Preface

The aim of these notes is to describe, in a unified fashion, a set of methods for the simplification of a wide variety of problems that all share the common feature of possessing multiple scales.[1] The mathematical methods we study are often referred to as the methods of **averaging** and of **homogenization**. The methods apply to partial differential equations (PDEs), stochastic differential equations (SDEs), ordinary differential equations (ODEs), and Markov chains. The unifying principle underlying the collection of techniques described here is the approximation of **singularly perturbed** linear equations. The unity of the subject is most clearly visible in the application of perturbation expansions to the approximation of these singular perturbation problems. A significant portion of the notes is devoted to such perturbation expansions. In this context we use the term **Result** to describe the conclusions of a formal perturbation argument. This enables us to derive important approximation results without the burden of rigorous proof, which can sometimes obfuscate the main ideas. However, we will also study a variety of tools from analysis and probability, used to place the approximations derived on a rigorous footing. The resulting theorems are proved using a range of methods, tailored to different settings. There is less unity to this part of the subject. As a consequence, considerable background is required to absorb the entire rigorous side of the subject, and we devote a significant part of the book to this background material.

The first part of the notes is devoted to the **Background**; the second to the **Perturbation Expansions**, which provide the unity of the subject matter; and the third to the **Theory** justifying these perturbative techniques. We do not necessarily recommend that the reader covers the material in this order. A natural way to get an overview of the subject is to read through Part II of the book on perturbation

[1] In this book we will apply the general methodology to problems with two widely separated characteristic scales. The extension to systems with many separated scales is fairly straightforward and will be discussed in a number of the Discussion and Bibliography sections, which conclude each chapter. In all cases, the important assumption will be that of scale separation.

expansions, referring to the background material as needed. The theory can then be studied, after the form of the approximations is understood, on a case-by-case basis.

Part I (Background) contains the elements of the theory of analysis, probability, and stochastic processes, as required for the material in these notes, together with basic introductory material on ODEs, Markov chains, SDEs, and PDEs. Part II (Perturbation Expansions) illustrates the use of ideas from averaging and homogenization to study ODEs, Markov chains, SDEs, and PDEs of elliptic, parabolic, and transport type; invariant manifolds are also discussed and are viewed as a special case of averaging. Part III (Theory) contains illustrations of the rigorous methods that may be employed to establish the validity of the perturbation expansions derived in Part II. The chapters in Part III relate to those in Part II in a one-to-one fashion. It is possible to pick particular themes from this book and cover subsets of chapters devoted only to those themes. The reader interested primarily in SDEs should cover Chapters 6, 10, 11, 17, and 18. Markov chains are covered in Chapters 5, 9, and 16. The subject of homogenization for elliptic PDEs is covered in Chapters 12 and 19. Homogenization and averaging for parabolic and transport equations are covered in Chapters 13, 14, 20, and 21.

The subject matter in this set of notes has, for the most part, been known for several decades. However, the particular presentation of the material here is, we believe, particularly suited to the pedagogical goal of communicating the subject to the wide range of mathematicians, scientists, and engineers who are currently engaged in the use of these tools to tackle the enormous range of applications that require them. In particular we have chosen a setting that demonstrates quite clearly the wide applicability of the techniques to PDEs, SDEs, ODEs, and Markov chains, as well as highlighting the unity of the approach. Such a wide-ranging setting is not undertaken, we believe, in existing books, or is done so less explicitly than in this text. We have chosen to use the phrasing **Multiscale Methods** in the title of the book because the material presented here forms the backbone of a significant portion of the amorphous field that now goes by that name. However, we recognize that there are vast parts of the field we do not cover. In particular, scale separation is a fundamental requirement in all of the perturbation techniques presented in this book. Many applications, however, possess a continuum of scales, with no clear separation. Furthermore, many of the problems arising in multiscale analysis are concerned with the interfacing of different mathematical models appropriate at different scales (such as quantum, molecular, and continuum); the tools presented in these notes do not directly address problems arising in such applications, as our starting point is a single mathematical model in which scale separation is present.

These notes are meant to be an introduction, aimed primarily toward graduate students. Part I of the book (where we lay the theoretical foundations) and Part III (where we state and prove theorems concerning simplified versions of the models studied in Part II) are necessarily terse; otherwise it would be impossible to present the wide range of applications of the ideas and illustrate their unity. Extensions and generalizations of the results presented in these notes, as well as references to the literature, are given in the Discussion and Bibliography section at the end of each chapter. With the exception of Chapter 1, all chapters are supplemented with exercises.

We hope that the format of the book will make it appropriate for use both as a textbook and for self-study.

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