Preface

This monograph, based on the author's 1997 EMS Lectures given at the University of Helsinki in May/June 1997, outlines the Loeb measure construction (a way to construct rich measure spaces using Robinson's nonstandard analysis) and discusses recent applications in stochastic fluid mechanics, stochastic calculus of variations ("Malliavin calculus" and related topics), and mathematical finance theory.

The four lectures in Helsinki were designed for a general audience, as is the expanded version presented here. No previous knowledge of either nonstandard analysis or the fields of application is assumed, beyond the general knowledge of the working mathematician.

The aim in Chapter 1 is to provide a brief but coherent account of the fundamentals of nonstandard analysis (NSA) and the Loeb construction that is sufficient to make sense of the applications of the later chapters. For each of these we have endeavoured to provide sufficient by way of introduction to the topics concerned to enable even the reader unfamiliar with them to appreciate the basic ideas of the field and then the particular contributions that can be made using NSA and Loeb measures. In fact, one of the major contributions that NSA has made to many fields of application is to aid in understanding of the basic ideas of that field[1] – and it is hoped that among other things this will come over in this monograph.

To cover both an introduction to NSA and Loeb measures together with applications to three advanced and diverse fields of current research in four lectures (and now in four corresponding chapters) is somewhat ambitious. Necessarily the treatment will omit many details. The present volume should be seen then as something of a trailer for in depth study of both Loeb measures and the way in which they can make useful contributions to mathematical research. The topics chosen for discussion in Chapters 2–4 are drawn mainly from work of the author in collaboration others, and bring together material that has mostly been published elsewhere but is scattered. In each

[1] A classic example of this is Anderson's construction of Brownian motion as an infinitesimal random walk, discussed in Chapter 3; at a more elementary level is Robinson's original discovery of how NSA can be used to develop real analysis rigorously, using infinitesimals to make precise the informal ideas of differentiation and integration.

of the areas the applications include results that represent advances in the *standard* theory.

The applications in Chapters 2–4 are in the three seemingly unrelated areas mentioned in the opening paragraph. The link between them, from the point of view of this monograph, is the common methodology of Loeb measure techniques. This stems of course from the fact that each field involves measures and integration – and in most cases there is the more specific common feature of stochastic analysis in a variety of guises. But there is also a less obvious unifying factor that is harder to pin down precisely. This involves the idea of *passage to a limit* in a very generalised way. NSA facilitates this because, having constructed or defined an object X_n, say, for each finite n in the standard world we have *automatically* in the nonstandard world an object X_N for *infinite natural number* N (the meaning of this will be made precise in Chapter 1). Such N is called *hyperfinite* – that is, finite from the point of view of NSA but infinite in that $N > n$ for all $n \in \mathbb{N}$.[2] The work is then to find some standard (real world) object associated with X_N which will provide the solution to the problem in question.

In the applications to fluid mechanics, for example (Chapter 2), the fundamental equations (PDEs and stochastic PDEs) are normally solved by solving finite dimensional approximations, which are ODEs and stochastic DEs, and then "passing to the limit". In our approach, the passage to the limit is achieved by taking X_N (in the terminology above) to be a solution for the (infinite but) hyperfinite dimensional approximation of dimension N. From the nonstandard solution X_N a standard solution is obtained.

The "Malliavin" calculus – treated in Chapter 3 – is at heart a kind of differential calculus for the Wiener space $C_0[0,1]$, thought of as a subspace of $\mathbb{R}^{[0,1]}$ which is itself viewed as a product space generalising $\mathbb{R}^n$, with its associated differential calculus. Conventional expositions do not make this so clear. In our approach we achieve the "passage to the limit" from $\mathbb{R}^n$ to $C_0[0,1]$ by considering ${}^*\mathbb{R}^N$ for infinite hyperfinite N. The Malliavin calculus can then be seen as a suitable projection of classical calculus in ${}^*\mathbb{R}^N$ onto $C_0[0,1]$. Among other applications, this shows how Wiener measure "is" simply the uniform probability measure on the sphere $S^\infty(\sqrt{\infty})$, and allows a precise formulation of the experts' intuition that the infinite dimensional Ornstein–Uhlenbeck process "is" Brownian motion on $S^\infty(\sqrt{\infty})$.

The final applications, in Chapter 4, are in the field of modern mathematical finance theory, which, in common with the the previous topics, has stochastic analysis (particulary Brownian motion and Itô integration) at its foundation. Here there is great interest in connecting the approach using financial models based on a discrete model of time, with the other main

[2] As the reader may be aware, the starting point of NSA is to construct a field ${}^*\mathbb{R} \supset \mathbb{R}$ that contains both infinitesimal and infinite elements – and this contains a corresponding extension ${}^*\mathbb{N}$ of $\mathbb{N}$. Then an infinite hyperfinite number N is an element of ${}^*\mathbb{N} \setminus \mathbb{N}$.

approach, using continuous time. The latter is in some sense obtained by "passing to the limit" in discrete-time models – and again the NSA framework greatly facilitates this. In essence, if we have for each $n \in \mathbb{N}$ a discrete financial model $\mathcal{M}_n$ then we immediately have $\mathcal{M}_N$ for infinite hyperfinite N. Then we can show that the continuous model is obtained as a suitable projection of $\mathcal{M}_N$, and as a result obtain some new powerful convergence results.

Acknowledgments

On a personal note, I should like to thank

- the EMS for the kind invitation to give the 1997 EMS Lectures;

- the Mathematics Department of the University of Helsinki, for their generous hospitality;

- my collaborators Marek Capiński, Jerry Keisler, Siu-Ah Ng, Ekkehard Kopp and Walter Willinger, who have worked jointly with him on much of the work that is described in these lectures.

Many thanks too to my wife Mary, who has been so supportive, patient and encouraging over the many years during which the material surveyed here was developed.

Hull, November 1999 *Nigel Cutland*