Preliminaries

Introduction

This memoir is a thoroughly revised and updated version of the *Subiaco Notebook*, which since early 1998 has been available on the web page of the senior author at `http://www-math.cudenver.edu/~spayne/`.

Our goal is to give a fairly complete and nearly self-contained treatment of the known (infinite families of) generalized quadrangles arising from q-clans, i.e., flocks of quadratic cones in $PG(3,q)$, with $q = 2^e$. Our main interest is in the construction of the generalized quadrangles, a determination of the associated ovals, and then a complete determination of the groups of automorphisms of these objects. A great deal of general theoretical material of related interest has been omitted. However, we hope that the reader will find the treatment here to be coherent and complete as a treatment of one major part of the theory of flock generalized quadrangles.

Since the appearance of the Subiaco Notebook the Adelaide q-clans have been discovered, generalizing the few examples of "cyclic" q-clans found first by computer (see [PPR97]). The revised treatment given here of the cyclic q-clans, which is a slight improvement of that given in [COP03], allows much of the onerous computation in the Subiaco Notebook to be avoided while at the same time allowing a more unified approach to the general subject. However, a great deal of computation is still unavoidable.

Most of the work done on the Adelaide examples, especially our study of the Adelaide ovals, and major steps in the clarification of the connection between the so-called Magic Action of O'Keefe and Penttila (see [OP02]) and the Fundamental Theorem of q-clan geometry, took place while the senior author was a visiting research professor at the Universities of Naples, Italy, and Ghent, Belgium, during the winter and spring of the year 2002. This was made possible by a semester-long sabbatical provided by the author's home institution, the University of Colorado at Denver.

During the two months he spent in Italy, at the invitation of Professor Dr. Guglielmo Lunardon (with the collaboration of Prof. Laura Bader), he received generous financial support from the GNSAGA and the University of Naples, along

with a great deal of personal support from his colleagues there. Also, much of the material on the cyclic q-clans derives from the reports [Pa02a] and [Pa02b] and has appeared in [CP03].

During his two months in Belgium, at the invitation of Prof. Dr. Joseph A. Thas, he was generously supported by the Research Group in Incidence Geometry at Ghent University. As always, it was a truly great pleasure to work in the stimulating and friendly atmosphere provided by his colleagues there. All the material on the Adelaide ovals was adapted from [PT05].

It was in Naples during the trimester February-March 2002 that the second author became familiar with the Four Lectures in Naples [Pa02a] and the idea of working with the senior author to complete the present memoir first occured to us.

The final steps in the clarification of the connections between the Magic action and the Fundamental Theorem, together with a revision of all the extracts from the Subiaco Notebook, were taken while the second author was visiting the Department of Mathematics at the University of Colorado at Denver in the Fall of 2002. During that period the present work was essentially completed. It was a great pleasure to study and work in the friendly, enjoyable and stimulating atmosphere provided by all our colleagues at the University of Colorado at Denver. We wish to thank in particular our friend and colleague Dr. William Cherowitzo for several fruitful mathematical conversations and for his many suggestions concerning the use of LaTeX. We hope that this memoir will become a standard source of information for those who are interested in q-clan geometries when q is even.

Finite Generalized Quadrangles

For a thorough introduction to finite generalized quadrangles, see the monograph [PT84]. However, for the convenience of the reader we review here a few definitions and elementary results without proof.

A *finite generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint (finite, non-empty) sets of objects called *points* and *lines*, respectively, and for which $I \subseteq (\mathcal{P} \times \mathcal{B}) \cup (\mathcal{B} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line;

(ii) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point;

(iii) If x is a point and L is a line not incident with x, there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $xIMIyIL$.

