IRRESISTIBLE INTEGRALS

Symbolics, Analysis and Experiments in the Evaluation of Integrals

GEORGE BOROS
Formerly of Xavier University of Lousiana

VICTOR MOLL
Tulane University

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2004

Printed in the United States of America

Typefaces Times Roman 10.25/13 pt. and Courier *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Boros George, 1947–

Irresistible integrals : symbolics, analysis and experiments in the evaluation of integrals /
George Boros, Victor H. Moll.

p. cm

Includes bibliographical references and index.

ISBN 0-521-79186-3 – ISBN 0-521-79636-9 (pbk)

1. Definite integrals. I. Moll, Victor. II. Title.

QA308.B67 2004
515′.43 – dc22 2003069574

ISBN 0 521 79186 3 hardback
ISBN 0 521 79636 9 paperback

Contents

Preface *page* xi

1 Factorials and Binomial Coefficients 1
- 1.1 Introduction 1
- 1.2 Prime Numbers and the Factorization of $n!$ 3
- 1.3 The Role of Symbolic Languages 7
- 1.4 The Binomial Theorem 10
- 1.5 The Ascending Factorial Symbol 16
- 1.6 The Integration of Polynomials 19

2 The Method of Partial Fractions 25
- 2.1 Introduction 25
- 2.2 An Elementary Example 30
- 2.3 Wallis' Formula 32
- 2.4 The Solution of Polynomial Equations 36
- 2.5 The Integration of a Biquadratic 44

3 A Simple Rational Function 48
- 3.1 Introduction 48
- 3.2 Rational Functions with a Single Multiple Pole 49
- 3.3 An Empirical Derivation 49
- 3.4 Scaling and a Recursion 51
- 3.5 A Symbolic Evaluation 53
- 3.6 A Search in Gradshteyn and Ryzhik 55
- 3.7 Some Consequences of the Evaluation 56
- 3.8 A Complicated Integral 58

4 A Review of Power Series 61
- 4.1 Introduction 61
- 4.2 Taylor Series 65
- 4.3 Taylor Series of Rational Functions 67

5 The Exponential and Logarithm Functions 73
5.1 Introduction 73
5.2 The Logarithm 74
5.3 Some Logarithmic Integrals 81
5.4 The Number e 84
5.5 Arithmetical Properties of e 89
5.6 The Exponential Function 91
5.7 Stirling's Formula 92
5.8 Some Definite Integrals 97
5.9 Bernoulli Numbers 99
5.10 Combinations of Exponentials and Polynomials 103
6 The Trigonometric Functions and π 105
6.1 Introduction 105
6.2 The Basic Trigonometric Functions and the Existence of π 106
6.3 Solution of Cubics and Quartics by Trigonometry 111
6.4 Quadratic Denominators and Wallis' Formula 112
6.5 Arithmetical Properties of π 117
6.6 Some Expansions in Taylor Series 118
6.7 A Sequence of Polynomials Approximating $\tan^{-1} x$ 124
6.8 The Infinite Product for $\sin x$ 126
6.9 The Cotangent and the Riemann Zeta Function 129
6.10 The Case of a General Quadratic Denominator 133
6.11 Combinations of Trigonometric Functions and Polynomials 135
7 A Quartic Integral 137
7.1 Introduction 137
7.2 Reduction to a Polynomial 139
7.3 A Triple Sum for the Coefficients 143
7.4 The Quartic Denominators: A Crude Bound 144
7.5 Closed-Form Expressions for $d_l(m)$ 145
7.6 A Recursion 147
7.7 The Taylor Expansion of the Double Square Root 150
7.8 Ramanujan's Master Theorem and a New Class of Integrals 151
7.9 A Simplified Expression for $P_m(a)$ 153
7.10 The Elementary Evaluation of $N_{j,4}(a;m)$ 159
7.11 The Expansion of the Triple Square Root 160
8 The Normal Integral 162
8.1 Introduction 162
8.2 Some Evaluations of the Normal Integral 164

