subset \mathbb{C}$ *be compact and* $f : K \to \mathbb{C}$ *be continuous. Then* f *is uniformly continuous on* K.

PROOF. A continuous mapping from a compact Hausdorff space to a metric space is uniformly continuous. □

DEFINITION 2.21. Given a sequence of functions $\{f_n\}$ all defined on the same set S in $\mathbb{C}$, we say that $\{f_n\}$ *converges uniformly* to a function f on S if for all $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{>0}$ such that

$$|f(z) - f_n(z)| < \epsilon \ \text{ for all } z \in S \text{ and all } n > N \ .$$

REMARK 2.22. $\{f_n\}$ converges uniformly on S if and only if for all $\epsilon > 0$, there exists an $N \in \mathbb{Z}_{>0}$ such that

$$|f_n(z) - f_m(z)| < \epsilon \ \text{ for all } z \in S \text{ and all } n \text{ and } m > N \ .$$

THEOREM 2.23. *Let* $\{f_n\}$ *be a sequence of functions on* S. *If*

(1) $\{f_n\}$ *converges uniformly on* S, *and*

(2) *each* f_n *is continuous on* S,

then the function f *defined by*

$$f(z) = \lim_{n \to \infty} f_n(z), \ z \in S \ ,$$

is continuous on S and $\{f_n\}$ converges uniformly to f on S.

PROOF. We start with two points z and ζ in S. Then for each n,

$$|f(z) - f(\zeta)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(\zeta)| + |f_n(\zeta) - f(\zeta)| \ .$$

Fix z and $\epsilon > 0$. By (1), the first and third term on the right-hand side are less than $\frac{\epsilon}{3}$ for n large. Fix n. By (2), the second term is less than $\frac{\epsilon}{3}$ as soon as ζ is close enough to z. □

DEFINITION 2.24. A *domain* or *region* in $\mathbb{C}$ is a subset of $\mathbb{C}$ that is open and connected.

REMARK 2.25. Note that a domain in $\mathbb{C}$ could also be defined as an open arcwise-connected subset of $\mathbb{C}$.

2.2. Differentiability and holomorphic mappings

The definition of the derivative of a complex-valued function of a complex variable mimics that for the derivative of a real-valued function of a real variable. We shall see that the properties of the two classes of functions are quite different.

DEFINITION 2.26. Let f be a function defined in some ball about $\zeta \in \mathbb{C}$. Assume $h \in \mathbb{C}$. We say that f is *(complex) differentiable at* ζ if and only if

$$\lim_{h \to 0} \frac{f(\zeta + h) - f(\zeta)}{h}$$

exists. In this case the limit is denoted by

$$f'(\zeta),\ \frac{df}{dz}(\zeta),\ \left.\frac{df}{dz}\right|_{z=\zeta},\ (Df)(\zeta)\ ,$$

and is called the *derivative of f at ζ*.

REMARK 2.27. (1) It is important that h is an arbitrary *complex* number (of small nonzero modulus) in the above definition.

(2) If f is differentiable at ζ, then f is continuous at ζ.

NOTATION 2.28. If the function f is differentiable on a domain D (that is, at each point of D), then it defines a function $f' : D \to \mathbb{C}$.

Thus for every $n \in \mathbb{Z}_{\geq 0}$, we can define inductively $f^{(n)}$, the n-th derivative of f, as follows:
$f^{(0)} = f$, and if $f^{(n)}$ is defined for $n \geq 0$, then we set $f^{(n+1)} = \left(f^{(n)}\right)'$ whenever the appropriate limits exist.

It is customary to abbreviate $f^{(2)}$ and $f^{(3)}$ by f'' and f''', respectively.

DEFINITION 2.29. Let f be a function defined in a neighborhood of $\zeta \in \mathbb{C}$. Then f is *holomorphic* or *analytic at* ζ if it is differentiable in a neighborhood (perhaps smaller) of ζ. A function defined on an (open) set U is *holomorphic* or *analytic on* U if it is holomorphic at each point of U.

A function f is *anti-holomorphic* if $\overline{f}$ is holomorphic.

