HODGE THEORY AND COMPLEX ALGEBRAIC GEOMETRY II

CLAIRE VOISIN
Institut de Mathématiques de Jussieu

Translated by Leila Schneps

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10/13 pt *System* LaTeX 2ε [TB]

A catalogue record for this book is available from the British Library

ISBN 0 521 80283 0 hardback

Contents

1

The Lefschetz Theorem on Hyperplane Sections

This chapter is devoted to a presentation of Morse theory on affine varieties, and its application to the proof of the famous Lefschetz theorem on hyperplane sections, which is the following statement.

Theorem 1.1 *Let X be a projective n-dimensional variety, and let $j : Y \hookrightarrow X$ be a hyperplane section such that $U = X - Y$ is smooth. Then the restriction map*

$$j^* : H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$$

is an isomorphism for $k < n - 1$ and is injective for $k = n - 1$.

This Lefschetz theorem follows from the vanishing of the cohomology with compact support in degree $> n$ of a smooth n-dimensional affine variety. Thus, most of this chapter concerns the study of the topology of affine varieties; the most general result we present is the following (see Andreotti & Frankel 1959; Milnor 1963).

Theorem 1.2 *A smooth affine variety X of (complex) dimension n has the homotopy type of a CW-complex of (real) dimension $\leq n$.*

This statement is obtained by applying the results of Morse theory to the square of the distance function $h_0(x) = d(x, 0)^2$ on X, where the metric is deduced from a Hermitian metric on the ambient space. The essential point is the following result.

Proposition 1.3 *The Morse index of the function h_0 at a non-degenerate critical point is at most equal to n.*

Theorem 1.2 then follows from this proposition and the following basic theorem of Morse theory.

Theorem 1.4 *If $f : X \to \mathbb{R}$ is a Morse function, and λ is a critical value corresponding to a unique critical point of index r, then the level set $X_{f \leq \lambda+\epsilon}$ has the homotopy type of the union of the level set $X_{f \leq \lambda-\epsilon}$ with a ball B^r.*

We give an introduction to Morse theory in the first section of this chapter, and this theorem is proved there. In the second section, we study the case of the square of the distance function on affine varieties, and deduce the Lefschetz theorem on hyperplane sections. Finally, in the last section, we give another proof of this result using Hodge theory and the vanishing theorems. This proof gives the result for rational cohomology under the hypothesis that Y and X are smooth.

1.1 Morse theory

1.1.1 Morse's lemma

Let X be a differentiable variety, and let f be a differentiable function on X. Assume that everything is $\mathcal{C}^\infty$, although in fact the result still holds under weaker hypotheses. We say that $0 \in X$ is a critical point of f if $df(0) = 0$. The value $f(0) \in \mathbb{R}$ is then called a critical value of f.

Let $x_1, \ldots, x_n$ be local coordinates on X centred at 0. The vector fields $\frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$ give a basis of $T_{X,x}$ for every x in a neighbourhood of 0.

Definition 1.5 *The Hessian of f at the point 0 is the bilinear form on $T_{X,0}$ defined by*

$$\mathrm{Hess}_0 f\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)(0).$$

The formula for the derivatives of a composition of maps (the chain rule) shows immediately that $\mathrm{Hess}_0 f$ does not depend on the choice of coordinates, since 0 is a critical point of f. Moreover, the symmetry of partial derivatives shows that $\mathrm{Hess}_0 f$ is symmetric.

Definition 1.6 *We say that $0 \in X$ is a non-degenerate critical point of f if the quadratic (or symmetric bilinear) form $\mathrm{Hess}_0 f$ on $T_{X,0}$ is non-degenerate.*

Such a quadratic form Q can be diagonalised in a suitable basis $u_1, \ldots, u_n$ of $T_{X,0}$, i.e. there is a basis such that

$$Q(u_i, u_j) = \delta_{ij}\epsilon_i$$

with $\epsilon_i = \pm 1$.

Definition 1.7 *The Morse index of f at 0, written $\mathrm{ind}_0 f$, is the index of the quadratic form $\mathrm{Hess}_0 f$, i.e. the number of ϵ_i equal to -1 in any diagonalisation as above.*

The non-degenerate critical points are classified by their index up to diffeomorphism, as shown in the following proposition.

Proposition 1.8 (Morse's lemma) *If 0 is a non-degenerate critical point of a function f, then in a neighbourhood of 0, there exist coordinates $x_1, \ldots, x_n$ centred at 0 such that for $x = (x_1, \ldots, x_n)$, we have*

$$f(x) = f(0) - \sum_{i=1}^{r} x_i^2 + \sum_{i=r+1}^{n} x_i^2$$

with $r = \mathrm{ind}_0 f$.

The following result is a first corollary of proposition 1.8.