8.3 Formulae from Gradshteyn and Rhyzik (G & R) 171
8.4 An Integral of Laplace 171
9 Euler's Constant 173
9.1 Existence of Euler's Constant 173
9.2 A Second Proof of the Existence of Euler's Constant 174
9.3 Integral Forms for Euler's Constant 176
9.4 The Rate of Convergence to Euler's Constant 180
9.5 Series Representations for Euler's Constant 183
9.6 The Irrationality of γ 184
10 Eulerian Integrals: The Gamma and Beta Functions 186
10.1 Introduction 186
10.2 The Beta Function 192
10.3 Integral Representations for Gamma and Beta 193
10.4 Legendre's Duplication Formula 195
10.5 An Example of Degree 4 198
10.6 The Expansion of the Loggamma Function 201
10.7 The Product Representation for $\Gamma(x)$ 204
10.8 Formulas from Gradshteyn and Rhyzik (G & R) 206
10.9 An Expression for the Coefficients $d_l(m)$ 207
10.10 Holder's Theorem for the Gamma Function 210
10.11 The Psi Function 212
10.12 Integral Representations for $\psi(x)$ 215
10.13 Some Explicit Evaluations 217
11 The Riemann Zeta Function 219
11.1 Introduction 219
11.2 An Integral Representation 222
11.3 Several Evaluations for $\zeta(2)$ 225
11.4 Apery's Constant: $\zeta(3)$ 231
11.5 Apery Type Formulae 235
12 Logarithmic Integrals 237
12.1 Polynomial Examples 238
12.2 Linear Denominators 239
12.3 Some Quadratic Denominators 241
12.4 Products of Logarithms 244
12.5 The Logsine Function 245
13 A Master Formula 250
13.1 Introduction 250
13.2 Schlomilch Transformation 251
13.3 Derivation of the Master Formula 252
13.4 Applications of the Master Formula 253
13.5 Differentiation Results 257

13.6 The Case $a = 1$ 258
13.7 A New Series of Examples 263
13.8 New Integrals by Integration 266
13.9 New Integrals by Differentiation 268
Appendix: The Revolutionary WZ Method 271
A.1 Introduction 271
A.2 An Introduction to WZ Methods 272
A.3 A Proof of Wallis' Formula 273

Bibliography 276

Index 299

1

Factorials and Binomial Coefficients

1.1. Introduction

In this chapter we discuss several properties of factorials and binomial coefficients. These functions will often appear as results of evaluations of definite integrals.

Definition 1.1.1. A function $f : \mathbb{N} \to \mathbb{N}$ is said to satisfy a **recurrence** if the value $f(n)$ is determined by the values $\{f(1), f(2), \cdots, f(n-1)\}$. The recurrence is of **order** k if $f(n)$ is determined by the values $\{f(n-1), f(n-2), \cdots, f(n-k)\}$, where k is a fixed positive integer. The notation f_n is sometimes used for $f(n)$.

For example, the **Fibonacci numbers** F_n satisfy the second-order recurrence

$$F_n = F_{n-1} + F_{n-2}. \tag{1.1.1}$$

Therefore, in order to compute F_n, one needs to know only F_1 and F_2. In this case $F_1 = 1$ and $F_2 = 1$. These values are called the **initial conditions** of the recurrence. The Mathematica command

```
F[n_]:= If[n==0,1, If[n==1,1, F[n-1]+ F[n-2]]]
```

gives the value of F_n. The modified command

```
F[n_]:= F[n]= If[n==0,1, If[n==1,1, F[n-1]+F[n-2]]]
```

saves the previously computed values, so at every step there is a single sum to perform.

Exercise 1.1.1. Compare the times that it takes to evaluate

$$F_{30} = 832040 \tag{1.1.2}$$

using both versions of the function F.