The usual rules of differentiation hold. Let f and g be functions defined in a neighborhood of $\zeta \in \mathbb{C}$, let k be a function defined in a neighborhood of $f(\zeta)$, and let $c \in \mathbb{C}$. Then (recall Remark 2.15)

(a) $(f + cg)'(\zeta) = f'(\zeta) + cg'(\zeta)$,
(b) $(fg)'(\zeta) = f(\zeta)g'(\zeta) + f'(\zeta)g(\zeta)$,
(c) $(k \circ f)'(\zeta) = k'(f(\zeta))f'(\zeta)$,
(d) $\left(\frac{1}{f}\right)'(\zeta) = -\frac{f'(\zeta)}{f(\zeta)^2}$ provided $f(\zeta) \neq 0$, and
(e) for $f(z) = z^n$ ($n \in \mathbb{Z}$), $f'(z) = n\, z^{n-1}$ (for $n < 0$, $z \neq 0$).

DEFINITION 2.30. A function is called *entire* if it is holomorphic on $\mathbb{C}$.

EXAMPLE 2.31. (1) Every polynomial (in one complex variable) is entire.

(2) A rational function $R = \frac{P}{Q}$, where P and Q are polynomials with Q not the zero polynomial, is holomorphic on $\mathbb{C} - \{\text{zeros of } Q\}$. The polynomial Q has only finitely many zeros (the number of zeros, properly counted, equals the *degree* of Q; see Exercise 3.17).

(3) A special case of Example 2.31.2 is $R(z) = \frac{az+b}{cz+d}$ with a, b, c, and d fixed complex numbers satisfying $ad - bc = 1$. These rational functions are called *fractional linear transformations* or *Möbius transformations*, and they will be studied in detail in Section 8.1.

Convention. Whenever we write $z = x + \imath y$ for variables and $f = u + \imath v$ for functions, then we automatically mean that $x = \Re z$, $y = \Im z$, $u = \Re f$, and $v = \Im f$. We write $u = u(x, y)$ and $v = v(x, y)$ as well as $u = u(z)$ and $v = v(z)$.

THEOREM 2.32. *If $f = u + \imath v$ is differentiable at $c = a + \imath b$, then u and v have partial derivatives with respect to x and y at c, and these satisfy the Cauchy–Riemann equations (to be abbreviated CR):*

$$u_x(a, b) = v_y(a, b), \quad u_y(a, b) = -v_x(a, b). \tag{CR}$$

Proof. First take $h = \alpha$ and then $h = \imath\beta$ (with α and β real) in the definition of differentiability (2.26) and compute

$$f'(c) = u_x(a,b) + \imath\, v_x(a,b) = -\imath\, u_y(a,b) + v_y(a,b)\ .$$

□

We will also use the obvious notation $f_x = u_x + \imath v_x$ and $f_y = u_y + \imath v_y$.

Remark 2.33. The CR equations are not sufficient for differentiability. To see this, define

$$f(z) = \begin{cases} z^5\,|z|^{-4} & \text{for } z \neq 0, \\ 0 & \text{for } z = 0. \end{cases}$$

The function f is continuous on $\mathbb{C}$ and its real and imaginary parts satisfy the Cauchy–Riemann equations at $z = 0$, but it is not differentiable at $z = 0$. For α real and nonzero we have $\frac{f(\alpha)}{\alpha} = 1$, and for β real and nonzero we have $\frac{f(\imath\beta)}{\imath\beta} = 1$. Hence the CR equations are satisfied. Thus if the CR equations implied differentiability, we would conclude that $f'(0) = 1$. Now take $h = (1+\imath)\gamma$ with γ real and nonzero and observe that $\frac{f(h)}{h} = -1$ so that $f'(0)$ would be -1.

In Exercise 2.8, we introduce the complex partial derivatives f_z and $f_{\overline{z}}$ of C^1-complex-valued functions[6] f defined on a region in the complex plane. These partials not only simplify the notation: For example, the two equations given in (CR) are written as the single equation

$$f_{\overline{z}} = 0\,, \qquad \text{(CR complex)}$$

but they allow us to produce more concise arguments (and as we shall see later prettier formulas), as illustrated in the proof of the lemma below. We also use the notation $\frac{\partial f}{\partial \overline{z}}$ interchangeably with $f_{\overline{z}}$.

