Part I

Relevant logic and its semantics

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Excerpt
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1 What is relevant logic and why do we need it?

1.1 Non-sequiturs are bad

The central aim of relevant logicians has been to give a more intuitive characterisation of deductive inference. Consider the following example. Since 1993, when Andrew Wiles completed his difficult proof of Fermat's Last Theorem, mathematicians have wanted a shorter, easier proof. Suppose when someone addressing a conference of number theorists, suggests the following proof of the theorem:

The sky is blue.

$\therefore$ There is no integer n greater than or equal to 3 such that for any non-zero integers $x, y, z,\ x^n = y^n + z^n$.

This proof would not be well received. It is a non-sequitur – its conclusion does not, in any intuitive sense, follow from its premise. It is a bad proof.

But let's think about this a little harder. According to the standard definition of 'validity', an argument is valid if and only if it is impossible for the premises all to be true and the conclusion false. The conclusion is not just true; it is necessarily true. All truths of mathematics are necessary truths. Thus, it is impossible for the conclusion to be false. So, it is impossible for the premise to be true and the conclusion to be false. Therefore the argument is valid. Moreover, the proof is sound, since its premise is also true. But it is clear that it is a bad argument and no one would take it seriously. Therefore, we need some notion other than the standard definition of validity to determine what is wrong with this proof. I suggest that what is wrong is that the standard notion of validity is too weak to provide a vertebrate distinction between good and bad arguments. It allows too many non-sequiturs to be classified as good arguments.

The standard notion of validity is at the heart of the logical theory known as *classical logic*. Classical logic is the sort of logic that students learn in introductory symbolic logic courses. Since we deal almost exclusively with propositional logic in this book, it will suffice here to discuss only the classical propositional calculus. The method of determining validity that we teach first to students is the method of truth tables. We first list the propositional variables that occur in the premises and conclusion and then we list each of the premises and the conclusion, all along a top row. Then we list in each subsequent row one combination of truth values for the propositional variables. The following is a schematic truth table for an inference with two propositional variables, premises $A_1, \ldots, A_n$ and conclusion B:

p	q	$A_1, \ldots A_n$	B
T	T		
T	F	$\vdots$	$\vdots$
F	T		
F	F		

Each row of the truth table defines a 'possibility'[1] and determines the truth value of each premise and conclusion in each of these possibilities. An argument is valid, according to the classical propositional calculus, if and only if in every row in which every premise is true the conclusion is also true.

Now, if we have a conclusion that is true in every possibility, then the argument is valid, regardless of what the premises say. Consider, then, a statement that we can prove to be true in every possibility on the truth tables, such as, $q \vee {\sim}q$. On the classical account of validity, the following argument is valid:

$$\frac{p}{\therefore q \vee {\sim}q}$$

Thus, if we set p to mean 'my dog barks at rubbish collectors' and q to mean 'it is raining in Bolivia right now' we find out that the inference

My dog barks at rubbish collectors
———
∴ Either it is raining in Bolivia right now or it is not.

is valid. Since the premise is true, the argument is also sound. But like the first inference that we investigated, it is a very bad argument. Thus, the classical notion of validity does not agree with our pre-logical intuitions about where the division between good arguments and non-sequiturs should be. Classical logic allows connections between premises and conclusions in valid arguments that are extremely loose. There needs to be more of a connection between the content of the premises and conclusion in an argument that we are prepared to call 'valid'.

[1] A possibility here is not a possible world. The rows of truth table are far too under-specified to determine a possible world. We will look at possible worlds in some depth in chapter 2 below.

1.2 The classical theory of consequence

We can avoid the problem given above if we think about the issue in syntactic, rather than semantic terms. In a mathematical proof, for example, we could demand that only obvious axioms and a few primitive rules are to be used. If we take this line, then the 'proof' of Fermat's last theorem given above would be rejected. Similarly, we could demand that an argument such as the one from 'My dog barks at rubbish collectors' to 'Either it is raining in Bolivia right now or it is not' be recast in terms of a system of proof, such as a natural deduction system, that also contains only a few rules.

But there are still problems with the classical view of consequence, even treated in this form. To see these problems, let's consider a standard proof theory for classical logic, of the sort that we teach to students in an introductory course on symbolic logic. The system that we consider here is a 'Fitch-style' natural deduction system (Fitch 1952). Readers who are familiar with this type of natural deduction system can skim this section. But don't skip it altogether, for there are some points made in it that we will build on later.

The structures that we construct in a natural deduction system are proofs. The first step in a proof is a formula that is a hypothesis. Every subsequent step in the proof is also a hypothesis or it is a formula that is derived from previous steps by means of one of the rules of the system. Each hypothesis introduces a subproof of the proof. The notion of a proof and a subproof are relative. Subproofs themselves can have subproofs. Subproofs here are indicated by means of vertical lines, as we shall see below.