The integers s and t are the *parameters* of the GQ, and $\mathcal{S}$ is said to have *order* (s,t). If $s = t$, $\mathcal{S}$ is said to have order s. If $\mathcal{S}$ has order (s,t), then $|\mathcal{P}| = (1+s)(1+st)$

and $|\mathcal{B}| = (1+t)(1+st)$. If $\mathcal{S}$ is a GQ of order (s,t), then the incidence structure $\mathcal{S}^D$ having as *lines*, the points of $\mathcal{S}$ and as *points* the lines of $\mathcal{S}$, is a GQ of order (t,s) called the *point-line dual* of $\mathcal{S}$.

The classical GQs of order q are $W(q)$, which arise as the absolute points and lines of a symplectic polarity of $PG(3,q)$; and $Q(4,q)$, which arise as the points and lines of a non-singular quadric of $PG(4,q)$. By ([PT84], 3.2.1), $W(q)$ is isomorphic to the dual of $Q(4,q)$ and is self-dual for q even. Another class of GQs of order q are the $T_2(\mathcal{O})$ constructed by J. Tits and first appearing in [De68] (see [PT84], 3.1.2). Let $\mathcal{O}$ be an oval of $PG(2,q)$, and let $\pi_\infty = PG(2,q)$ be embedded as a hyperplane in $PG(3,q)$. The *points* of $T_2(\mathcal{O})$ are: (i) the points of $PG(3,q) \setminus \pi_\infty$, called the *affine points*; (ii) the planes of $PG(3,q)$ meeting π_∞ in a single point of $\mathcal{O}$; and (iii) a symbol (∞). The *lines* of $T_2(\mathcal{O})$ are: (a) the lines of $PG(3,q)$, not in π_∞, that meet π_∞ in a single point of $\mathcal{O}$; and (b) the points of $\mathcal{O}$. Incidence is as follows: a point of type (i) is incident only with the lines of type (a) that contain it; a point of type (ii) is incident with all the lines of type (a) contained in it and with the unique line of type (b) on it; the point of type (iii) is incident with all lines of type (b) and no line of type (a).

The GQ $T_2(\mathcal{O})$ has a slightly simpler description that is easily seen to be equivalent to the one given above *when q is even.* Let N be the nucleus of $\mathcal{O}$. Here the points of $T_2(\mathcal{O})$ are the points of $PG(3,q) \setminus \pi_\infty$ and the planes of $PG(3,q)$ that contain the point N. The lines of $T_2(\mathcal{O})$ are the lines of $PG(3,q)$ that meet $\mathcal{O}$ in a single point. Incidence is that inherited from $PG(3,q)$. For example, the plane π_∞ as a point of $T_2(\mathcal{O})$ is incident with the lines of π_∞ that are tangent to $\mathcal{O}$, i.e., the lines of π_∞ incident with N.

There is a natural analog $T_3(\Omega)$ of the preceding construction also given by J. Tits that yields GQ of order (q,q^2) starting from an ovoid Ω in $PG(3,q)$ (i.e., q^2+1 points no three on a line). Let Ω be an ovoid of $\Sigma = PG(3,q)$ which is embedded as a hyperplane in $PG(4,q)$. Construct the point-line incidence geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, I)$ with pointset $\mathcal{P}$, lineset $\mathcal{B}$, and incidence I as follows:

The points of $\mathcal{S}$, i.e., elements of $\mathcal{P}$, are of three types:

(i) The points of $PG(4,q) \setminus \Sigma$;

(ii) The hyperplanes of $PG(4,q)$ meeting Σ in a plane tangent to $\mathcal{O}$.

(iii) The symbol (∞).

The lines of $\mathcal{S}$ are of two types:

(a) The lines of $PG(4,q)$ that are not contained in Σ and meet Σ in a (necessarily unique) point of Ω.

(b) The points of Ω.

Incidence in $T_3(\Omega)$ is defined by the following: the point (∞) is incident with the $1+q^2$ lines of type (b). Suppose $\triangle$ is a solid of $PG(4,q)$ meeting Σ in the plane T_p tangent to Ω at the point p. Then $\triangle$ is incident with p (as a line of type (b)) and with the q^2 lines of $\triangle$ through p but not contained in Σ. Each point x of $PG(4,q) \setminus \Sigma$ is incident with the $1+q^2$ lines px, $p \in \Omega$.

