Preface

One of the fundamental problems in any area of mathematics is to determine the distinct variants of an object under consideration. As for complex-functional analysis, one is interested, for example, in studying the equivalent representations of the conformally invariant classes of holomorphic functions. This problem is addressed here for the holomorphic Q classes.

For $p \in [0, \infty)$ and dm – the element of the two dimensional Lebesgue measure, we say that f, a holomorphic function in the unit disk $\mathbf{D}$, is of the class $\mathcal{Q}_p$ provided

$$E_p(f) = \sup\left\{\left(\int_{\mathbf{D}} |f'(z)|^2 \left(\log\left|\frac{1-\bar{w}z}{w-z}\right|\right)^p dm(z)\right)^{1/2} : w \in \mathbf{D}\right\} < \infty.$$

It is clear that each $\mathcal{Q}_p$ can serve as a sample of the conformally invariant classes in the sense of: $E_p(f \circ \sigma) = E_p(f)$ for all $f \in \mathcal{Q}_p$ and $\sigma \in Aut(\mathbf{D})$ – the group of all conformal automorphisms of $\mathbf{D}$.

The goal of this monograph is to bring the major features of $\mathcal{Q}_p$ to light, in particular, to characterize $\mathcal{Q}_p$ in different terms. More precisely:

Chapter 1 contains some (most) basic properties of $\mathcal{Q}_p$ such as Möbius boundedness, image area and higher derivatives. The aim is to show that the classes $\mathcal{Q}_p$, $p \in (0, 1)$ are of independent interest.

Chapter 2 discusses the problem of how a $\mathcal{Q}_p$ can be embedded into a Bloch-type space and vice versa. This problem will be solved by considering boundedness and compactness of a composition operator acting between two spaces.

Chapter 3 describes the coefficients of either Taylor or random series of functions from $\mathcal{Q}_p$. In particular, we will see that there is a big difference between the cases: $p \in (0, 1)$ and $p = 1$.

Chapter 4 exhibits a geometric way to understand $\mathcal{Q}_p$, that is, p-Carleson measure characterization of $\mathcal{Q}_p$. This simple but important property induces certain deep relations between $\mathcal{Q}_p$ and the mean Lipschitz spaces, as well as the Besov spaces which are conformally invariant, too.

Chapter 5 characterizes the inner and outer functions in $\mathcal{Q}_p$ by means of p-Carleson measure and other two conformally invariant measures: Poisson measure and hyperbolic measure.

Chapter 6 gives the boundary value behavior of a $\mathcal{Q}_p$-function for $p \in (0, 1)$. This allows us to study $\mathcal{Q}_p$ via those non-holomorphic functions on the unit circle $\mathbf{T}$ and even on the exterior of the unit disk $\mathbf{C} \setminus \mathbf{D}$, and hence leads to a consideration of harmonic analysis.

Chapter 7 explores a list of properties of $\mathcal{Q}_p(\mathbf{T})$ (i.e. Q class on $\mathbf{T}$). Specially, the $\mathcal{Q}_p(\mathbf{T})$-solutions of the $\bar{\partial}$-equation produce a decomposition of $\mathcal{Q}_p$ through the bounded functions on $\mathbf{T}$. As applications, the corona theorem and interpolation theorem related to $\mathcal{Q}_p$ are established.

Chapter 8 deals with a localization of $\mathcal{Q}_p(\mathbf{T})$ based on the dyadic partitions of all subarcs of $\mathbf{T}$. The results enable us to recognize $\mathcal{Q}_p$ from mean oscillation to dyadic model, and finally to wavelet basis.

The exposition is at as elementary a level as possible, and it is intended to be accessible to graduate students with a basic knowledge of complex-functional-real analysis. The material of this monograph has been collected from a series of talks that I gave over the past six years most in Canada, China, Finland, Germany, Greece and Sweden, but also from a lecture course at University of La Laguna in the fall semester of 1999. The selection of topics is rather arbitrary, but reflects the author's preference for the analytic approach. There is no attempt to cover all recent advances (for instance, Q classes of higher dimensions), and yet, it is hoped that the reader will be intrigued by this monograph and will, at some point, read the notes presented at the end of each chapter as well as the papers listed in the references, and proceed to a further research.

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