Preface

Grothendieck duality theory on noetherian schemes, particularly the notion of a dualizing sheaf, plays a fundamental role in contexts as diverse as the arithmetic theory of modular forms [**DR**], [**M**] and the study of moduli spaces of curves [**DM**]. The goal of the theory is to produce a trace map in terms of which one can formulate duality results for the cohomology of coherent sheaves. In the 'classical' case of Serre duality for a proper, smooth, geometrically connected, n-dimensional scheme X over a field k, the trace map amounts to a canonical k-linear map $t_X : \mathrm{H}^n(X, \Omega^n_{X/k}) \to k$ such that (among other things) for any locally free coherent sheaf $\mathscr{F}$ on X with dual sheaf $\mathscr{F}^\vee = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$, the cup product yields a pairing of finite-dimensional k-vector spaces

$$\mathrm{H}^i(X, \mathscr{F}) \otimes \mathrm{H}^{n-i}(X, \mathscr{F}^\vee \otimes \Omega^n_{X/k}) \to \mathrm{H}^n(X, \Omega^n_{X/k}) \xrightarrow{t_X} k$$

which is a perfect pairing for all i. In particular, using $\mathscr{F} = \mathscr{O}_X$ and $i = 0$, we see that $\dim_k \mathrm{H}^n(X, \Omega^n_{X/k}) = 1$ and t_X is non-zero, so t_X must be an isomorphism. Grothendieck duality extends this to a relative situation, but even the relative case where the base is a discrete valuation ring is highly non-trivial. The foundations of Grothendieck duality theory, based on residual complexes, are worked out in Hartshorne's *Residues and Duality* (hereafter denoted [**RD**]). These foundations make the duality theory quite computable in terms of differential forms and residues, and such computability can be very useful (e.g., see Berthelot's thesis [**Be**, VII, §1.2] or Mazur's pioneering work on the Eisenstein ideal [**M**, II, p.121]).

In the construction of this theory in [**RD**] there are some essential compatibilities and explications of abstract results which are not proven and are quite difficult to verify. The hardest compatibility in the theory, and also one of the most important, is the base change compatibility of the trace map in the case of proper Cohen-Macaulay morphisms with pure relative dimension (e.g., flat families of semistable curves). *Ignoring* the base change question, there are simpler methods for obtaining duality theorems in the *projective* CM case (see [**AK1**], [**K**], which also have results in the projective non-CM case). However, there does not seem to be a published proof of the duality theorem in the general proper CM case over a locally noetherian base, let alone an analysis of its behavior with respect to *base change.* For example, the rather important special case of compatibility of the trace map with respect to base change to a geometric fiber is not at all obvious, even if we restrict attention to duality for projective *smooth* maps. This was our original source of motivation in this topic and (amazingly) even this special case does not seem to be available in the published literature.

The aim of this book is to prove the hard unproven compatibilities in the foundations given in [**RD**], particularly base change compatibility of the trace map, and to explicate some important consequences and examples of the abstract theory. This book should be therefore be viewed as a companion to [**RD**], and is by no means a logically independent treatment of the theory from the very beginning. Indeed, we often appeal to results proven in [**RD**] along the way, rather than reprove everything from scratch (and we are careful to avoid any circular reasoning). More precisely, we will give the definitions of most of the basic constructions we need from **[RD]** (aside from a few cases in which the definitions are very elaborate, in which case we refer to specific places in **[RD]** for the relevant definitions), and we will sometimes refer to **[RD]** for proofs of various properties of these basic constructions. It is our hope that by providing a detailed explanation of some of the more difficult aspects of the foundations, Grothendieck's work on duality for coherent sheaves will be better understood by a wider audience.

There is a different approach to duality, and particularly the base change problem for the trace map, which should be mentioned. In **[LLT]**, Lipman works out a vast generalization of Grothendieck's theory, using Deligne's abstract construction of a trace map [**RD**, Appendix] in place of Grothendieck's 'concrete' approach via residual complexes (as in the main text of [**RD**]). Lipman's theory requires a lot more preliminary work with derived categories than is needed in [**RD**], but it yields a more general theory without noetherian conditions or boundedness hypotheses on derived categories (though the 'old' theory in [**RD**] is adequate for nearly all practical purposes). In these terms, Lipman says that he can deduce the base change compatibility of the trace map in the proper Cohen-Macaulay case. However, it seems unwise to ignore the foundations based on residual complexes, because of their usefulness in calculations.

In any case, it is unlikely that Lipman's powerful abstract methods lead to a much shorter proof that the trace map is compatible with base change. The reason is that ultimately one wants to have statements in terms of sheaves of differentials or (at least in the projective case) their $\mathscr{E}xt$'s, with *concrete* base change maps. Translating Deligne's more abstract approach into these terms is a non-trivial matter which cannot be ignored, and this appears to cancel out any appearance of brevity in the proofs. Either one builds the concreteness directly into the foundations (as in [**RD**] and this book) and then one needs to check a lot of commutative diagrams, or else one uses abstract foundations and has to do a lot of hard work to make the results concrete. To quote Lipman on the issue of the choice of foundations,

> " ... The abstract approach of Deligne and Verdier, and the more recent one of Neeman, seem on the surface to avoid many of the grubby details; but when you go beneath the surface to work out the concrete interpretations of the abstractly defined dualizing functors, it turns out to be not much shorter. I don't know of any royal road ... "