Corollary 1.9 *If 0 is a non-degenerate critical point of f, then 0 is an isolated critical point of f, and $f(0)$ is an isolated critical value of f restricted to a neighbourhood of 0.*

This corollary also follows immediately from the fact that $\mathrm{Hess}_0 f$ can be viewed as the differential of the map χ defined (using coordinates) by

$$x \mapsto \chi(x) = df_x \in (\mathbb{R}^n)^*.$$

If this differential is an isomorphism, i.e. when the Hessian is non-degenerate, the local inversion theorem shows that in a neighbourhood of 0, the set $\chi^{-1}(0)$ of critical points of f is reduced to $\{0\}$.

Proof of proposition 1.8 We proceed by induction on n. If $n > 0$, then clearly we can find a hypersurface $Y \subset X$ passing through 0, defined in the neighbourhood of 0, and smooth at 0, such that $f_{|Y}$ admits 0 as a non-degenerate critical point. Indeed, for this last condition to be satisfied, it suffices that the non-degenerate quadratic form $\mathrm{Hess}_0 f$ remain non-degenerate on the hyperplane $T_{Y,0} \subset T_{X,0}$. By hypothesis, there thus exist coordinates $x_1, \ldots, x_{n-1}$ on Y such that for $x = (x_1, \ldots, x_{n-1}) \in Y$, we have

$$f(x) = f(0) - \sum_{i=1}^{r'} x_i^2 + \sum_{i=r'+1}^{n-1} x_i^2$$

with $r' = \mathrm{ind}_0 f_{|Y}$. (Here r' can be equal to r or $r-1$.) The functions x_i can be extended to functions also called x_i on X. The function

$$f - f(0) - \left(-\sum_{i=1}^{r'} x_i^2 + \sum_{i=r'+1}^{n-1} x_i^2\right)$$

on X is thus $\mathcal{C}^\infty$, and vanishes along Y.

Hadamard's lemma then shows that if t is an equation defining Y, there exists a $\mathcal{C}^\infty$ function g such that

$$f - f(0) - \left(-\sum_{i=1}^{r'} x_i^2 + \sum_{i=r'+1}^{n-1} x_i^2\right) = tg.$$

The fact that 0 is a non-degenerate critical point of f is then expressed by the fact that the function g vanishes at 0 and has non-zero differential at 0.

Lemma 1.10 *If $g(0) = 0$, and if the functions $x_1, \ldots, x_{n-1}, t$ give a system of coordinates centred at 0, then there exist $\mathcal{C}^\infty$ functions α_i, $i = 1\ldots, n-1$ and ϕ such that*

$$g(x_1, \ldots, x_{n-1}, t) = 2\sum_{i=1}^{n-1} x_i\alpha_i + t\phi.$$

Temporarily admitting this lemma, let us now set

$$x_i' = x_i + \epsilon_i\alpha_i t$$

for $i \leq n-1$, where ϵ_i is equal to -1 for $i \leq r'$ and to 1 otherwise. Then for $x = (x_i, t)$, we have

$$f(x) = f(0) - \sum_{i=1}^{r'} x'^2_i + \sum_{i=r'+1}^{n-1} x'^2_i + t^2\psi,$$

where ψ is a $\mathcal{C}^\infty$ function. The fact that 0 is a non-degenerate critical point of f then immediately implies that $\psi(0) \neq 0$, so in a neighbourhood of 0, we can write $t^2\psi = \pm(x'_n)^2$. Clearly the x'_i form a system of coordinates on X centred at 0 in which f has the desired expression (possibly after permuting the coordinates). $\square$

Proof of lemma 1.10 We show it by induction, applying Hadamard's lemma. The lemma holds on Y by the induction hypothesis, so there exist $\mathcal{C}^\infty$ functions α_i on X such that $g - 2\sum_{i=1}^{n-1} x_i\alpha_i$ vanishes on Y. Then, by Hadamard's lemma, we have

$$g = 2\sum_{i=1}^{n-1} x_i\alpha_i + t\phi.$$

$\square$

Now let X be a topological space, and $f : X \to \mathbb{R}$ a continuous function.

Definition 1.11 *We say that f is an exhaustion function if for every element $M \in \mathbb{R}$, the closed subset $f^{-1}(]-\infty, M]) \subset X$ is compact.*

Such a map is in particular proper, and the fibres $X_a := f^{-1}(a)$ are compact. In what follows, we will write $X_{\leq M}$ for the subset $f^{-1}(]-\infty, M])$ for every $M \in \mathbb{R}$, and $X_{[M_1,M_2]}$ for the subset $f^{-1}([M_1, M_2])$, where $M_1 \leq M_2 \in \mathbb{R}$. These sets are called level sets.

If f is a differentiable exhaustion function having only non-degenerate critical points, then every critical value corresponds to a finite number of critical points, and the set of critical values is discrete by corollary 1.9. In particular, there exists only a finite number of critical values in each interval $]-\infty, M]$, $M \in \mathbb{R}$.

Such a function is called a Morse function. We sometimes require that every fibre X_λ have at most one critical point, and that the indices $r(\lambda)$ for a critical value λ increase with λ.