Lemma 2.34. *If f is a C^1-complex-valued function defined in a neighborhood of $c \in \mathbb{C}$, then for $z \in \mathbb{C}$ with $|z-c|$ small,*

$$f(z) - f(c) = (z-c)f_z(c) + \overline{(z-c)}f_{\overline{z}}(c) + |z-c|\,\varepsilon(z,c), \qquad (2.1)$$

where $\varepsilon(z,c)$ is a complex-valued function of z and c with

$$\lim_{z \to c} \varepsilon(z,c) = 0\ .$$

[6] C^1-complex-valued functions may be defined as functions whose real and imaginary parts have continuous first partial derivatives.

PROOF. As usual we write $z = x + \imath y$, $c = a + \imath b$, and $f = u + \imath v$ and abbreviate $\triangle u = u(z) - u(c)$, $\triangle x = x - a$, $\triangle y = y - b$, and $\triangle z = z - c = \triangle x + \imath \triangle y$.

By hypothesis, the real-valued function u has continuous first partial derivatives defined in a neighborhood of c, and we can define ε_1 by

$$\varepsilon_1(z,c) = \frac{\triangle u - u_x(c)\triangle x - u_y(c)\triangle y}{|\triangle z|} \ .$$

We show that

$$\lim_{z\to c} \varepsilon_1(z,c) = 0 \ . \tag{2.2}$$

If we rewrite $\triangle u$ as

$$\triangle u = [u(x,y) - u(x,b)] + [u(x,b) - u(a,b)] \ ,$$

it follows from the (real) mean value theorem that

$$\text{RHS} = u_y(x,y_0)\triangle y + u_x(x_0,b)\triangle x \ ,$$

where y_0 is between y and b and x_0 is between x and a. Thus

$$\varepsilon_1(z,c) = \frac{[u_y(x,y_0) - u_y(a,b)]\triangle y + [u_x(x_0,b) - u_x(a,b)]\triangle x}{|\triangle z|} \ .$$

Hence we see that

$$|\varepsilon_1(z,c)| \le |u_y(x,y_0) - u_y(a,b)| + |u_x(x_0,b) - u_x(a,b)| \ .$$

Thus we have shown that

$$u(z) - u(c) = (x-a)u_x(a,b) + (y-b)u_y(a,b) + |z-c|\,\varepsilon_1(z,c) \ ,$$

with (2.2).

Similarly,

$$v(z) - v(c) = (x-a)v_x(a,b) + (y-b)v_y(a,b) + |z-c|\,\varepsilon_2(z,c) \ ,$$

with

$$\lim_{z\to c} \varepsilon_2(z,c) = 0. \tag{2.3}$$

With obvious notational conventions,

$$\begin{aligned}\triangle f &= \triangle u + \imath\triangle v = \\ &(u_x(a,b) + \imath v_x(a,b))\,\triangle x + (u_y(a,b) + \imath v_y(a,b))\,\triangle y + |\triangle z|\,\varepsilon(z,c) \\ &= \frac{\triangle z + \overline{\triangle z}}{2} f_x(c) + \imath\frac{\overline{\triangle z} - \triangle z}{2} f_y(c) + |\triangle z|\,\varepsilon(z,c) \\ &= \triangle z f_z(c) + \overline{\triangle z} f_{\bar z}(c) + |\triangle z|\,\varepsilon(z,c).\end{aligned}$$

Since $\varepsilon(z,c) = \varepsilon_1(z,c) + \imath\,\varepsilon_2(z,c)$, equalities (2.2) and (2.3) imply that

$$\lim_{z\to c} \varepsilon(z,c) = 0.$$

□

THEOREM 2.35. *If the function f has continuous first partial derivatives in a neighborhood of c that satisfy the CR equations at c, then f is (complex) differentiable at c.*

PROOF. The theorem is an immediate consequence of (2.1) since in this case $f_{\bar{z}}(c) = 0$ and hence $f'(c) = f_z(c)$. □

COROLLARY 2.36. *If the function f has continuous first partial derivatives in an open neighborhood U of $c \in \mathbb{C}$ and the CR equations hold at each point of U, then f is holomorphic at c (in fact on U).*

REMARK 2.37. The converse is also true. It will take us some time to prove it.