These GQ constructed by J. Tits first appeared in [De68] (also see [PT84]).

The GQ that are of primary interest in this memoir are those with parameters (q^2, q) arising from a q-clan, i.e., from a flock of a quadratic cone in $PG(3,q)$. These will be introduced in Chapter 1. If Ω is an elliptic quadric in Σ, it turns out that $T_3(\Omega)$ is isomorphic to the point-line dual of a GQ arising from a *linear flock*.

As this monograph is an extended and updated version of the Subiaco Notebook, it seems appropriate to present here an updated version of the Prolegomena from the Subiaco Notebook.

Prolegomena

At a conference in Han-Sur-Lesse, Belgium, held in 1979, W. M. Kantor [Ka80] surprised the finite geometry community with the construction of a family of generalized quadrangles (GQ) of order (q^2, q) for each prime power q with $q \equiv 2 \pmod 3$. His discovery of these examples began with a classical generalized hexagon, but his description of them introduced the following very important general method.

Let G be a finite group of order s^2t, $1 < s$, $1 < t$, together with a family $\mathcal{J} = \{A_i : 0 \le i \le t\}$ of $1+t$ subgroups of G, each of order s. In addition, suppose that for each $A_i \in \mathcal{J}$ there is a subgroup A_i^* of G, of order st, containing A_i. Put $\mathcal{J}^* = \{A_i^* : 0 \le i \le t\}$ and define as follows a point-line geometry $\mathcal{S} = (P, B, I) = \mathcal{S}(G, \mathcal{J})$ with pointset P, lineset B, and incidence I.

Points are of three kinds:

(i) the elements of G;
(ii) the right cosets A_i^*g, (for all $A_i^* \in \mathcal{J}^*$, $g \in G$);
(iii) a symbol (∞).

Lines are of two kinds:

(a) the right cosets A_ig, (for all $A_i \in \mathcal{J}$, $g \in G$);
(b) the symbols $[A_i]$, $A_i \in \mathcal{J}$.

A point g of type (i) is incident with each line A_ig, $A_i \in \mathcal{J}$; a point A_i^*g of type (ii) is incident with $[A_i]$ and with each line A_ih contained in A_i^*g; the point (∞) is incident with each line $[A_i]$ of type (b).

Kantor [Ka80] was the first to recognize that $\mathcal{S}(G, \mathcal{J})$ is a GQ of order (s,t) if and only if the following two conditions are satisfied:

K1: $A_iA_j \cap A_k = \{e\}$, for distinct i, j, k, and

K2: $A_i^* \cap A_j = \{e\}$, for $i \ne j$.

If the conditions K1 and K2 are satisfied, then

$$A_i^* = \cup\{A_i g : A_i g = A_i \text{ or } A_i g \cap A_j = \emptyset \text{ for all } A_j \in \mathcal{J}\},$$

so that A_i^* is uniquely defined by A_i and $\mathcal{J}$.

Suppose K1 and K2 are satisfied. For any $h \in G$ define θ_h by $g^{\theta_h} = gh$, $(A_i g)^{\theta_h} = A_i gh$, $(A_i^* g)^{\theta_h} = A_i^* gh$, $[A_i]^{\theta_h} = [A_i]$, $(\infty)^{\theta_h} = (\infty)$, for $g \in G$, $A_i \in \mathcal{J}$, $A_i^* \in \mathcal{J}^*$. Then θ_h is an automorphism of $\mathcal{S}(G, \mathcal{J})$ which fixes the point (∞) and all lines of type (b). If $G' = \{\theta_h : h \in G\}$, then clearly G' is a group isomorphic to G, and G' acts regularly on the points of type (i).

