Introduction

In this work, we will deal with *standard determinantal ideals*, *determinantal ideals*, and *symmetric determinantal ideals*, i.e., ideals generated by the maximal minors of a homogeneous polynomial matrix, by the minors (not necessarily maximal) of a homogeneous polynomial matrix, and by the minors of a homogeneous symmetric polynomial matrix, respectively. Some classical ideals that can be constructed in this way are the homogeneous ideal of Segre varieties, the homogeneous ideal of rational normal scrolls, and the homogeneous ideal of Veronese varieties.

Standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals have been a central topic in both commutative algebra and algebraic geometry and they also have numerous connections with invariant theory, representation theory and combinatorics. Due to their important role, their study has attracted many researchers and has received considerable attention in the literature. Some of the most remarkable results are due to J.A. Eagon and M. Hochster [20] and to J.A. Eagon and D.G. Northcott [21]. J.A. Eagon and M. Hochster proved that generic determinantal ideals are Cohen–Macaulay while the Cohen–Macaulayness of symmetric determinantal ideals was proved by R. Kutz in [62, Theorem 1]. J.A. Eagon and D.G. Northcott constructed a finite free resolution for any standard determinantal ideal and as a corollary they got that standard determinantal ideals are Cohen–Macaulay. In [85], B. Sturmfels uses the Knuth–Robinson–Schensted (KRS) correspondence for the computation of Gröbner bases of determinantal ideals. The application of the KRS correspondence to determinantal ideals has also been investigated by S.S. Abhyankar and D.V. Kulkarni in [1] and [2]. Furthermore, variants of the KRS correspondence can be used to study symmetric determinantal ideals (see [17]) or ideals generated by Pfaffians of skew symmetric matrices (see [47], [5], and [18]). Many other authors have made important contributions to the study of standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals without even being mentioned here and we apologize to those whose work we may have failed to cite properly.

In this book, we will mainly restrict our attention to standard determinantal ideals and we will attempt to address the following three crucial problems.

(1) CI-liaison class and G-liaison class of standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals.

(2) The multiplicity conjecture for standard determinantal ideals, determinantal ideals, and symmetric determinantal ideals.

(3) Unobstructedness and dimension of families of standard determinantal schemes, determinantal schemes, and symmetric determinantal schemes.

Given the extensiveness of the subject, it is not possible to go into great detail in every proof. Still, it is hoped that the material that we choose will be beneficial and illuminating for the reader. The reader can refer [10], [54], [56], [59], [60], [70], [9], and [22] for background, history, and a list of important papers.

Let us now briefly describe the contents of each single chapter of this book. We start out in Chapter 1 by fixing notation and providing the basic concepts in the field. Minimal free resolutions, arithmetically Cohen–Macaulay (ACM) schemes, and arithmetically Gorenstein (AG) schemes are the topics of Section 1.1. In Section 1.2, we recall the definition and basic facts on standard determinantal ideals, determinantal ideals and symmetric determinantal ideals, we also provide the results on good and standard determinantal schemes $X \subset \mathbb{P}^n$ and the associated complexes, used later on. In particular, we recall the definition of generalized Koszul complexes which provide minimal free R-resolutions of the homogeneous ideal $I(X)$ of X and of the canonical module K_X of X. Section 1.3 is devoted to the definition of CI-liaison and G-liaison and to overview the known results on liaison theory needed later on. In particular, we present G-liaison theory as a theory of divisors on arithmetically Cohen–Macaulay schemes which collapses to the setting of CI-liaison theory as a theory of generalized divisors on a complete intersection scheme. In order for meaningful applications of G-liaison to be found, we need useful constructions of Gorenstein ideals. We end this chapter describing the method that has been used either directly or at least indirectly in most of the results about G-liaison discovered in the last years (see Theorems 1.3.11 and 1.3.12).