1.1.2 Local study of the level sets

Let us consider the function

$$f(x) = -\sum_{i=1}^{r} x_i^2 + \sum_{i=r+1}^{n} x_i^2$$

defined on $\mathbb{R}^n$. Let B^η denote the ball of radius η centred at 0. Let $B_\epsilon^r \subset \mathbb{R}^n$

and $S_\epsilon^{r-1} \subset \mathbb{R}^n$ denote the ball and the sphere defined by

$$\begin{aligned} B_\epsilon^r &:= \left\{(x_1,\ldots,x_n)\in B^\eta \;\middle|\; x_i=0,\ i>r,\ \sum_{i\le r} x_i^2 \le \epsilon\right\} \subset \mathbb{R}^n_{\le\epsilon}, \\ S_\epsilon^{r-1} &:= \left\{(x_1,\ldots,x_n)\in B^\eta \;\middle|\; x_i=0,\ i>r,\ \sum_{i\le r} x_i^2 = \epsilon\right\} \subset \mathbb{R}^n_{\le-\epsilon}. \end{aligned} \tag{1.1}$$

We easily see that for $\epsilon \le \eta^2$, $B_\epsilon^r < B^\eta$, for $\epsilon = \eta^2$, we have

$$S_\epsilon^{r-1} = \partial B_\epsilon^r = B^\eta_{\le-\epsilon}.$$

Let B denote the ball of radius $\sqrt{2\epsilon}$ in $\mathbb{R}^n$, and S its boundary, the sphere of radius $\sqrt{2\epsilon}$ in $\mathbb{R}^n$. We propose to show the following result.

Proposition 1.12 *There exists a retraction by deformation of $B_{\le\epsilon}$ onto the union $B_{\le-\epsilon} \cup_{S_\epsilon^{r-1}} B_\epsilon^r$, which induces a retraction by deformation of $S_{\le\epsilon}$ onto $S_{\le-\epsilon}$.*

More precisely, we will exhibit a homotopy

$$(H_t)_{t\in[0,1]} : B_{\le\epsilon} \to B_{\le\epsilon}$$

such that

(i) $H_1 = \mathrm{Id},\ H_t = \mathrm{Id}$ on $B_{\le-\epsilon} \bigcup_{S_\epsilon^{r-1}} B_\epsilon^r$.
(ii) H_0 has values in $B_{\le-\epsilon} \bigcup_{S_\epsilon^{r-1}} B_\epsilon^r$.
(iii) On $S_{[-\epsilon,\epsilon]}$, the homotopy H_t is given by a trivialisation of the fibration $f : S_{[-\epsilon,\epsilon]} \to [-\epsilon,\epsilon]$, i.e. a diffeomorphism

$$C = (C_0,\ f) : S_{[-\epsilon,\epsilon]} \cong S_{-\epsilon} \times [-\epsilon,\epsilon],$$

and by the retraction by deformation K'_t of the segment $[-\epsilon,\epsilon]$ onto the point $-\epsilon$ given by $K'_t(\alpha) = -(1-t)\epsilon + t\alpha$.

We have the following lemma.

Lemma 1.13 *For $\epsilon = \eta^2$, there exists a homotopy*

$$(R_t)_{t\in[0,1]} : B^\eta_{\le\epsilon} \to B^\eta_{\le\epsilon}$$

such that $R_1 = \mathrm{Id}$, and R_t is the identity on the ball B_ϵ^r defined in (1.1). Finally, $\mathrm{Im}\, R_0$ is contained in the ball B_ϵ^r, so R_0 is a retraction onto B_ϵ^r.

Note that as $B^\eta_{\le-\epsilon} = S_\epsilon^{r-1}$, the above homotopy is the identity on the level subset $B^\eta_{\le-\epsilon}$.

Proof Writing $x = (X_1, X_2)$ with $X_1 = (x_1, \ldots, x_r)$ and $X_2 = (x_{r+1}, \ldots, x_n)$, consider the map $R_t : B^\eta \to B^\eta$ given by

$$R_t(x) = (X_1, tX_2).$$

□

We will now show that up to increasing the radius of the ball, we can construct a retraction by deformation as above, which moreover preserves the boundary. Let us return to the ball B of radius $\sqrt{2\epsilon}$ in $\mathbb{R}^n$, and its boundary S, the sphere of radius $\sqrt{2\epsilon}$ in $\mathbb{R}^n$.

Lemma 1.14 *The restriction of the map f to S does not admit any critical points in the set*

$$S_{[-\epsilon,\epsilon]} := \{x \in S \mid -\epsilon \leq f(x) \leq \epsilon\}.$$

Proof A critical point of f in S is such that the functions $f = -f_1 + f_2$ and $\| \|^2 = f_1 + f_2$ have proportional differentials, where $f_1(x) = \|X_1\|^2$ and $f_2(x) = \|X_2\|^2$. This implies that df_1 and df_2 are proportional, and clearly this is not possible unless df_1 or $df_2 = 0$. But then f_1 or $f_2 = 0$, so $|f| = \| \|^2$. Now, this is impossible on $S_{[-\epsilon,\epsilon]}$, since on S, we have $\| \|^2 = 2\epsilon$. □

By Ehresmann's theorem (see vI, prop. 9.3), there exists a trivialisation

$$C = (C_0, f) : S_{[-\epsilon,\epsilon]} \cong S_{-\epsilon} \times [-\epsilon, \epsilon],$$

where $C_0 : S_{[-\epsilon,\epsilon]} \to S_{-\epsilon}$ is a differentiable map which induces a diffeomorphism $S_t \cong S_{-\epsilon}$ for every $t \in [-\epsilon, \epsilon]$.