A recurrence can also be used to *define* a sequence of numbers. For instance

$$D_{n+1} = n(D_n + D_{n-1}),\ n \geq 2 \tag{1.1.3}$$

with $D_1 = 0,\ D_2 = 1$ defines the **derangement numbers**. See Rosen (2003) for properties of this interesting sequence.

We now give a recursive definition of the factorials.

Definition 1.1.2. The **factorial** of $n \in \mathbb{N}$ is defined by

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1. \tag{1.1.4}$$

A recursive definition is given by

$$\begin{aligned} 1! &= 1 \\ n! &= n \times (n-1)!. \end{aligned} \tag{1.1.5}$$

The first exercise shows that the recursive definition characterizes $n!$. This technique will be used throughout the book: in order to prove some identity, you check that both sides satisfy the same recursion and that the initial conditions match.

Exercise 1.1.2. Prove that the factorial is the unique solution of the recursion

$$x_n = n \times x_{n-1} \tag{1.1.6}$$

satisfying the initial condition $x_1 = 1$. **Hint**. Let $y_n = x_n/n!$ and use (1.1.5) to produce a trivial recurrence for y_n.

Exercise 1.1.3. Establish the formula

$$D_n = n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{1.1.7}$$

Hint. Check that the right-hand side satisfies the same recurrence as D_n and then check the initial conditions.

The first values of the sequence $n!$ are

$$1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \tag{1.1.8}$$

and these grow very fast. For instance

$$\begin{aligned} 50! = {} & 30414093201713378043612608166064768844377641568960512 \\ & 000000000000 \end{aligned}$$

and 1000! has 2568 digits.

Mathematica 1.1.1. The Mathematica command for $n!$ is `Factorial [n]`. The reader should check the value 1000! stated above. The number of digits of an integer can be obtained with the Mathematica command `Length[IntegerDigits[n]]`.

The next exercise illustrates the fact that the extension of a function from $\mathbb{N}$ to $\mathbb{R}$ sometimes produces unexpected results.

Exercise 1.1.4. Use Mathematica to check that $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$.

The exercise is one of the instances in which the factorial is connected to π, the fundamental constant of trigonometry. Later we will see that the growth of $n!$ as $n \to \infty$ is related to e: the base of natural logarithms. These issues will be discussed in Chapters 5 and 6, respectively. To get a complete explanation for the appearance of π, the reader will have to wait until Chapter 10 where we introduce the **gamma function**.

1.2. Prime Numbers and the Factorization of *n*!

In this section we discuss the factorization of $n!$ into prime factors.

Definition 1.2.1. An integer $n \in \mathbb{N}$ is **prime** if its only divisors are 1 and itself.

The reader is refered to Hardy and Wright (1979) and Ribenboim (1989) for more information about prime numbers. In particular, Ribenbiom's first chapter contains many proofs of the fact that there are infinitely many primes. Much more information about primes can be found at the site

`http://www.utm.edu/research/primes/`

The set of prime numbers can be used as building blocks for all integers. This is the content of the **Fundamental Theorem of Arithmetic** stated below.

Theorem 1.2.1. *Every positive integer can be written as a product of prime numbers. This factorization is unique up to the order of the prime factors.*

The proof of this result appears in every introductory book in number theory. For example, see Andrews (1994), page 26, for the standard argument.

Mathematica 1.2.1. The Mathematica command `FactorInteger[n]` gives the complete factorization of the integer n. For example `FactorInteger[1001]` gives the prime factorization $1001 = 7 \cdot 11 \cdot 13$. The concept of prime factorization can now be extended to rational numbers by allowing negative exponents. For example

$$\frac{1001}{1003} = 7 \cdot 11 \cdot 13 \cdot 17^{-1} \cdot 59^{-1}. \tag{1.2.1}$$

The efficient complete factorization of a large integer n is one of the basic questions in computational number theory. The reader should be careful with requesting such a factorization from a symbolic language like Mathematica: the amount of time required can become very large. A safeguard is the command

```
FactorInteger[n, FactorComplete -> False]
```

which computes the small factors of n and leaves a part unfactored. The reader will find in Bressoud and Wagon (2000) more information about these issues.

