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# Osserman Manifolds in Semi-Riemannian Geometry



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*To Fernanda, Meltem and Maica*

# Preface

The notion of curvature is one of the central concepts of differential geometry, perhaps the central one, which distinguishes the geometric core of the subject from the others that are analytic, algebraic or topological. It has always been a pursuit of great interest to understand to what extent the sectional curvatures of a semi-Riemannian manifold can provide information about the curvature and metric tensors. One question of this kind that has been under much scrutiny in recent years, which will be our central theme, is whether the Osserman condition (involving sectional curvatures, and more precisely, the Jacobi operator) determines the curvature and metric tensors. In this research monograph, our goal is to expound the recent developments in the exploration of the answer to this question in Riemannian, Lorentzian, semi-Riemannian and affine differential geometry.

The question in its original form is known as the Osserman conjecture in Riemannian geometry. Significant progress has been made in search of a complete answer to this question by Chi [39], [40], [41], albeit it still remains largely open. Later the conjecture was questioned in Lorentzian geometry in its Lorentzian setting. An affirmative answer to the Osserman conjecture was obtained in a sequence of papers by García-Río and Kupeli [58], García-Río, Kupeli and Vázquez-Abal [59] and Blažić, Bokan and Gilkey [16]. Finally the conjecture appealed in semi-Riemannian geometry in its semi-Riemannian forms. This time, however, several counterexamples were found to the conjecture in [23], [31], [61], which diverted the research to the direction of understanding the semi-Riemannian Osserman manifolds. On the other hand, Osserman condition can be stated in affine differential geometry naturally as well and the affine Osserman manifolds were lately investigated in [60].

We plan to present all these developments in six chapters. Preliminaries needed in this monograph are intermediate level aspects of differential geometry. Instead of collecting them in a single chapter, we prefer to introduce them briefly as they are encountered in each section. We expect that this approach will lead the reader quickly into the subject and enable them to read the chapters independently without having to know all the prerequisites before hand.

In Chapter 1, we give the main definitions of this monograph in their general setting in semi-Riemannian geometry. Corresponding definitions for the Riemannian and Lorentzian counterparts are special cases of them. We then prove that the timelike and spacelike Osserman conditions are equivalent, with respect to which we define a semi-Riemannian manifold to be Osserman if the characteristic polynomials of the Jacobi operators are independent of the vectors in both the timelike and spacelike unit tangent bundles of the manifold. We illustrate some examples that serve as model spaces of semi-Riemannian Osserman manifolds and prove a result to relate the pointwise and (global) Osserman conditions, where the pointwise Osserman condition is the version of the global definition when the “tangent bundle” is replaced by “tangent space at each point”.

In Chapter 2, we study the known results toward an affirmative answer to the Osserman conjecture in Riemannian geometry, which states that a Riemannian Osserman manifold is either flat or rank-one symmetric. We demonstrate the solution to this conjecture in full for the dimensions  $n \neq 4m$ ,  $m \geq 1$ , and outline the solutions for the other dimensions under certain additional assumptions.

In Chapter 3, we provide the complete solution to Osserman conjecture in Lorentzian geometry. That is, we show that a Lorentzian Osserman manifold is a real space form, and hence is either flat or rank-one symmetric. The proof of this fact is different from what has appeared in the literature in that we explore the equivalence between the timelike and spacelike Osserman conditions to make the proof rather simple and short. We also introduce the null Osserman condition and statements equivalent to it, and show that it yields certain warped product decomposition theorems under additional assumptions.

In Chapter 4, we study semi-Riemannian Osserman manifolds of signature  $(2, 2)$ . First we construct counterexamples to the Osserman conjecture in semi-Riemannian geometry by showing the existence of nonsymmetric semi-Riemannian Osserman manifolds of signature  $(2, 2)$ , some of which are not even locally homogeneous. The second part of this chapter is devoted to a study of those semi-Riemannian manifolds satisfying the Osserman condition by following the work of Blažić, Bokan and Rakić [18]. Here the Osserman curvature tensors are classified into four types according to the properties of the minimal and characteristic polynomials of the Jacobi operators. (In fact, this is a generalization to the semi-Riemannian setting of the equivalence between pointwise Osserman and Einstein self-dual 4-manifolds previously pointed out in the Riemannian case.) Based on this information we obtain a classification of semi-Riemannian manifolds with metric tensors of signature  $(2, 2)$  whose Jacobi operators are diagonalizable. They must be either flat or locally rank-one symmetric, which is contrary to the nondiagonalizable case where two kinds of rank-two symmetric spaces are allowed.

Included in Chapter 5 are new classification results of ours for higher-dimensional semi-Riemannian Osserman manifolds. Due to the existence of non-symmetric examples in any signature  $(p, q)$ ,  $p, q > 1$ , we focus on the examination of the simplest cases of semi-Riemannian Osserman manifolds. Following the work of Chi [41] in the Riemannian case, we consider semi-Riemannian Osserman manifolds with exactly two distinct eigenvalues whose associated eigenspaces satisfy a kind of infinitesimal Hopf fibration property and call them semi-Riemannian special Osserman manifolds. The main objective of this chapter is to prove that such manifolds are either locally complex, quaternionic, paracomplex or paraquaternionic space forms, or locally isometric to the Cayley planes over the octonians or the anti-octonians. This shows that, besides the space forms, semi-Riemannian special Osserman manifolds correspond to the simplest semi-Riemannian manifolds from the viewpoint of their curvatures.

In Chapter 6, we review certain Osserman-related conditions. Following [133], [67] and [76] we introduce semi-Riemannian generalized Osserman manifolds. The fact that the space forms are the only semi-Riemannian generalized Osserman rank-one symmetric spaces shows that such a condition is much more restrictive than the Osserman condition. Along a different vein, the Osserman condition finds its natural setting in affine differential geometry as well. The notion of the affine Osserman condition is originated from our effort to supply new examples of semi-Riemannian Osserman manifolds via the construction called the Riemann extension [60]. It turns out that the affine Osserman condition does not generalize the semi-Riemannian ones since all eigenvalues of the Jacobi operators can be shown to be zero. It seems that the affine Osserman condition is the only reasonable notion of Osserman type in the general affine case where the unit sphere cannot be defined. Also, we relate  $\mathcal{C}$ -spaces and the Riemannian manifolds with isoparametric geodesic spheres to Riemannian Osserman manifolds here in connection with harmonic manifolds. To conclude, a kind of Osserman condition is studied for skew-symmetric curvature operators. Such a condition is shown to be the characteristic of space forms and certain special classes of Robertson-Walker type warped products in many cases. We only indicate here the basic notion and refer to [74] and [85] for the proofs and further references.

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