THEOREM 2.38. *If f is holomorphic and real valued on a domain D, then f is constant.*

PROOF. As usual we write $f = u + \imath\,v$; in this case $v = 0$. The CR equations say $u_x = v_y = 0$ and $u_y = -v_x = 0$. Thus u is constant. □

THEOREM 2.39. *If f is holomorphic and $f' = 0$ on a domain D, then f is constant.*

PROOF. As above $f = u + \imath\,v$ and $f' = u_x + \imath v_x = 0$. The last equation together with the CR equations say $0 = u_x = v_y$ and $0 = v_x = -u_y$. Thus both u and v are constant. □

Exercises

2.1. (a) Let $\{z_n\}$ be a sequence of complex numbers and assume

$$|z_n - z_m| < \frac{1}{1 + |n-m|}\ , \quad \text{for all } n \text{ and } m\ .$$

Show that the sequence converges.
Do you have enough information to evaluate $\lim_{n\to\infty} z_n$?
What more can you say about this sequence?

(b) Let $\{z_n\}$ be a sequence with $\lim_{n\to\infty} z_n = 0$, and let $\{w_n\}$ be a bounded sequence. Show that

$$\lim_{n\to\infty} w_n z_n = 0\ .$$

2.2. Verify the Cauchy–Riemann equations for the function $f(z) = z^3$ by splitting f into its real and imaginary parts.

2.3. Let $x = r\cos\theta,\ y = r\sin\theta$. Show that the Cauchy–Riemann equations in polar coordinates for $F = U + \imath V$, $(U = U(r,\theta), V = V(r,\theta))$ are

$$r\,\frac{\partial U}{\partial r} = \frac{\partial V}{\partial \theta} \text{ and } r\,\frac{\partial V}{\partial r} = -\frac{\partial U}{\partial \theta}.$$

Alternatively, one can write

$$rU_r = V_\theta, \text{ and } rV_r = -U_\theta.$$

2.4. Suppose $z = x + \imath\, y$. Define

$$f(z) = \frac{xy^2\,(x + \imath\, y)}{x^2 + y^4}\ ,$$

for $z \neq 0$, and $f(0) = 0$.

Show that

$$\lim \frac{f(z) - f(0)}{z} = 0$$

as $z \to 0$ along any straight line. Show that as $z \to 0$ along the curve $x = y^2$, the limit of the difference quotient is $\frac{1}{2}$, thus showing that $f'(0)$ does not exist.

2.5. Does there exist a holomorphic function f on $\mathbb{C}$ whose real part is

(a) $u\,(x, y) = e^x$?
(b) or $u\,(x, y) = e^x(x\cos y - y\sin y)$?

Justify your answer. That is, if yes, exhibit the holomorphic function(s); if not, prove it.

2.6. Prove the *Fundamental Theorem of Algebra*: If $a_0\,, \ldots\,, a_{n-1}$ are complex numbers $(n \geq 1)$ and $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$, then there exists a number $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Hints:

(a) Show there is an $M > 0$ and an $R > 0$ so that $|p(z)| \geq M$ for $|z| \geq R$.
(b) Show next that there is a $z_0 \in \mathbb{C}$ such that

$$|p(z_0)| = \min\{|p(z)|\,; z \in \mathbb{C}\}\ .$$

(c) By the change of variable $p(z + z_0) = g(z)$, it suffices to show that $g(0) = 0$.

(d) Write $g(z) = \alpha + z^m(\beta + c_1 z + \ldots + c_{n-m} z^{n-m})$ with $\beta \neq 0$. Choose γ such that

$$\gamma^m = -\frac{\alpha}{\beta}.$$

If $\alpha \neq 0$, obtain the contradiction $|g(\gamma z)| < |\alpha|$ for some z.