Definition 1.2.2. Let p be prime and $r \in \mathbb{Q}^+$. Then there are unique integers a, b, not divisible by p, and $m \in \mathbb{Z}$ such that

$$r = \frac{a}{b} \times p^m. \tag{1.2.2}$$

The p-**adic valuation** of r is defined by

$$\nu_p(r) = p^{-m}. \tag{1.2.3}$$

The integer m in (1.2.2) will be called the **exponent of** p **in** m and will be denoted by $\mu_p(r)$, that is,

$$\nu_p(r) = p^{-\mu_p(r)}. \tag{1.2.4}$$

Extra 1.2.1. The p-adic valuation of a rational number gives a new way of measuring its size. In this context, a number is small if it is divisible by a large power of p. This is the basic idea behind *p-adic Analysis*. Nice introductions to this topic can be found in Gouvea (1997) and Hardy and Wright (1979).

Exercise 1.2.1. Prove that the valuation ν_p satisfies

$$\nu_p(r_1 r_2) = \nu_p(r_1) \times \nu_p(r_2),$$
$$\nu_p(r_1/r_2) = \nu_p(r_1)/\nu_p(r_2),$$

and

$$\nu_p(r_1 + r_2) \leq \text{Max}\left(\nu_p(r_1), \nu_p(r_2)\right),$$

with equality unless $\nu_p(r_1) = \nu_p(r_2)$.

Extra 1.2.2. The p-adic numbers have many surprising properties. For instance, a series converges p-adically if and only if the general term converges to 0.

Definition 1.2.3. The **floor** of $x \in \mathbb{R}$, denoted by $\lfloor x \rfloor$, is the smallest integer less or equal than x. The Mathematica command is `Floor[x]`.

We now show that the factorization of $n!$ can be obtained *without* actually computing its value. This is useful considering that $n!$ grows very fast—for instance 10000! has 35660 digits.

Theorem 1.2.2. *Let p be prime and $n \in \mathbb{N}$. The exponent of p in $n!$ is given by*

$$\mu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor. \tag{1.2.5}$$

Proof. In the product defining $n!$ one can divide out every multiple of p, and there are $\lfloor n/p \rfloor$ such numbers. The remaining factor might still be divisible by p and there are $\lfloor n/p^2 \rfloor$ such terms. Now continue with higher powers of p. □

Note that the sum in (1.2.5) is finite, ending as soon as $p^k > n$. Also, this sum allows the fast factorization of $n!$. The next exercise illustrates how to do it.

Exercise 1.2.2. Count the number of divisions required to obtain

$$50! = 2^{47} \cdot 3^{22} \cdot 5^{12} \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47,$$

using (1.2.5).

Exercise 1.2.3. Prove that every prime $p \leq n$ appears in the prime factorization of $n!$ and that every prime $p > n/2$ appears to the first power.

There are many expressions for the function $\mu_p(n)$. We present a proof of one due to Legendre (1830). The result depends on the expansion of an integer in base p. The next exercise describes how to obtain such expansion.

Exercise 1.2.4. Let $n, p \in \mathbb{N}$. Prove that there are integers $n_0, n_1, \cdots, n_r$ such that

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r \tag{1.2.6}$$

where $0 \leq n_i < p$ for $0 \leq i \leq r$. **Hint**. Recall the division algorithm: given $a, b \in \mathbb{N}$ there are integers q, r, with $0 \leq r < b$ such that $a = qb + r$. To obtain the coefficients n_i first divide n by p.