In fact, it is easy to explicitly produce such a trivialisation. With notation as above, we take

$$C_0(x) = (\alpha(x)X_1, \beta(x)X_2),$$

where the positive functions $\alpha,\ \beta$ must satisfy the following conditions:

$$\begin{aligned} \alpha^2 f_1 + \beta^2 f_2 &= 2\epsilon, \\ -\alpha^2 f_1 + \beta^2 f_2 &= -\epsilon. \end{aligned} \tag{1.2}$$

The conditions (1.2) simply state that $C_0(x) \in S$ and $f(C_0(x)) = -\epsilon$. It is easy to see that the equations (1.2) have unique solutions in $S_{[-\epsilon,\epsilon]}$.

Using the above trivialisation, we construct a retraction by deformation $(K_t)_{t\in[0,1]}$ of $S_{[-\epsilon,\epsilon]}$ onto $S_{-\epsilon}$, compatible with f. This means that K_t induces the identity on $S_{-\epsilon}$ for every t, $K_1 = \mathrm{Id}$, and the image of K_0 is contained in $S_{-\epsilon}$. (The compatibility condition also says that $K_t(S_\alpha) \subset S_{K'_t(\alpha)}$ for a certain retraction by deformation K'_t of the segment $[-\epsilon, \epsilon]$ onto the point ϵ.) We simply set

$$K_t = C^{-1} \circ (C_0, (1-t)(-\epsilon) + tf).$$

In other words, in the trivialisation C, K_t is induced by the affine retraction K'_t of the segment $[-\epsilon, \epsilon]$ onto the point $-\epsilon : K'_t(u) = (1-t)(-\epsilon) + tu$. Note that K_t is of the form

$$K_t(x) = (\alpha_t(x)X_1, \beta_t(x)X_2), \tag{1.3}$$

where the positive functions α_t, β_t for $t \in [0, 1]$ are determined by the conditions

$$\begin{aligned} \alpha_t^2 f_1 + \beta_t^2 f_2 &= 2\epsilon, \\ -\alpha_t^2 f_1 + \beta_t^2 f_2 &= (1-t)(-\epsilon) + tf. \end{aligned}$$

□

Proof of proposition 1.12 We construct the homotopy H_t by setting $H_t = R_t$ in the ball B^η of radius $\sqrt{\epsilon}$, and $H_t = K_t$ on $S_{[-\epsilon,\epsilon]}$. We then look for H_t of the form

$$H_t(x) = (\alpha'_t(x)X_1, \beta'_t(x)X_2)$$

in $B_{[-\epsilon,\epsilon]}$, where the functions α'_t, β'_t now satisfy the conditions

$$\alpha_t'^2 f_1 + \beta_t'^2 f_2 \le 2\epsilon, \quad -\epsilon \le -\alpha_t'^2 f_1 + \beta_t'^2 f_2 \le (-f_1 + f_2), \tag{1.4}$$

i.e. $H_t(x) \in B$, $f(H_t(x)) \le f(x)$, and must coincide with α_t, β_t of (1.3) on $S_{[-\epsilon,\epsilon]}$ and with $1, t$ in $B^\eta_{\le\epsilon}$. We set $H_t = \mathrm{Id}$ in $B_{\le -\epsilon}$.

It is easy to check that we can construct such a pair (α'_t, β'_t), which also satisfies the conditions

$$\mathrm{Im}\, H_0 \subset B_{\le -\epsilon} \cup B^r_\epsilon$$

and

$$H_{t\,|B_{-\epsilon} \cup_{S^{r-1}_\epsilon} B^r_\epsilon} = \mathrm{Id},$$

already satisfied in B^η and $S_{[-\epsilon,\epsilon]}$. □

1.1.3 Globalisation

Now let X be a differentiable variety, and $f : X \to \mathbb{R}$ a differentiable exhaustion function. Let λ be a critical value, and let $\epsilon > 0$ be such that λ is the only critical value of f in $[\lambda - \epsilon, \lambda + \epsilon]$. Let 0_i, $i = 1, \ldots, r$ be the critical points of f on X_λ, and let r_i be their Morse indices. The local analysis above, together with proposition 1.8, allows us to prove the following theorem.