Chapter 2 is devoted to study the CI-liaison class and G-liaison class of standard determinantal ideals. Liaison theory has its roots dating more than a century ago although the greatest activity has been in the last 30 years, beginning with the work of C. Peskine and L. Szpiro [75], where they established liaison theory as a modern discipline and they gave a rigorous proof of Gaeta's theorem. The goal of Section 2.1 is to sketch a proof of Gaeta's theorem: every arithmetically Cohen–Macaulay codimension 2 subscheme X of $\mathbb{P}^n$ can be CI-linked in a finite number of steps to a complete intersection subscheme; i.e., X is licci. Since it is well known for subschemes of codimension 2 of $\mathbb{P}^n$ that arithmetically Cohen–Macaulay subschemes are standard determinantal (Hilbert–Burch theorem) and that arithmetically Gorenstein subschemes are complete intersections, Gaeta's theorem can be viewed as a first result about the CI-liaison and G-liaison of standard determinantal schemes. In Section 2.2, we prove that in the CI-liaison context Gaeta's theorem does not generalize well to subschemes $X \subset \mathbb{P}^n$ of higher codimension. More precisely, we introduce some graded modules which are CI-liaison invariants and we use them to prove the existence of infinitely many different CI-liaison classes containing standard determinantal curves $C \subset \mathbb{P}^4$ (see Remark 2.2.14). The purpose of Section 2.3 is to extend Gaeta's theorem, viewed as a statement on standard determinantal subschemes of codimension 2, to arbitrary codimension and to prove that any standard determinantal subscheme X of $\mathbb{P}^n$ can be G-linked in a finite number of steps to a complete intersection subscheme; i.e., X is glicci.

In Chapter 3, we study the relation between the graded Betti numbers of a homogeneous standard determinantal ideal $I \subset R = K[x_1, \ldots, x_n]$ and the multiplicity of I, $e(R/I)$. The motivation for comparing the multiplicity of a graded R-module M to products of the shifts in a graded minimal free R-resolution of M comes from a paper of C. Huneke and M. Miller [51]. They focused on homogeneous Cohen–Macaulay ideals $I \subset R$ of codimension c with a pure resolution,

$$0 \longrightarrow \oplus R(-d_c)^{a_c} \longrightarrow \cdots \longrightarrow R(-d_1)^{a_1} \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

and they proved for such ideals the following beautiful formula for the multiplicity of R/I:

$$e(R/I) = \frac{\prod_{i=1}^{c} d_i}{c!}.$$

Since then there has been a considerable effort to bound the multiplicity of a homogeneous Cohen–Macaulay ideal $I \subset R$ in terms of the shifts in its graded minimal free R-resolution; and J. Herzog, C. Huneke, and H. Srinivasan have made the following conjecture (*multiplicity conjecture*) which relates the multiplicity $e(R/I)$ to the maximum and the minimum degree shifts in the graded minimal free R-resolution of R/I.

Conjecture 0.0.1. *Let $I \subset R$ be a graded Cohen–Macaulay ideal of codimension c. We consider the minimal graded free R-resolution of R/I:*

$$0 \longrightarrow \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{p,j}(R/I)} \longrightarrow \cdots \longrightarrow \oplus_{j\in\mathbb{Z}} R(-j)^{\beta_{1,j}(R/I)} \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Set

$$m_i(I) := \min\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}$$

and

$$M_i(I) = \max\{j \in \mathbb{Z} \mid \beta_{i,j}(R/I) \neq 0\}.$$

Then, we have

$$\frac{\prod_{i=1}^{c} m_i}{c!} \leq e(R/I) \leq \frac{\prod_{i=1}^{c} M_i}{c!}.$$

There is a growing body of the literature proving special cases of Conjecture 0.0.1. For example, it holds for complete intersection ideals [46], powers of complete intersection ideals [37], perfect ideals with a pure resolution (i.e., $m_i = M_i$) [51], perfect ideals with a quasi-pure resolution (i.e., $m_i \geq M_{i-1}$) [46], perfect ideals of codimension 2 [46], and Gorenstein ideals of codimension 3 [67]. We devote

Chapter 3 to prove that Conjecture 0.0.1 works for standard determinantal ideals of arbitrary codimension (see Theorem 3.2.6). We end Chapter 3, proving that the ith total Betti number $\beta_i(R/I)$ of a standard determinantal ideal I can be bounded above by a function of the maximal shifts $M_i(I)$ in the minimal graded free R-resolution of R/I as well as bounded below by a function of both the maximal shifts $M_i(I)$ and the minimal shifts $m_i(I)$.