Theorem 1.2.3. *The exponent of p in n! is given by*

$$\mu_p(n!) = \frac{n - s_p(n)}{p-1}, \tag{1.2.7}$$

where $s_p(n) = n_0 + n_1 + \cdots + n_r$ *is the sum of the base-p digits of n. In particular,*

$$\mu_2(n!) = n - s_2(n). \tag{1.2.8}$$

Proof. Write n in base p as in (1.2.6). Then

$$\mu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = (n_1 + n_2 p + \cdots + n_r p^{r-1}) \\ + (n_2 + n_3 p + \cdots + n_r p^{r-2}) + \cdots + n_r,$$

so that

$$\begin{aligned} \mu_p(n!) &= n_1 + n_2(1+p) + n_3(1+p+p^2) + \cdots + n_r(1+p+\cdots+p^{r-1}) \\ &= \frac{1}{p-1}\big(n_1(p-1) + n_2(p^2-1) + \cdots + n_r(p^r-1)\big) \\ &= \frac{n - s_p(n)}{p-1}. \end{aligned}$$

□

Corollary 1.2.1. *The exponent of p in n! satisfies*

$$\mu_p(n!) \leq \frac{n-1}{p-1}, \tag{1.2.9}$$

with equality if and only if n is a power of p.

Mathematica 1.2.2. The command `IntegerDigits[n,p]` gives the list of numbers n_i in Exercise 1.2.4.

Exercise 1.2.5. Define

$$A_1(m) = (2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1). \tag{1.2.10}$$

Prove that, for any prime $p \neq 2$,

$$\mu_p(A_1(m)) \geq \mu_p(m!). \tag{1.2.11}$$

Hint. Let $a_m = \prod_{k=1}^{m}(4k-1)$ and $b_m = \prod_{k=1}^{m}(4k+1)$ so that a_m is the product of the least m positive integers congruent to 1 modulo 4. Observe that for $p \geq 3$ prime and $k \in \mathbb{N}$, exactly one of the first p^k positive integers congruent to 3 modulo 4 is divisible by p^k and the same is true for integers congruent to 1 modulo 4. Conclude that $A_1(m)$ is divisible by the odd part of $m!$. For instance,

$$\frac{A_1(30)}{30!} = \frac{359937762656357407018337533}{2^{24}}. \tag{1.2.12}$$

The products in (1.2.10) will be considered in detail in Section 10.9.

1.3. The Role of Symbolic Languages

In this section we discuss how to use Mathematica to conjecture general closed form formulas. A simple example will illustrate the point.

Exercise 1.2.3 shows that $n!$ is divisible by a large number of consecutive prime numbers. We now turn this information around to empirically suggest closed-form formulas. Assume that in the middle of a calculation we have obtained the numbers

$$\begin{aligned} x_1 &= 5356234211328000 \\ x_2 &= 102793666719744000 \\ x_3 &= 2074369080655872000 \\ x_4 &= 43913881247588352000 \\ x_5 &= 973160803270656000000, \end{aligned}$$

and one hopes that these numbers obey a simple rule. The goal is to obtain a function $x : \mathbb{N} \to \mathbb{N}$ that interpolates the given values, that is, $x(i) = x_i$ for $1 \leq i \leq 5$. Naturally this question admits more than one solution, and we will use Mathematica to find one. The prime factorization of the data is

$$\begin{aligned} x_1 &= 2^{23} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \\ x_2 &= 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17^2 \\ x_3 &= 2^{18} \cdot 3^{12} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \\ x_4 &= 2^{16} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19^3 \\ x_5 &= 2^{22} \cdot 3^8 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \end{aligned}$$

and a moment of reflection reveals that x_i contains all primes less than $i + 15$. This is also true for $(i + 15)!$, leading to the consideration of $y_i = x_i/(i + 15)!$. We find that

$$\begin{aligned} y_1 &= 256 \\ y_2 &= 289 \\ y_3 &= 324 \\ y_4 &= 361 \\ y_5 &= 400, \end{aligned}$$

so that $y_i = (i + 15)^2$. Thus $x_i = (i + 15)^2 \times (i + 15)!$ is one of the possible rules for x_i. This can be then tested against more data, and if the rule still holds, we have produced the conjecture

$$z_i = i^2 \times i!, \tag{1.3.1}$$

where $z_i = x_{i+15}$.

