Preface

In the last five years there was an essential progress in the development of the theory of transcendental numbers. A new approach to the arithmetic properties of values of modular forms and theta-functions was found. First and up to a recent time there was a unique result of this type established in 1941 by Th. Schneider. It states that the modular function $j(\tau)$ has transcendental values at any algebraic point τ of the complex upper-half plane, distinct from the imaginary quadratic irrationalities. It is well known that the value of the modular function at any imaginary quadratic argument τ with $\mathrm{Im}\tau > 0$, is an algebraic number.

In 1995, K. Barré-Sirieix, G. Diaz, F. Gramain and G. Philibert proved that the value of the function $J(z)$, connected to $j(\tau)$ by the relation $j(\tau) = J(e^{2\pi i\tau})$, at any non-zero algebraic point z of the unit disk is transcendental. In 1969, this assertion was proposed by K. Mahler as a conjecture. The proof rests heavily on the functional modular equations connecting the functions $J(z)$ and $J(z^n)$ over $\mathbf{C}$ for any natural number n. Simultaneously the p-adic analogue of this theorem (conjecture formulated by Yu. Manin, 1971) was proved and it has important consequences in the theory of algebraic numbers.

The solution of the Mahler-Manin's problem was an impulse for further intensive researches. In 1996 Yu. Nesterenko generalized this result, he proved the expected lower bound for the transcendence degree of the field generated over $\mathbf{Q}$ by the values of the Eisenstein series $E_2(\tau)$, $E_4(\tau)$, $E_6(\tau)$ and the exponential function $e^{2\pi i\tau}$. The fact that Eisenstein series satisfy an algebraic system of differential equations is fundamental to the proof. In particular, an effective bound for the multiplicities of zeros for polynomials in the functions involved played an important rôle.

This result of algebraic independence has a number of remarkable consequences. First it implies that the three numbers π, e^π, $\Gamma(\frac{1}{4})$ are algebraically independent (over $\mathbf{Q}$) and, in particular, the two numbers π and e^π are algebraically independent. This last assertion was an old problem in transcendence theory. The theorem implies an even more general assertion : for any natural number D the real numbers π and $e^{\pi\sqrt{D}}$ are algebraically independent. One more consequence is the algebraic independence of the numbers $J(q)$, $J'(q)$ and $J''(q)$ for any algebraic number q satisfying $0 < |q| < 1$. D. Bertrand and independently D. Duverney, Ke. Nishioka, Ku. Nishioka, I. Shiokawa (DNNS) deduced results on algebraic independence of the values of theta-functions at algebraic points and in particular derived the transcendence of the sums $\sum_{n=1}^{\infty} q^{n^2}$ for any algebraic q satisfying $0 < |q| < 1$. J. Liouville introduced in 1851 this series with $q = \ell^{-1}$ for $\ell \in \mathbf{Z}$, $\ell > 1$ as an example for which his method only proved irrationality. No assertion on transcendence of these sums was known. DNNS have also proved transcendence of the values of the Dedekind eta-function $\eta(q)$ at algebraic points q, $0 < |q| < 1$ and several other interesting corollaries.

P. Philippon introduced a general class of K-functions containing in particular the functions $E_{2k}(\frac{\log z}{2\pi i})$, and he proposed a general approach to study the algebraic independence of the values of these functions. In this way, new results on the algebraic independence of the values of the so-called Mahler's functions at transcendental arguments were obtained. He also found, that the algebraic independence of the numbers π and $e^{\pi\sqrt{D}}$ can be shown using an older measure of algebraic independence of couples of numbers, connected to periods and quasi-periods of elliptic functions with complex multiplications.

It should be stressed, that the above mentioned results were established following a general approach for proofs of algebraic independence of numbers developped in the seventies and eighties. This method uses tools from commutative algebra for establishing zeros and multiplicities estimates (Yu. Nesterenko, W.D. Brownawell, D.W. Masser, G. Wüstholz, P. Philippon) and elimination theory in the form of a criterion for algebraic independence of P. Philippon. It had already been successfully applied before to the study of the values of functions satisfying addition theorems (exponential, elliptic, abelian functions), linear differential equations over $\mathbf{C}(z)$ (the so-called E-functions of Siegel) or special functional equations introduced by K. Mahler (which find applications in the study of dynamical systems).

An instructional conference on algebraic independence was held from September 29 till October 3, 1997 in Luminy (France) co-organized by M. Waldschmidt, R. Tijdeman and Yu. Nesterenko. The aim of this course was to enable graduate students and post-docs to get an acquaintance with the new methods and results in transcendence theory. The lectures delivered on this occasion made the basis for the present book. Parts of this book has been written with the help of the grant INTAS-RFBR, IR97-1904.

This book is ideally divided into four unequal parts.

The first part, consisting of the first four chapters, presents the latest results in transcendence and algebraic independence theory obtained for modular functions and modular forms. It serves as an introduction to and a motivation for reading the three other parts. Chapter 1 establishes the differential equations satisfied by Eisenstein series or modular theta functions as well as Masser's period relations, which play a fundamental rôle in chapter 3 (and 4). It also gives results and conjectures on the values of theta functions and their derivatives. Chapter 2 gives a proof of the transcendance of the values of the modular invariant function J, which has been seminal for the generalisation presented in chapter 3. It also discusses the quantitative aspect of the question with measures of transcendence and conjectures on the values of modular and exponential functions overlapping with those of chapter 1. Chapter 3 gives the proof of the algebraic independence of the values of modular forms and details several remarkable corollaries. It rests on the preliminaries of chapter 1 as well as a fundamental zeros estimate, the proof of which is postponed until chapter 10. Chapter 4 sums up the tools involved in this proof and shows a short-cut to the proof of algebraic independence of π, e^{π} and $\Gamma(1/4)$ (resp. π, e^{π} and $\Gamma(1/3)$) using a known measure of algebraic independence of π

and $\Gamma(1/4)$ (resp. $\Gamma(1/3)$) obtained with the help of elliptic functions, rather than modular ones.