Theorem 1.15 *There exists a retraction by deformation of the level set $X_{\leq\lambda+\epsilon}$ onto the union of $X_{\leq\lambda-\epsilon}$ with r_i-dimensional balls B^{r_i} glued on $X_{\leq\lambda-\epsilon}$ along their boundaries, which are disjoint $(r_i - 1)$-dimensional spheres S^{r_i-1}.*

Proof As f is a fibration over $[\lambda-\epsilon, \lambda[$ and $]\lambda, \lambda+\epsilon]$, Ehresmann's theorem shows that it suffices to prove the theorem for very small ϵ. Morse's lemma then shows that in the neighbourhood of each 0_i, there exists a ball B_i in which f can be written as in the preceding subsection with $r = r_i$. We may of course assume that these balls are disjoint. By the preceding subsection, in each of these balls B^i, we now have a retraction by deformation H_t^i of $B^i_{\leq\lambda+\epsilon}$ onto $B^i_{\leq\lambda-\epsilon} \cup_{S_\epsilon^{r_i-1}} B_\epsilon^{r_i}$, which has the property of being induced by a trivialisation of the fibration f in the neighbourhood of $S^i_{[\lambda-\epsilon,\lambda+\epsilon]}$, where $S^i = \partial B^i$, and by the affine retraction of the segment $[\lambda - \epsilon, \lambda + \epsilon]$ onto the point $\lambda - \epsilon$. But since over the segment $[\lambda - \epsilon, \lambda + \epsilon]$, the restriction of f to $X - \bigcup_i B_0^i$ is a fibration of manifolds with boundary (where B_0^i denotes the interior of B^i), the trivialisations of the fibration f in the neighbourhood of $S^i_{[\lambda-\epsilon,\lambda+\epsilon]}$ extend to a trivialisation of the fibration f on $X - \bigcup_i B_0^i$ over $[\lambda - \epsilon, \lambda + \epsilon]$:

$$C = (C_0, f) : \left(X - \bigcup_i B_0^i\right)_{[\lambda-\epsilon,\lambda+\epsilon]} \cong \left(X - \bigcup_i B_0^i\right)_{\lambda-\epsilon} \times [\lambda - \epsilon, \lambda + \epsilon].$$

The $(H_t^i)_{|S^i} = K_t^i$ then extend to a retraction by deformation of $(X - \bigcup_i B_0^i)_{[\lambda-\epsilon,\lambda+\epsilon]}$ onto $(X - \bigcup_i B_0^i)_{\lambda-\epsilon}$, given in the trivialisation above by

$$H_t(x) = (C_0(x), K_t'(f(x))).$$

Clearly, H_t can be glued together with the H_t^i in B^i and with Id in $X_{\leq\lambda-\epsilon}$, which yields the desired retraction by deformation. □

1.2 Application to affine varieties

1.2.1 Index of the square of the distance function

Definition 1.16 *A (smooth) affine variety is a (smooth) closed analytic subvariety of $\mathbb{C}^N$ for some integer N.*

Let X be such a smooth, connected variety and let n denote its complex dimension. If h is a Hermitian metric on $\mathbb{C}^N$ and $0 \in \mathbb{C}^N$, we obtain a $\mathcal{C}^\infty$ function $f_0 : X \to \mathbb{R}$ by setting $f_0(x) = h(\overrightarrow{0x})$.

More generally, we can define such a 'square of the distance' function for any differentiable subvariety X of a Euclidean space $\mathbb{R}^N$. First consider the general situation. Obviously, f is always an exhaustion function.

Lemma 1.17 *Let $X \subset \mathbb{R}^N$ be a differentiable subvariety. Then for a general point* 0 *of* $\mathbb{R}^N$*, the corresponding function* f_0 *is a Morse function.*

Proof We have $df_{0,x}(u) = 2\langle\overrightarrow{0x}, u\rangle$ for $u \in T_{X,x}$. Thus, x is a critical point of f when $\overrightarrow{0x}$ is orthogonal to $T_{X,x}$. Let $Z \subset X \times \mathbb{R}^N$ be the set

$$Z = \{(x, 0) \in X \times \mathbb{R}^N \mid \overrightarrow{0x} \perp T_{X,x}\}.$$

Clearly, we have $\dim Z = N$. The second projection $\pi : Z \to \mathbb{R}^N$ is thus submersive at a point if and only if it is immersive at that point.

Lemma 1.18 *Let* $(x, 0) \in Z$*, and let* $u \in T_{X,x}$*. Then the tangent vector* $(u, 0) \in T_{X,x} \times T_{\mathbb{R}^N,0}$ *lies in* $T_{Z,(x,0)}$ *if and only if* $u \in T_{X,x}$ *lies in the kernel of the quadratic form* $\mathrm{Hess}_x f_0$.