Definition 1.3.1. Given a sequence of numbers $\{a_k : k \in \mathbb{N}\}$, the function

$$T(x) = \sum_{k=0}^{\infty} a_k x^k \tag{1.3.2}$$

is the **generating function** of the sequence. If the sequence is finite, then we obtain a **generating polynomial**

$$T_n(x) = \sum_{k=0}^{n} a_k x^k. \tag{1.3.3}$$

The generating function is one of the forms in which the sequence $\{a_k : 0 \leq k \leq n\}$ can be incorporated into an analytic object. Usually this makes it easier to perform calculations with them. Mathematica *knows* a large number of polynomials, so if $\{a_k\}$ is part of a known family, then a symbolic search will produce an expression for T_n.

Exercise 1.3.1. Obtain a closed-form for the generating function of the Fibonacci numbers. **Hint**. Let $f(x) = \sum_{n=0}^{\infty} F_n x^n$ be the generating function. Multiply the recurrence (1.1.1) by x^n and sum from $n = 1$ to ∞. In order to manipulate the resulting series observe that

$$\begin{aligned} \sum_{n=1}^{\infty} F_{n+1} x^n &= \sum_{n=2}^{\infty} F_n x^{n-1} \\ &= \frac{1}{x} \left(f(x) - F_0 - F_1 x \right). \end{aligned}$$

The answer is $f(x) = x/(1 - x - x^2)$. The Mathematica command to generate the first n terms of this is

```
list[n_]:= CoefficientList
[Normal[Series[ x/(1-x-x^{2}), {x,0,n-1}]],x]
```

For example, `f[10]` gives {0, 1, 1, 2, 3, 5, 8, 13, 21, 34}.

It is often the case that the answer is expressed in terms of more complicated functions. For example, Mathematica evaluates the polynomial

$$G_n(x) = \sum_{k=0}^{n} k! x^k \tag{1.3.4}$$

as

$$G_n(x) = -\frac{e^{-1/x}}{x} \left\{ \Gamma(0, -\tfrac{1}{x}) + (-1)^n \Gamma(n+2) \Gamma(-1-n, -\tfrac{1}{x}) \right\}, \tag{1.3.5}$$

where e^u is the usual **exponential function**,

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt \tag{1.3.6}$$

is the **gamma function**, and

$$\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t}\, dt \tag{1.3.7}$$

is the **incomplete gamma function**. The exponential function will be discussed in Chapter 5, the gamma function in Chapter 10, and the study of $\Gamma(a, x)$ is postponed until Volume 2.

1.4. The Binomial Theorem

The goal of this section is to recall the binomial theorem and use it to find closed-form expressions for a class of sums involving binomial coefficients.

Definition 1.4.1. The **binomial coefficient** is

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!}, \quad 0 \leq k \leq n. \tag{1.4.1}$$

Theorem 1.4.1. *Let $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$. Then*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k. \tag{1.4.2}$$

Proof. We use induction. The identity $(a+b)^n = (a+b) \times (a+b)^{n-1}$ and the induction hypothesis yield

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n-k} b^k + \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n-k-1} b^{k+1} \\ &= a^n + \sum_{k=1}^{n-1} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] a^{n-k} b^k + b^n.\end{aligned}$$

The result now follows from the identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \tag{1.4.3}$$

that admits a direct proof using (1.4.1). □

Exercise 1.4.1. Check the details.