Note. We will later have a simpler proof of this theorem using results from complex analysis. See Theorem 5.15. See also the April 2006 issue of *The American Mathematical Monthly* for still other proofs of this fundamental result.

2.7. Using the Fundamental Theorem of Algebra stated in Exercise 2.6, prove *Frobenius Theorem*: If F is a field containing the reals and such that the dimension of F as a real vector space is finite, then either F is the reals or F is (isomorphic to) $\mathbb{C}$.

Hints:

(a) Assume $\dim_{\mathbb{R}} F = n > 1$. Show that for θ in $F - \mathbb{R}$ there exists a nonzero real polynomial p with leading coefficient 1 and such that $p(\theta) = 0$.

b) Show that there exist real numbers β and γ such that

$$\theta^2 - 2\beta\theta + \gamma = 0.$$

c) Show that there exists a positive real number δ such that $(\theta - \beta)^2 = -\delta^2$, and therefore,

$$\sigma = \frac{\theta - \beta}{\delta}$$

is an element of F satisfying $\sigma^2 = 1$.

d) The field

$$G = \mathbb{R}(\sigma) = \{x + y\sigma : x, y \in \mathbb{R}\} \subseteq F$$

is isomorphic to $\mathbb{C}$, so without loss of generality, assume $\sigma = \imath$ and $G = \mathbb{C}$. Conclude by showing that any element of F is the root of a complex polynomial with leading coefficient 1 and is therefore a complex number.

2.8. Let f be a complex-valued function defined on a region in the complex plane, and assume that both f_x and f_y exist in this region. Define:

$$f_z = \frac{1}{2}\left(f_x - \imath f_y\right)$$

and

$$f_{\overline{z}} = \frac{1}{2}\left(f_x + \imath f_y\right).$$

Show that for C^1-functions f,

$$f \text{ is holomorphic if and only if } f_{\overline{z}} = 0\,,$$

and that in this case $f_z = f'$.

2.9. Let R and Φ be two real-valued C^1-functions of a complex variable z. Show that $f = Re^{\imath\Phi}$ is holomorphic if and only if

$$\frac{R_{\overline{z}}}{R} = -\imath\Phi_{\overline{z}}.$$

2.10. Show that if f and g are C^1-functions, then the (complex) chain rule is expressed as follows (here $w = f(z)$ and g is viewed as a function of w):

$$(g \circ f)_z = g_w\, f_z + g_{\overline{w}}\, \overline{f}_z$$

and

$$(g \circ f)_{\overline{z}} = g_w\, f_{\overline{z}} + g_{\overline{w}}\, \overline{f}_{\overline{z}}\ .$$

2.11. Let p be a complex-valued polynomial of two real variables:

$$p(z) = \sum a_{ij} x^i y^j\ .$$

Write

$$p(z) = \sum_{j \geq 0} P_j(z) \overline{z}^j,$$

where each P_j is of the form $P_j(z) = \sum b_{ij} z^i$. Prove that p is an entire function if and only if

$$0 \equiv P_1 \equiv P_2 \equiv \ldots .$$

2.12. (a) Given two points z_1, z_2 such that $|z_1| < 1$ and $|z_2| < 1$, show that for every point $z \neq 1$ in the closed triangle with vertices z_1, z_2 and 1,

$$\frac{|1 - z|}{1 - |z|} \leq K,$$

where K is a constant that depends only on z_1 and z_2.

(b) Determine the smallest value of K for $z_1 = \frac{1+\imath}{2}$ and $z_2 = \frac{1-\imath}{2}$.

2.13. Deduce the analogs of the CR equations for anti-holomorphic functions, in rectangular, polar, and complex coordinates.

2.14. Let D be an arbitrary (nonempty) open set in $\mathbb{C}$. Describe the class of complex-valued functions on D that are both holomorphic and anti-holomorphic.

2.15. (a) Every automorphism of the real field is the identity.

(b) Every continuous automorphism of the complex field is either the identity or the conjugation.