Admitting this lemma, we see that the set of points 0 for which the function f_0 admits a degenerate critical point is the image under π of the set of points of Z where π is not an immersion, so not a submersion. By Sard's lemma (see Rudin 1966), this set has empty interior. □

Proof of lemma 1.18 The subset Z is defined in $X \times \mathbb{R}^N$ as the vanishing locus of the section σ of the vector bundle $\mathrm{pr}_1^*\Omega_X$ given by

$$\sigma_{(x,0)} = \langle\overrightarrow{0x}, \cdot\rangle = df_{0,x}.$$

Taking local coordinates x_i on X, σ can be written

$$\sum_i \frac{\partial f_0}{\partial x_i} dx_i.$$

The tangent space to Z at $(x, 0)$ is thus described by

$$T_{Z,(x,0)} = \left\{ (u, w) \in T_{X,x} \times T_{\mathbb{R}^N,0} \;\middle|\; d_u \frac{\partial f_0}{\partial x_i} + d_w \frac{\partial f_0}{\partial x_i} = 0, \ \forall i \right\}.$$

Writing $u = \sum_i u_i \frac{\partial}{\partial x_i}$, the first term is equal to $\sum_j u_j \frac{\partial^2 f_0}{\partial x_i \partial x_j}$. Thus, the vector $(u, 0)$ lies in $T_{Z,(x,0)}$ if and only if we have $\sum_j u_j \frac{\partial^2 f_0}{\partial x_i \partial x_j} = 0$ for every i, which means by definition that u lies in the kernel of $\mathrm{Hess}_x f_0$. □

Let us now return to the case where X is an n-dimensional complex analytic subvariety of $\mathbb{C}^N$, and the metric is Hermitian.

Proposition 1.19 *The index of the function $f = f_0$ is less than or equal to n at every critical point of f.*

Proof Let us introduce the second fundamental form

$$\Phi : S^2 T_{X,x} \to \mathbb{C}^N / T_{X,x},$$

which can be defined as the differential of the Gauss map

$$X \to \mathrm{Grass}(n, N), \ x \mapsto T_{X,x}.$$

By definition, $\Phi(u, v)$ can be computed as follows. Let V be a vector field on X defined in the neighbourhood of x, and whose value at x is v. Then, as $T_X \subset (T_{\mathbb{C}^N})_{|X}$, we can see V as a map with values in $\mathbb{C}^N$, and set

$$\Phi(u, v) = d_u V \bmod T_{X,x}.$$

One knows that this depends only on the vector v and not on the vector field V; see vI, lemma 10.7.

Let h be the Hermitian form, and $Q = \Re h = \langle \, , \rangle$ the corresponding Euclidean scalar product. The formula

$$df_x(u) = 2\langle \overrightarrow{0x}, u \rangle, \quad u \in T_{X,x}$$

shows immediately by differentiation that for a critical point x of f, we have

$$\mathrm{Hess}_x f(u, v) = 2(\langle \overrightarrow{0x}, \Phi(u, v)\rangle + \langle u, v \rangle).$$

To diagonalise $\mathrm{Hess}_x f$, it thus suffices to diagonalise the quadratic form

$$Q(u, v) = \langle \overrightarrow{0x}, \Phi(u, v)\rangle$$

in an orthonormal basis for $\langle\, ,\, \rangle$. But since X is a complex subvariety of $\mathbb{C}^N$, the second fundamental form Φ is $\mathbb{C}$-bilinear for the complex structures on $T_{X,x}$ and $\mathbb{C}^N/T_{X,x}$, so that Q is the real part of the $\mathbb{C}$-bilinear symmetric form $H(u, v) = h(\Phi(u, v), \overrightarrow{0x})$ on $T_{X,x} \cong \mathbb{C}^n \subset \mathbb{C}^N$.

Lemma 1.20 *Let H be a $\mathbb{C}$-bilinear symmetric form on $\mathbb{C}^n$, and let $\langle\, ,\, \rangle = \Re h$ be the Euclidean product associated to a Hermitian form on $\mathbb{C}^n$. Then the eigenvalues of the form $Q = \Re H$ with respect to the Euclidean form constitute a set which is stable under the involution $\lambda \mapsto -\lambda$ (including multiplicities).*

Admitting this lemma, we conclude that the eigenvalues of $\mathrm{Hess}_x f$ with respect to the Euclidean metric are of the form $2(1+\lambda_i),\ 2(1-\lambda_i),\ i = 1, \ldots, n$. Thus, at most n of these eigenvalues are negative. □

Proof of lemma 1.20 The form H, and thus also the form $Q = \Re H$, are multiplied by -1 under the automorphism of $\mathbb{C}^n$ induced by multiplication by i (which is unitary). Thus, the set of eigenvalues is stable under multiplication by -1. The statement concerning the multiplicities follows easily, since every form $Q = \Re H$ is a specialisation of a form of the type $\Re H$ with distinct eigenvalues. □

Definition 1.21 *A compact CW-complex is a topological space which can be written as a finite union of closed 'cells' homeomorphic to closed balls of $\mathbb{R}^k$. The largest integer k appearing in this cellular decomposition is called the dimension of the CW-complex.*

We define a CW-complex as a topological space which is the countable union of increasing open sets U_i such that the closure $\overline{U_i}$ is a compact CW-complex. The following result is a consequence of theorem 1.15 and proposition 1.19 (see Andreotti & Frankel 1959; Milnor 1963).