If K1 and K2 are satisfied, we say that $\mathcal{J}$ is a 4-*gonal family for* G, or that $(G, \mathcal{J}, \mathcal{J}^*)$ is a *Kantor family*.

The appearance of the new GQ in Kantor [Ka80] inspired the development in Payne [Pa80] of a more specific recipe for constructing GQ. Then in Kantor [Ka86] the conditions in Payne [Pa80] for q odd were shown to be equivalent to having what today we call a q-clan. This in turn inspired an analogous interpretation in Payne [Pa85] for $q = 2^e$, and the discovery of a new infinite family of GQ with order (q^2, q), $q = 2^e$, e odd. The term q-*clan* was first used in Payne [Pa89]. When $q = 2^e$, along with each q-clan comes a collection of ovals in $PG(2, q)$. The ovals in Payne [Pa85] were new, but only exist for q a nonsquare. It was still true that the only examples of ovals not related to conics that were known for q a square were associated with the translation ovals (first constructed by B. Segre [Se57]) or with the single example of Lunelli-Sce [LS58] in $PG(2, 16)$. Then in 1993 W. Cherowitzo, T. Penttila, I. Pinneri and G. Royle discovered the Subiaco q-clans with their new ovals and GQ. By the time their paper [CPPR96] appeared, a great deal of work on the Subiaco geometries had been done. In particular we mention [Pa94], [BLP94], [PPP95], and [OKT96]. Work on the Subiaco GQ also directly inspired [Pa96] and [Pa95], and the latter contains additional results on the Subiaco ovals.

With the computationally efficient tensor product notation of [Pa95] having been found only after most the computations for the Subiaco GQ had been done, with so much work on the Subiaco GQ involving computations only sketched in the published articles, and with the discovery of the Adelaide examples, we feel that there should exist a single, coherent treatment of the general body of work related especially to the GQ arising from q-clans and their associated ovals. Especially we wanted to see one treatment determining the collineation groups of the GQ and of the associated ovals. Hence these notes!

The scope of this replacement for the Subiaco Notebook remains similar to that of the Subiaco Notebook (but of course the Adelaide examples were not known when the Subiaco Notebook was written). We have included an essentially self-contained introduction to q-clan geometry (i.e., flock GQ, etc.) with $q = 2^e$, giving a detailed construction of the known examples and a determination of their groups. Moreover, even though we attempt to give many details here and always enough for the energetic reader to fill in what we have left out, still there are many computations that we have found it feasible only to sketch.

The other geometries associated with the Subiaco q-clans, for example spreads of $PG(3,q)$ and translation planes, are also mentioned briefly, but the main goal is to give a complete, self-contained treatment of the GQ and ovals.

For a broad survey of these topics, including geometries over finite and infinite fields of arbitrary characteristic, see the survey by N. L. Johnson and S. E. Payne [JP97]. For an excellent survey of the geometries related to finite q-clans, especially flocks of quadratic cones known prior to the discovery of the Subiaco examples, see Chapers 7 and 9 by J. A. Thas in the *Handbook* [Bu95].

The most exciting development not covered in the present work is no doubt the geometric construction by J. A. Thas of flock GQ for all characteristics, along with his work on Property (G). See [Th98] and its references for this major work that succeeded in proving that when q is odd, Property (G) at a point completely characterizes the flock GQ. His work left the characteristic 2 case not quite finished, but more recently M. R. Brown (see [BBP06], [Br07] and [Br06]) has finished this case as well, giving new insight to the constructions.

The present work is not really an introduction to the general theory of finite generalized quadrangles per se, or even to *elation generalized quadrangles*. And although in the preceding section we have repeated Kantor's construction of elation generalized quadrangles as group coset geometries, we assume that the reader has access to some other source of basic concepts and proofs related to finite GQ, for example the monograph by S. E. Payne and J. A. Thas [PT84]. The most thorough recent treatment of related topics is surely the new monograph [TTVM06].