The second part consists of chapters 5 to 9, it develops the necessary tools from commutative algebra already presented in chapter 3. Chapter 5 establishes the basic facts from elimination theory which enable one to attach a form to a variety. Its originality is in its multi-homogeneous setting which is new in this context. After proving the elimination theorem and the principality of the eliminating ideal it shows the link with the geometry of multi-projective spaces and studies the so-called resulting form, the use of which is crucial in this multi-homogeneous setting, for example for specialisations. Chapters 6 and 7 introduce, in the homogeneous and multi-homogeneous case respectively, the notions and results from diophantine geometry (height, distance, Bézout theorems, ...) which follow from the study of eliminating and resulting forms. The refined definition of height adopted here is inspired from the calculus in Arakelov theory. In the multi-homogeneous case, the notion of distance from a point to a variety is shown to be more subtle than in the homogeneous case (chapter 7, § 4). Chapter 8 derives from the previous machinery several criteria for algebraic independence which are used in proofs of algebraic independence. In particular, these results include a multi-projective criteria which allows one to take advantage of the different behaviors of numbers in a proof of algebraic independence. One will find another criterion in chapter 13, § 4, involving multiplicities. Finally, we have included (chapter 9) in this part an upper bound for the classical Hilbert function of an homogeneous (or bi-homogeneous) polynomial ideal whose proof rests on a clever use of Bertini's theorem for reducing to the case of dimension 1 which is easy.

The third part (chapters 10 and 11) deals with zeros estimates. Chapter 10 is devoted to the multiplicity estimate used in the proof of the main result of chapter 3, more generally the results here address solutions of algebraic differential equations which satisfy a special property, called D-property. The proof of the general estimate depends on the upper bound for Hilbert functions of ideals given in chapter 9 and then it takes a careful study of the differential system satisfied by the Eisenstein series considered in chapter 3 to show that the D-property is valid for them. Chapter 11 establishes zeros estimate on group varieties which are necessary for the proofs of algebraic independence in chapter 13 and 14. Here one has to control the multiplicities of polynomials at points of a finite type subgroup of the ambiant algebraic group. The degeneracies depend on the distribution of the points with respect to algebraic subgroups and one exhibits a so-called obstructing subgroup which in a way gives the worst degeneracy. The proof, based essentially on a control of degree and multiplicities in intersection by suitable hypersurfaces is presented here in a geometric setting. Yet another multiplicity estimate is to be found in chapter 12.

The fourth part (chapters 12 to 16) is concerned with applications of the tools developed so far. Chapter 12 shows measures of algebraic independence of values of Mahler type functions in one variable (that is satisfying a functional equation for exponentiation of the variable) as well as the zeros estimate for this type of functions. It should be noted that the quality of the measure obtained depends on the precise type of functional equation involved. Chapters 13 and 14 address algebraic independence of two and more numbers respectively in commutative algebraic groups, especially the values of the exponential and elliptic functions. In particular, chapter 13 presents a proof of algebraic independence of two numbers through interpolation determinants which rests on a generalisation of Gelfond's criterion using multiplicities, not covered in chapter 8. Chapter 14 deals with large transcendence degrees of the values of the exponential function. It presents several conjectures in one and several analytic variables and makes some historical comments discussing in particular the so-called technical hypothesis. It gives the proof of the almost best known result towards Gelfond-Schneider's conjecture. Chapter 15 is devoted to metric results, it shows that almost all families of m complex numbers have an order of measure of algebraic independence equal to $m + 1$. The proof uses a kind of "transfer theorem" between measures of algebraic independence and measure of approximation by algebraic numbers. It should be noted that this differs from the approximation properties introduced in chapter 4 only in that here one assumes *a priori* that the numbers are close enough to some variety in order to find good approximations by algebraic numbers. Finally, chapter 16 presents an alternative approach to algebraic independence through effective versions of the Nullstellensatz, which is more direct and enables one to weaken the technical hypothesis. Actually, further developments are mentioned such as Łojasievitch inequalities and algorithmic methods which are relevant for the interpolation problems underlying algebraic independence theory in many ways.

The reader will find thereafter the list of contributors which includes their addresses and the chapters they are responsible for. Chapters authors names are also repeated at the bottom of the first page of each chapter. Statements are numbered afresh in each chapter according to their section of occurrence, and when they are referred to from a different chapter an indication of the chapter number is added. On a different footing, formulas are numbered at a stretch throughout the book. There is a dual system of bibliographical references, one concerns the references that we (rather arbitrarily) declared "in" the subject, the list of which may be found at the end of the book, the other deals with references that we considered "on the rim", and those appear as footnotes on the pages where they show up. However, if such a foot-reference appears twice or more in the same chapter the reader is then prompted to the page of its first occurrence. Finally, a short index is provided after the bibliography, at the very end of the book.