Theorem 1.22 *An affine variety X of complex dimension n has the homotopy type of a CW-complex of real dimension $\leq n$.*

Proof Let $X \subset \mathbb{C}^N$, and take a Morse function on X of the form $h_0(x) = d(x, 0)^2$, for a Hermitian metric on $\mathbb{C}^N$. Then X can be written as the countable union of the interiors X_k^0 of the level sets $X_k = \{x \in X \mid h_0(x) \leq k\}$, and assuming that the integers k are not critical values, theorem 1.15 and proposition 1.19 show that each level set X_{k+1} has the homotopy type of the union of X_k with a finite number of balls of dimension $\leq k$. □

1.2.2 Lefschetz theorem on hyperplane sections

Let X be a smooth complex projective variety, and $Y \overset{j}{\hookrightarrow} X$ a smooth hyperplane section. The cohomology class $[Y] \in H^2(X, \mathbb{Z})$ is equal to $h := c_1(\mathcal{O}_X(1))$ (see vI, Thm 11.33). Let us first show that we have the identity

$$j^* \circ j_* = h_Y \cup : H^k(Y, \mathbb{Z}) \to H^{k+2}(Y, \mathbb{Z}), \tag{1.5}$$

where $h_Y := h_{|Y} = [Y]_{|Y}$.

This formula follows immediately from the description of the Gysin morphism j_* as the composition $k_* \circ (\cup \eta_Y) \circ \pi^*$, where

$$\begin{tikzcd} T \arrow[r, "\pi"] \arrow[d, "k"'] & Y \\ X & \end{tikzcd}$$

is an open tubular neighbourhood of Y in X, and $\eta_Y \in H_c^2(T, \mathbb{Z})$ is the cohomology class with compact support of Y in T. Indeed, forgetting the torsion we know that the Gysin morphism $j_* : H^k(Y, \mathbb{Z}) \to H^{k+2}(X, \mathbb{Z})$ is the Poincaré dual of the restriction morphism

$$j^* : H^{2n-k-2}(X, \mathbb{Z}) \to H^{2n-k-2}(Y, \mathbb{Z}).$$

Passing to real coefficients, we must check that for a closed k-form β on Y and a closed $(2n-k-2)$-form α on X, we have

$$\int_Y \beta \wedge j^*\alpha = \int_X \tilde{\eta}_Y \wedge \pi^*\beta \wedge \alpha.$$

Here, the form $\tilde{\eta}_Y$ is a de Rham representative of η_Y: it is a closed 2-form with support in T such that $\pi_*\tilde{\eta}_Y = 1_Y$ (see vI.11.1.2). The form $\tilde{\eta}_Y \wedge \pi^*\beta$ on T is extended to X by 0, and thus the right-hand term is equal to $\int_T \tilde{\eta}_Y \wedge \pi^*\beta \wedge \alpha$. Now, as T can be retracted by deformation onto Y by π, there exists a closed $(2n-k-2)$-form α' on Y and a $(2n-k-3)$-form α'' on T such that

$$\alpha_{|T} = \pi^*\alpha' + d\alpha''.$$

But then, applying Stokes and the fact that $\pi_*\tilde{\eta}_Y = 1_Y$, we find

$$\int_T \tilde{\eta}_Y \wedge \pi^*\beta \wedge \alpha = \int_T \tilde{\eta}_Y \wedge \pi^*\beta \wedge \pi^*\alpha'$$
$$= \int_Y \beta \wedge \alpha' = \int_Y \beta \wedge j^*\alpha,$$

which proves the equality $j_* = k_* \circ (\cup\eta_Y) \circ \pi^*$.

As h_Y is a Kähler class, we can now apply the hard Lefschetz theorem (vI, Thm 6.25) to conclude that:

The cup-product

$$\cup h_Y : H^k(Y, \mathbb{Q}) \to H^{k+2}(Y, \mathbb{Q})$$

is injective for $k < n - 1 := \dim Y$ *and surjective for* $k + 2 > n - 1 = \dim Y$.

As $\cup h_Y = j^* \circ j_*$, it follows that

$$j_* : H^k(Y, \mathbb{Q}) \to H^{k+2}(X, \mathbb{Q})$$

is injective for $k < n - 1$ and

$$j^* : H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$$

is surjective for $k > n - 1$.

Moreover, we also have the equality

$$j_* \circ j^* = [Y]\cup = h\cup : H^k(X, \mathbb{Z}) \to H^{k+2}(X, \mathbb{Z}), \tag{1.6}$$

which can be proved by the same argument as above.

Applying the Lefschetz theorem cited above to X, we conclude that $h\cup = j^* \circ j_*$ is injective on $H^k(X, \mathbb{Q})$ for $k < n = \dim X$, which implies that

$$j^* : H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$$

is injective for $k < n = \dim X$.

We will now use the topological analysis developed in the preceding section to obtain a much stronger statement, namely the Lefschetz theorem on hyperplane sections.

Theorem 1.23 *Let* $X \subset \mathbb{P}^N$ *be a (not necessarily smooth)* n*-dimensional algebraic subvariety, and let* $Y = \mathbb{P}^{N-1} \cap X$ *be a hyperplane section such that*

$U := X - Y$ is smooth and n-dimensional. Then the restriction morphism

$$j^* : H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$$

is an isomorphism for $k < n - 1$ and is injective for $k = n - 1$.

