Preface

Properties of products of conjugacy classes of finite groups are an old branch of finite group theory. This topic was intensively studied in the 1980's. The book [22] "Products of Conjugacy Classes in Groups," edited by Z. Arad and M. Herzog, gives a comprehensive picture of the results obtained during this period.

It was realized by several authors that this research could be extended to products of irreducible characters. We refer the reader to the papers [1,2,11, 13–16,21,23,35,40,51,52,65].

In several of these papers the authors found an analogy between products of conjugacy classes and products of irreducible characters which led to the notion of *table algebra*, introduced by H.I. Blau and Z. Arad in [7], in order to study in a uniform way the decomposition of products of conjugacy classes and irreducible characters of finite groups. Since then, the theory of table algebras was extensively developed in papers of Z. Arad, H. Arisha, H. Blau, F. Bünger, D. Chillag, M.R. Darafsheh, J. Erez, E. Fisman, V. Miloslavsky, M. Muzychuk, A. Rahnamai, C. Scopolla and B. Xu [3–5,7–10,12,17–20,25,29–33,35,41].

Table algbras, as defined, may be considered a special class of C-algbras introduced by Y. Kawada [49] and G. Hoheisel [48]. More precisely, a table algebra is a C-algebra where the structure constants are nonnegative. Each finite group yields two natural table algebras: the table algebra of conjugacy classes and the table algebra of generalized characters.

Both table algebras arriving from group theory have an additional property: their structure constants and degrees are nonnegative integers (we refer the reader to the Introduction where these notions are defined). Such algebras were defined in [30] as *integral table algebras* (briefly, ITA). *Generalized table algebras* (briefly, GT-algebras were introduced in [20]. They generalize properties of such well-known objects, e.g., homogeneous coherent algebras, Iwahori-Hecke algebras, etc.

Each integral table algebra may be rescaled to a homogeneous one [32], *i.e.,* an algebra whose non-trivial degrees are equal. This common degree is a natural parameter which may be used for a classification of integral table algebras. The first result in this direction was obtained by Z. Arad and H. Blau in [7] where homogeneous table algebras of degree 1 were classified. The classification of homogeneous integral table algebras of degree 2 with a faithful element was obtained by H. Blau in [31]. This research was continued in [10] where a complete classification of homogeneous integral table algebras of degree 3 with a faithful element was obtained provided that the algebra does not contain linear elements.

Another important class of ITA is comprised of so-called standard integral table algebras (briefly, SITA) which axiomatize the properties of Bose-Mesner algebras of commutative association schemes. The standard algebras are also involved in the study of homogeneous ITA.

Each element of a table algebra is contained in a unique table subalgebra which may be considered as a table subalgebra generated by this element. So it is natural to start the study of integral table algebras from those which are generated by a single element. Table algebras generated by an element of degree 2 were completely classified by H. Blau in [29] under the assumption that either a generating element is real or the algebra does not contain linear elements of degree a power of 2. If a table algebra is generated by an element of degree 3 or greater, then its structure is more complicated. If a generating element is real, then we are faced with a classification of P-polynomial table algberas which would imply powerful consequences for a classification of distance-regular graphs. In contrast, if a generating element is non-real and of small degree, then either a complete classification or important structure information may be obtained. For example, standard integral table algebras generated by a non-real element of degree 3 were classified in [5], [33] under the additional assumption that there is no nontrivial element of degree 1. In this volume we continue the investigation of integral standard table algebras generated by a non-real element of small valency. More precisely, we collect here the recent results about integral standard table algberas generated by a non-real element of degree 4 or 5.

In all the examples known to us of SITA generated by a non-real element of degree k, the degrees of all basis elements are bounded by some function $f(k)$. This gives evidence of the following

Conjecture 1 *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if a SITA is generated by a non-real element of degree k, then all degrees of the algebra are bounded by $f(k)$.*

The results of [29] show that this conjecture is true if $k = 2$. If $k = 3$ and a SITA does not contain nontrivial elements of degree 1, then all degrees are bounded by 6 and the conjecture is valid. The partial classification of standard ITA generated by an element of degrees $4, 5$ obtained in this volume also supports this conjecture. It is not difficult to show that the conjecture holds for the table algebras of generalized characters of a finite group even without the assumption of being non-real.

The book [22] and the paper [7] attracted many researchers from various countries to work on table algebras, products of conjugacy classes and related topics.

At Bar-Ilan University, Z. Arad and his students H. Arisha, V. Miloslavsky, and his former student E. Fisman, jointly with his colleague M. Muzychuk, performed extensive research on table algebras. In the academic year 1998/99, H. Blau from Northern Illinois University (deKalb) and two postdoctoral students, F. Bünger from Germany and M. Hirasaka from Japan, joined the Bar-Ilan University group in order to further advance the theory of table algebras. This volume, together with [5] and [33], collect most of the results obtained in this period at Bar-Ilan University.

This volume contains 5 chapters. The first chapter is an *Introduction*, which contains all necessary definitions and facts about table algebras. The second chapter, *Integral Table Algebras with a Faithful Nonreal Element of Degree* 4, deals with standard integral table algebras generated by a non-real element of degree 4. The contribution of one of its co-authors, H. Arisha, is a part of his Ph.D thesis. Another co-author, E. Fisman, was supported by the Emmy Noether Research Institute at Bar-Ilan University. The third chapter, *Standard Integral Table Algebras with a Faithful Nonreal Element of Degree* 5, and the fourth chapter, *Standard Integral Table Algebras with a Faithful Real Element of Degree* 5 *and Width* 3, are devoted to standard integral algebras generated by an element of degree 5. F. Bünger, one of the co-authors of these chapters, was supported by the Minerva Foundation in Germany through the Emmy Noether Research Institute at Bar-Ilan University. The last chapter, *The Enumeration of Primitive Commutative Association Schemes with a Non-symmetric Relation of Valency at Most* 4, classifies primitive commutative association schemes which contain a connected non-symmetric relation of valency 3 or 4. Its author, Mitsugu Hirasaka, was supported by the Japan Society for Promotion of Science, and worked in both the Graduate School of Mathematics at Kyushu University and the Emmy Noether Research Institute at Bar-Ilan University.

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