There are three rules of proof that we will use that do not involve any logical connective, such as conjunction, disjunction, implication, or negation. The first of these is a rule that allows us to introduce a formula as a premise, or hypothesis. The other two rules allow us to copy formulae in proofs. The rule of repetition (*rep*) allows us to copy a line of a proof within that proof and the rule of reiteration (*reit*) allows us to copy a line from a proof into any of its subproofs (or subproofs of subproofs, and so on). This will all become clearer when we look at some examples of proofs. But before we do so, we need to discuss the rules governing the connectives.

Each connective has two rules associated with it – an introduction rule and an elimination rule. In this chapter, we will only be concerned with the implication connective ($\rightarrow$). The elimination rule for implication is[2]

> ($\rightarrow E$) From $A \rightarrow B$ and A to infer B.

The rule of implication introduction is a little more difficult. It says that

> ($\rightarrow I$) Given a proof of B from the hypothesis A,
> one may infer $A \rightarrow B$.

[2] I present rules of inference in the style of (Anderson *et al.* 1992).

Consider, for example, a proof of the formula $A \to ((A \to B) \to B)$:

1.	$\mid A$	*hyp*
2.	$\mid \; \mid A \to B$	*hyp*
3.	$\mid \; \mid A$	1, *reit*
4.	$\mid \; \mid B$	$2, 3, \to E$
5.	$\mid (A \to B) \to B$	$2-4, \to I$
6.	$A \to ((A \to B) \to B)$	$1-5, \to I$

Now let's take a look at an easier proof, this time for the scheme, $A \to A$:

1.	$\mid A$	*hyp*
2.	$\mid A$	1, *rep*
3.	$A \to A$	$1-2, \to I$

There seems to be nothing objectionable about this proof. But, according to the classical theory of consequence, we can convert it into a proof of $A \to A$ from the *irrelevant* hypothesis B as follows:

1.	$\mid B$	*hyp*
2.	$\mid \; \mid A$	*hyp*
3.	$\mid \; \mid A$	2, *rep*
4.	$\mid A \to A$	$2-3, \to I$

Thus, for example, we can prove from 'The sky is blue' that 'Ramsey is a dog implies that Ramsey is a dog.' The former has nothing to do with the latter.

What the above proof illustrates is that in the classical theory of consequence we can add any premise to a valid deduction and obtain another valid deduction. Thus we can add completely irrelevant premises. This seems to be far too generous a notion of consequence and the classical notion of implication seems to capture a far looser relationship between propositions than does our common notion of implication.

Let us go on to see how relevant logic is supposed to remedy this situation.

1.3 The real use of premises

The problem with non-sequiturs like the ones discussed above is that some of the premises of the inferences appear to have nothing to do with the conclusion. Relevant logic attempts to repair this problem by placing a constraint on proofs that the *premises really be used in the derivation of the conclusion.*

We will present this idea in the context of Anderson and Belnap's natural deduction system for relevant logic. The idea is pretty simple. Each premise, or rather hypothesis, in a proof is indexed by a number. The various steps in a proof are indexed by the numbers of the hypotheses which are used to derive the steps. For example, the following is a valid argument in this system:

1.	$A \to B_{\{1\}}$	*hyp*
2.	$\quad A_{\{2\}}$	*hyp*
3.	$\quad A \to B_{\{1\}}$	1, *reit*
4.	$\quad B_{\{1,2\}}$	$2, 3, \to E$

The numbers in curly brackets are indices. The indexing system allows us to do away with the scope lines, since the numbers indicate in which subproof a step is contained. Removing the scope lines, we get the following proof:[3]

1.	$A \to B_{\{1\}}$	*hyp*
2.	$A_{\{2\}}$	*hyp*
3.	$B_{\{1,2\}}$	$1, 2, \to E$

Here we can see, without scope lines, that there are two subproofs, since there are two indices (hence two hypotheses) used in the proof. Moreover, we do not need to use the reiteration rule in this proof. The indices make sure that when we use a step from a proof and one from a subproof in combination, as we have done here in the modus ponens that gives us step 3, we do so in the subproof. This is indicated by the fact that the number of the later hypothesis appears in the index of the conclusion.

Indices are important for tracking which conclusions depend on which hypotheses. They help to eliminate non-sequiturs, like the one presented in the introductory section above, and they guide the way in which premises can be discharged in conditional proofs, which are the topic of the next section.