Hilbert schemes are not just sets of objects; they are endowed with a scheme structure far of being well understood. In Chapter 4, we consider some aspects related to families of standard determinantal schemes; more precisely, we address the problem of determining the unobstructedness and the dimension of families of standard (resp., good) determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension c. The first important contribution to this problem is due to G. Ellinsgrud [25]; in 1975, he proved that every arithmetically Cohen–Macaulay, codimension 2 closed subscheme X of $\mathbb{P}^{n+2}$ is unobstructed (i.e., the corresponding point in the Hilbert scheme $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+2})$ is smooth) provided $n \geq 1$, and he also computed the dimension of the Hilbert scheme $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+2})$ at $[X]$. Recall also that the homogeneous ideal of an arithmetically Cohen–Macaulay, codimension 2 closed subscheme X of $\mathbb{P}^{n+2}$ is given by the maximal minors of a $t \times (t+1)$ homogeneous matrix, the Hilbert–Burch matrix; i.e., X is standard determinantal. The purpose of this chapter is to extend Ellingsrud's theorem, viewed as a statement on standard determinantal schemes of codimension 2, to arbitrary codimension. We also address the problem whether the closure of the locus of standard determinantal schemes in $\mathbb{P}^{n+c}$ is an irreducible component of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$, and when $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ is generically smooth along the determinantal locus (see Corollaries 4.2.37, 4.2.41, 4.2.43, and 4.2.44).

Given integers $b_1, \ldots, b_t$ and $a_0, a_1, \ldots, a_{t+c-2}$, we denote by

$$W(\underline{b}; \underline{a}) \subset \mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$$

the locus of good determinantal schemes $X \subset \mathbb{P}^{n+c}$ of codimension $c \geq 2$ defined by the maximal minors of a homogeneous matrix $\mathcal{A} = (f_{ji})^{i=1,\ldots,t}_{j=0,\ldots,t+c-2}$, where $f_{ji} \in K[x_0, x_1, \ldots, x_{n+c}]$ is a homogeneous polynomial of degree $a_j - b_i$. In Section 4.2, using induction on c by successively deleting columns of the largest possible degree and using repeatedly the Eagon–Northcott complexes and the Buchsbaum–Rim complexes associated with a standard determinantal scheme, we state an upper bound for the dimension of $W(\underline{b}; \underline{a})$ in terms of $b_1, \ldots, b_t$ and $a_0, a_1, \ldots, a_{t+c-2}$ (cf. Theorem 4.2.7 and Proposition 4.2.15). Using again induction on the codimension and the theory of Hilbert flag schemes, we analyze when the upper bound of $\dim W(\underline{b}; \underline{a})$, given in Theorem 4.2.7 and Proposition 4.2.15, is indeed the dimension of the determinantal locus. It turns out that the upper bound of $\dim W(\underline{b}; \underline{a})$, given in Theorem 4.2.7, is sharp in a number of instances. More precisely, for $2 \leq c \leq 3$, this is known (see [56], [25]), for $4 \leq c \leq 5$, it is a consequence of one of the main theorems of this section (see Corollaries 4.2.26 and 4.2.30), while for $c \geq 6$, we get the expected dimension formula for $W(\underline{b}; \underline{a})$ under more restrictive numerical assumptions (see Corollary 4.2.31). Finally, we study when the closure of $W(\underline{b};\underline{a})$ is an irreducible component of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ and when $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ is generically smooth along $W(\underline{b};\underline{a})$. In Theorem 4.2.35, we show that the closure of $W(\underline{b};\underline{a})$ is a generically smooth irreducible component provided the zero-degree pieces of certain Ext^1-groups vanish. The conditions of Theorem 4.2.35 can be shown to be satisfied in a wide number of cases which we make explicit. In particular, we show that the mentioned Ext^1-groups vanish if $3 \leq c \leq 4$ (Corollary 4.2.37). Similarly, in Corollaries 4.2.41, 4.2.43, and 4.2.44 and as a consequence of Theorem 4.2.35, we prove that under certain numerical assumptions the closure of $W(\underline{b};\underline{a})$ is indeed a generically smooth, irreducible component of $\mathrm{Hilb}^{p(t)}(\mathbb{P}^{n+c})$ of the expected dimension. In Examples 4.2.40 and 4.2.42, we show that this is not always the case, although the examples created are somewhat special because all the entries of the associated matrix are linear entries. We end Chapter 4 with two conjectures raised by these results and proved in many cases (see Conjectures 4.2.47 and 4.2.48).