Note 1.4.1. The binomial theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{1.4.4}$$

shows that $(1+x)^n$ is the generating function of the binomial coefficients

$$\left\{ \binom{n}{k} \,:\, 0 \leq k \leq n \right\}.$$

In principle it is difficult to predict if a given sequence will have a *simple* generating function. Compare (1.3.5) with (1.4.4).

We now present a different proof of the binomial theorem in which we illustrate a general procedure that will be employed throughout the text. The goal is to find and prove an expression for $(a+b)^n$.

a) **Scaling**. The first step is to write

$$(a+b)^n = b^n(1+x)^n \tag{1.4.5}$$

with $x = a/b$, so that it suffices to establish (1.4.2) for $a = 1$.

b) **Guessing the structure**. The second step is to formulate an educated guess on the form of $(1+x)^n$. Expanding $(1+x)^n$ (for any specific n) shows that it is a polynomial in x of degree n, with positive integer coefficients, that is,

$$(1+x)^n = \sum_{k=0}^{n} b_{n,k} x^k \tag{1.4.6}$$

for some undetermined $b_{n,k} \in \mathbb{N}$. Observe that $x = 0$ yields $b_{n,0} = 1$.

c) The next step is to find a way to **understand the coefficients** $b_{n,k}$.

Exercise 1.4.2. Differentiate (1.4.6) to produce the recurrence

$$b_{n,k+1} = \frac{n}{k+1} b_{n-1,k} \quad 0 \leq k \leq n-1. \tag{1.4.7}$$

Conclude that the numbers $b_{n,k}$ are determined from (1.4.7) and initial condition $b_{n,0} = 1$.

We now **guess the solution** to (1.4.7) by studying the list of coefficients

$$L[n] := \{b_{n,k} : 0 \leq k \leq n\}. \tag{1.4.8}$$

The list $L[n]$ can be generated symbolically by the command

```
term[n_,k_]:=If[n==0,1, If[ k==0, 1,
        n*term[n-1,k-1]/k]];
 L[n_]:= Table[ term[n,k], {k,0,n}];
```

that produces a list of the coefficients $b_{n,k}$ from (1.4.7). For instance,

$$\begin{aligned}
L[0] &= \{1\} \\
L[1] &= \{1,\ 1\} \\
L[2] &= \{1,\ 2,\ 1\} \\
L[3] &= \{1,\ 3,\ 3,\ 1\} \\
L[4] &= \{1,\ 4,\ 6,\ 4,\ 1\} \\
L[5] &= \{1,\ 5,\ 10,\ 10,\ 5,\ 1\} \\
L[6] &= \{1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1\}.
\end{aligned} \tag{1.4.9}$$

The reader may now recognize the binomial coefficients (1.4.1) from the list (1.4.9) and conjecture the formula

$$b_{n,k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \tag{1.4.10}$$

from this data. Naturally this requires *a priori* knowledge of the binomial coefficients. An alternative is to employ the procedure described in Section 1.3 to conjecture (1.4.10) from the data in the list $L[n]$.

The guessing of a closed-form formula from data is sometimes obscured by dealing with small numbers. Mathematica can be used to generate terms in the middle part of $L[100]$. The command

```
t:= Table[ L[100][[i]],{i,45,49}]
```

chooses the elements in positions 45 to 49 in $L[100]$:

$$\begin{aligned}
L[100][[45]] &= 49378235797073715747364762200 \\
L[100][[46]] &= 61448471214136179596720592960 \\
L[100][[47]] &= 73470998190814997343905056800 \\
L[100][[48]] &= 84413487283064039501507937600 \\
L[100][[49]] &= 93206558875049876949581681100,
\end{aligned} \tag{1.4.11}$$

and, as before, we examine their prime factorizations to find a pattern.