1.4 Implication

In natural deduction systems we do not usually merely display proofs with hypotheses. We discharge premises to prove theorems of a system. The key rule that we will use here is the rule of conditional proof, or $\to I$ (implication introduction), viz.,

> From a proof of B_α from the hypothesis $A_{\{k\}}$ to infer $A \to B_{\alpha-\{k\}}$, where k occurs in α.

The proviso that k occur in α is essential. It ensures that, in this case, A is really used in the derivation of B. And so relevant implication also captures this notion of the real use of a hypothesis in bringing about a conclusion. The

[3] Readers familiar with E. J. Lemmon's introductory logic book (Lemmon 1965), will recognise this style of proof. Lemmon's system, however, is set up for classical logic rather than relevant logic. One main difference is that the two systems have different conjunction introduction rules.

reason that this rule is key in proving theorems is that it allows us to remove numbers from an index. In the terminology of natural deduction, it allows us to discharge hypotheses.

This rule of implication introduction forces the antecedent and consequent of a provable implication to have something to do with one another. Suppose that $A \to B$ is provable in this system. Then, it can be shown that A and B have some 'non-logical content' in common. That is, they share at least one propositional variable. This is called the 'variable-sharing property' or the 'relevance property'.

In previous sections, we have been discussing the problem of logical relevance in terms of inference, but we can also think about it in terms of implication. Relevant logic was developed in part to avoid the so-called paradoxes of material and strict implication. These are formulae that are theorems of classical logic, but are counterintuitive when we think of the arrow as meaning 'implication' in any ordinary sense of the term, or any pre-logical philosophical sense.

Let us look first at the paradoxes of material implication. Among them are the following:

M1 $A \to (B \to A)$ (positive paradox);

M2 $\sim A \to (A \to B)$;

M3 $(A \to B) \vee (B \to A)$;

M4 $(A \to B) \vee (B \to C)$.

These schemes are paradoxical because they seem so very counterintuitive. For example, take paradox M3. If we accept the scheme $(A \to B) \vee (B \to A)$, then we are committed to holding that for any two propositions one implies the other. But this violates our preformal notion of implication. The problem with material implication *as a way of representing our pre-theoretical notion of implication* is that it is truth functional. The two truth values by themselves are not enough to determine when an implication holds. Something else is needed.

Could this something else be necessity? That is, should we follow C. I. Lewis in claiming that implication is really strict implication? It would seem not, for strict implication has its own menu of paradoxes. Among these are:

S1 $A \to (B \to B)$;

S2 $A \to (B \vee \sim B)$;

S3 $(A \wedge \sim A) \to B$ (ex falso quodlibet).

S1 and S2 are instances of the general rule regarding strict implication that all valid formulae follow from any proposition. But according to our pre-theoretical logical intuitions, it does not seem that we are justified in inferring any logical truth from any proposition. Similarly, it does not seem that we are justified in inferring any proposition from any logical falsehood. But this is exactly what ex falso quodlibet says.

Lewis and C. H. Langford give an 'independent argument' for ex falso quodlibet. Here I reproduce their argument with only minor notational changes.

From a proposition of the form $A \wedge \sim A$, any proposition whatever, B, may be deduced as follows:

Assume $A \wedge \sim A$ (1)

(1) $\rightarrow A$ (2)

If A is true and A is false, then A is true.

(1) $\rightarrow \sim A$ (3)

If A is true and A is false, then A is false.

(2) $\rightarrow (A \vee B)$ (4)

If, by (2), A is true, then at least one of the two, A and B, is true.

((3) $\wedge$ (4)) $\rightarrow B$

If, by (3), A is false; and, by (4), at least one of the two, A and B, is true; then B must be true. ((Lewis and Langford 1959), p. 250)

Is this argument good? I think not.

It seems that Lewis had already seen what was wrong with the argument in his 1917 article 'The Issues Concerning Material Implication' (Lewis 1917). There he sets out a dialogue between two characters – X and himself (L). Here is a relevant part of that dialogue:

L. But tell me: do you admit that 'Socrates was a solar myth' materially implies $2 + 2 = 5$?

X. Yes; but only because Socrates was not a solar myth.

L. Quite so. But if Socrates were a solar myth, would it be true that $2 + 2 = 5$? If you granted some paradoxer his assumption that Socrates was a solar myth, would you feel constrained to go on and grant that $2 + 2 = 5$?

X. I suppose you mean to harp on 'irrelevant' some more. ((Lewis 1917), p. 355)

Let's apply this line of reasoning to the Lewis and Langford argument given above. Suppose that we 'grant some paradoxer his assumption' that $A \wedge \sim A$ is true. We then have to take seriously what would happen in a context in which a contradiction is true. We will explore this issue in depth in chapters 5 and 10 below. We will argue in chapter 10 that it is the last step – the use of disjunctive syllogism – that is unwarranted. If we allow that a proposition and its negation can be both true in the same context, then we cannot infer from $\sim A$ and $A \vee B$ that B is true. For it might be that $A \vee B$ is made true by the fact that A is true alone, despite the fact that B fails to be true.