Throughout this book, we have mentioned various open problems. Some of them and further problems related to determinantal ideals and symmetric determinantal ideals are collected in the last chapter of the book. In fact, in Chapter 5 we address for determinantal ideals and symmetric determinantal ideals the problems addressed in the previous chapters for standard determinantal ideals; namely, the G-liaison class, the multiplicity conjecture, and the unobstructedness of determinantal ideals and symmetric determinantal ideals. We collect what is known, some open problems that naturally arise in this general setup, and we add some conjectures raised in the work.

More precisely, in Section 5.1, we determine the G-liaison class of symmetric determinantal subschemes (see Proposition 5.1.11 and Theorem 5.1.12) and we also sketch the proof of Gorla's theorem: Any determinantal subscheme is glicci (see Theorem 5.1.4). We devote Section 5.2 to prove the multiplicity conjecture for symmetric determinantal ideals of codimension $\binom{m-t+2}{2}$ defined by the $t \times t$ minors of an $m \times m$ homogeneous symmetric matrix for any $t = 1, m-1$, and m, and we left open the cases $2 \leq t \leq m-2$. In Section 5.2, we also show that the ith total Betti number $\beta_i(R/I)$ of a symmetric determinantal ideal, defined by the submaximal minors of a homogeneous symmetric matrix, is bounded above by a function of the maximal shifts $M_i(I)$ in the minimal graded free R-resolution of R/I as well as bounded below by a function of both the maximal $M_i(I)$ and the minimal shifts $m_i(I)$. In the last section of this work, we write down a lower bound for $\dim_{[X]} \mathrm{Hilb}^{p(t)}(\mathbb{P}^n)$, where $X \subset \mathbb{P}^n$ is a symmetric determinantal subscheme of codimension 3 defined by the submaximal minors of an $m \times m$ homogeneous symmetric matrix (see Theorem 5.3.5), and we analyze when the mentioned bound is sharp. This last result is a nice contribution to the classification problem of codimension r Cohen–Macaulay quotients of the polynomial ring $K[x_0, x_1, \ldots, x_n]$. There is, in our opinion, little hope of solving the above classification problem in full generality and for arbitrary codimension, and in this last section we have restricted our attention to codimension 3 arithmetically Cohen–Macaulay subschemes $X \subset \mathbb{P}^n$ with homogeneous ideal generated by the submaximal minors of an $m \times m$ homogeneous symmetric matrix.

We have tried hard to keep the text as self-contained as possible. The basics of algebraic geometry supplied by Hartshorne's book [39] suffices as a foundation for this text. Some familiarity with commutative algebra, as developed in Matsumura's book [64] and Bruns–Herzog's book [9], is helpful and the rudiments on Ext and Tor contained in every introduction to homological algebra will be used freely.

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