The prime factorization of $n = L[100][[45]]$ is

$$n = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97,$$

suggesting the evaluation of $n/97!$. It turns out that this the reciprocal of an integer of 124 digits. Its factorization

$$\frac{97!}{n} = 2^{91} \cdot 3^{43} \cdot 5^{20} \cdot 7^{13} \cdot 11^{8} \cdot 13^{7} \cdot 17^{5} \cdot 19^{4} \cdot 23^{3} \cdot 29^{2} \cdot 31^{2} \cdot 37^{2} \cdot 41^{2} \cdot 43^{2} \cdot 47 \cdot 53$$

leads to the consideration of

$$\frac{97!}{n \times 53!} = 2^{42} \cdot 3^{20} \cdot 5^{8} \cdot 7^{5} \cdot 11^{4} \cdot 13^{3} \cdot 17^{2} \cdot 19^{2} \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43,$$

and then of

$$\frac{97!}{n \times 53! \times 43!} = \frac{2^3 \times 3 \times 11}{5 \times 7}. \tag{1.4.12}$$

The numbers 97, 53 and 43 now have to be slightly adjusted to produce

$$n = \frac{100!}{56! \times 44!} = \binom{100}{56}. \tag{1.4.13}$$

Repeating this procedure with the other elements in the list (1.4.11) leads to the conjecture (1.4.10).

Exercise 1.4.3. Use the method described above to suggest an analytic expression for

$$\begin{aligned} t_1 &= 3342221319350328344531984006070010189011388869544 1\\ &\quad 601636800,\\ t_2 &= 478657831091859016375280557008832085120001822095 4\\ &\quad 9068756000,\\ t_3 &= 632735060180453305105552747288270827687799251445 37\\ &\quad 753208000,\\ t_4 &= 772186537259697948007105490934041043000570796994 19\\ &\quad 429079500. \end{aligned}$$

d) **Recurrences**. Finally, in order to prove that our guess is correct, define

$$a_{n,k} := \binom{n}{k}^{-1} b_{n,k} \tag{1.4.14}$$

and show that (1.4.7) becomes

$$a_{n,k+1} = a_{n-1,k}, \quad n \geq 1,\ 0 \leq k \leq n-1, \tag{1.4.15}$$

so that $a_{n,k} \equiv 1$.

Exercise 1.4.4. Check that $b_{n,k} = \binom{n}{k}$ by verifying that $\binom{n}{k}$ satisfies (1.4.7) and that this recurrence admits a unique solution with $b_{n,0} = 1$.

Note 1.4.2. The sequence of binomial coefficients has many interesting properties. We provide some of them in the next exercises. The reader will find much more in

`http://www.dms.umontreal.ca/~andrew/Binomial/index.htlm`

Exercise 1.4.5. Prove that the exponent of the **central binomial coefficients** $C_n = \binom{2n}{n}$ satisfies

$$\mu_p(C_n) = \frac{2s_p(n) - s_p(2n)}{p-1}. \tag{1.4.16}$$

Hint. Let $n = a_0 + a_1 p + \cdots + a_r p^r$ be the expansion of n in base p. Define λ_j by $2a_j = \lambda_j p + \nu_j$, where $0 \leq \nu_j \leq p-1$. Check that λ_j is either 0 or 1 and confirm the formula

$$\mu_p(C_n) = \sum_{j=0}^{r} \lambda_j. \tag{1.4.17}$$

In particular $\mu_p(C_n) \leq r+1$. Check that C_n is always even. When is $C_n/2$ odd?

The binomial theorem yields the evaluation of many finite sums involving binomial coefficients. The discussion on binomial sums presented in this book is nonsystematic; we see them as results of evaluations of some definite integrals. The reader will find in Koepf (1998) and Petkovsek et al. (1996). a more complete analysis of these ideas.

Exercise 1.4.6. Let $n \in \mathbb{N}$.
a) Establish the identities

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
$$\sum_{k=0}^{n} k\binom{n}{k} = 2^{n-1}n$$
$$\sum_{k=0}^{n} k^2\binom{n}{k} = 2^{n-2}n(n+1). \tag{1.4.18}$$