We will leave topics concerning negation to later. To understand fully the treatment of negation in relevant logic, we need to understand its semantics. Other paradoxes, such as positive paradox are treated by the relevant logician's insistence that the hypotheses in a proof really be used in that proof. Consider the following attempt at a proof of positive paradox:

1.	$A_{\{1\}}$	*hyp*
2.	$B_{\{2\}}$	*hyp*
3.	$A_{\{1\}}$	1, *reit.*
4.	$B \rightarrow A_{\{1\}}$	$2-3, \rightarrow I$
5.	$A \rightarrow (B \rightarrow A)_{\emptyset}$	$1-4, \rightarrow I$

The illegitimate move here is the use of implication introduction in the fourth step. 2 does not belong to {1} and so we cannot discharge the second hypothesis here.

1.5 Implication and entailment

In the preceding sections, we have been discussing the notion of implication. As we have said, the relation of implication holds between two propositions if the consequent follows from the antecedent. By itself, this statement does not give us a very good interpretation of implication. It is rather ambiguous. As we shall see throughout this book, there are many different ways in which propositions can follow from one another. The matter is made worse by the fact that various philosophers use the word 'implication' to refer to different relations.

In chapter 3 below, we will make our use of 'implication' quite precise. But for the moment, we will note that the relation of implication, as we use this term here, is a *contingent* relation. What we mean by a contingent relation can be understood most clearly if it is contrasted to the corresponding necessary relation, which is known as *entailment*. Consider the scheme,

The proposition p follows from $p \wedge q$.

The notion of 'following from' here can be taken to be a very strong relationship.[4] For we can derive by logical means alone the proposition p from the conjunction $p \wedge q$. The connection between p and $p \wedge q$ is a necessary connection. Now consider the following statement:

A violation of New Zealand law follows from not paying income tax on honoraria given for presenting seminars at other universities.

Here the relationship between not paying income tax on an honorarium and a violation of the New Zealand tax code is not a necessary connection. We can easily imagine a world in which the tax code were different such that it made honoraria tax exempt income. When we make claims like the one above, we do so assuming other facts that connect the failure to pay tax and a violation

[4] I use the phrase 'can be taken to be' here because if a strong relationship holds, then every weaker relation holds as well.

of the law, that is, particular facts about the New Zealand tax code. Thus, the way in which the violation of the law and the failure to pay tax are connected is contingent. We say that the relationship between the axioms of arithmetic and $2 + 2 = 4$ is that of *entailment* and the relationship between the failure to pay tax and the violation of the law is that of *implication*.

Like implication, entailment is a relevant conditional. Some philosophers have held that entailment is just necessary material implication. That is, the proposition that A entails B is interpreted as $\Box(A \supset B)$, where the box means 'it is necessity that' and the hook is material implication. But on this analysis every proposition entails every truth-table tautology and every proposition is entailed by every contradiction. These are no less counterintuitive in the case of entailment than they were in the case of implication.

In chapter 3 below, we develop an interpretation of implication that makes sense of the notion of propositions following one another contingently. The theory is called the theory of *situated inference*. The theory gives an interpretation of relevant implication in terms of inferential connections between parts of the world. Very briefly, it says that an implication, $A \rightarrow B$, is true in a part of the world if there is information in that situation that tells us that if A is true in some part of the world, then B is also true in some part of the world. These parts of the world are *situations*. I shall show how the various elements of the natural deduction system for relevant logic can be interpreted in this situational framework.

In chapter 6, we will use the theory of situated inference as a basis for a theory of entailment. I follow Anderson, Belnap and many others in holding that entailment incorporates implication and necessity. Thus, when we add modality to our semantics, we get a semantics for entailment as well. As we shall see, the issues involved are more complicated than this, but that is the essence of the view.

1.6 Relevance and conditionals

Although they are important notions, implication and entailment are related to a class of connectives that are much more prevalent in everyday discourse. These are the natural language conditionals. Natural language conditionals have very different properties from implication and entailment. Implication and entailment, for example, are transitive. Consider the following proof:

1.	$A \rightarrow B_{\{1\}}$	*hyp*
2.	$B \rightarrow C_{\{2\}}$	*hyp*
3.	$A_{\{3\}}$	*hyp*
4.	$B_{\{1,3\}}$	$1, 3, \rightarrow E$
5.	$C_{\{1,2,3\}}$	$2, 4, \rightarrow E$
6.	$A \rightarrow C_{\{1,2\}}$	$3-5, \rightarrow I$