Proof Let us admit the fact, which is proved using triangulations, that Y admits a fundamental system of neighbourhoods Y_i in X which can be retracted by deformation onto Y. It follows that

$$H^k(X, Y, \mathbb{Z}) \cong \lim_{\rightarrow} H^k(X, Y_i, \mathbb{Z}). \tag{1.7}$$

By excision, we also obtain isomorphisms

$$H^k(X, Y_i, \mathbb{Z}) \cong H^k(U, Y_i \cap U, \mathbb{Z}).$$

But since U is an oriented differentiable $2n$-dimensional variety, Poincaré duality gives an isomorphism (which is canonical, depending only on the orientation)

$$H^k(U, U - K, \mathbb{Z}) \cong H_{2n-k}(K, \mathbb{Z}) \tag{1.8}$$

for every compact set $K \subset U$ having the property that K is the retraction by deformation of an open set of U (see Spanier 1996, 6.2).

If we now apply the Poincaré duality (1.8) to $K_i := U - Y_i \cap U$, we obtain an isomorphism

$$H^k(X, Y, \mathbb{Z}) \cong \lim_{\rightarrow} H^k(U, Y_i \cap U, \mathbb{Z}) \cong \lim_{\rightarrow} H_{2n-k}(K_i, \mathbb{Z}).$$

Since every singular chain is contained in one of the compact sets $K_i \subset U$, it is clear that we have

$$H_{2n-k}(U, \mathbb{Z}) = \lim_{\rightarrow} H_{2n-k}(K_i, \mathbb{Z}).$$

In conclusion, we have a natural isomorphism

$$H^k(X, Y, \mathbb{Z}) \cong H_{2n-k}(U, \mathbb{Z}). \tag{1.9}$$

Returning to our proof, we consider the long exact sequence of relative cohomology of the pair (X, Y):

$$\cdots \to H^k(X, Y, \mathbb{Z}) \to H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z}) \to H^{k+1}(X, Y, \mathbb{Z}) \to \cdots.$$

It follows that theorem 1.23 is equivalent to the vanishing of the groups $H^k(X, Y, \mathbb{Z})$ for $k \leq n - 1 = \dim Y$. Applying the isomorphism (1.9), this is equivalent to the vanishing of the groups $H_k(U, \mathbb{Z})$ for $k \geq n + 1$.

Now, U is an affine variety embedded in $\mathbb{C}^N$. Let us equip $\mathbb{C}^N$ with a Hermitian metric, and let $0 \in \mathbb{C}^N$ be such that the function f_0 is a Morse function on U. U can be written as the union of the increasing level sets $U_{\leq M}$, $M \in \mathbb{Z}$, with $U_{\leq -1} = \emptyset$. It follows immediately that

$$H_k(U, \mathbb{Z}) = \varinjlim_{M \to \infty} H_k(U_{\leq M}, \mathbb{Z}),$$

and it suffices to show that for every level set $U_{\leq M}$, we have

$$H_k(U_{\leq M}, \mathbb{Z}) = 0 \quad \text{for} \quad k > n.$$

But this follows immediately, by induction on M, from theorem 1.15 and proposition 1.19. Indeed, assuming for simplicity that M and $M+1$ are not critical values of f_0, there exists a finite number of critical values λ_i of the function f_0 contained between M and $M+1$, i.e.

$$M < \lambda_1 < \cdots < \lambda_i < \cdots \lambda_k < M+1.$$

For $1 \leq i \leq k-1$, let us choose $\lambda_i' \in]\lambda_i, \lambda_{i+1}[$, and set $M = \lambda_0'$, $M+1 = \lambda_k'$. Then by theorem 1.15, each level set $U_{\leq \lambda_i'}$ has the homotopy type of the union of $U_{\leq \lambda_{i-1}'}$ with balls $B_{i,j}^{r_j}$ of dimension r_j equal to the index of the critical point $x_{i,j}$ of critical value λ_i, glued along their boundary $S_{i,j}^{r_j - 1}$. By proposition 1.19, all of these indices are at most equal to n. Thus, for each of these balls, we have

$$H_k(B, S, \mathbb{Z}) = 0 \quad \text{for} \quad k > n,$$

which by excision implies that

$$H_k(U_{\leq M+1}, U_{\leq M}, \mathbb{Z}) = 0 \quad \text{for} \quad k > n.$$

Thus, by the long exact sequence of relative homology of the pair (U_{M+1}, U_M), if

$$H_k(U_{\leq M}, \mathbb{Z}) = 0 \quad \text{for} \quad k > n,$$

then also $H_k(U_{\leq M+1}, \mathbb{Z}) = 0$ for $k > n$. As $U_{\leq -1} = \emptyset$, we thus have

$$H_k(U_{\leq M}, \mathbb{Z}) = 0 \quad \text{for} \quad k > n$$

for every M. This proves theorem 1.23. □

1.2.3 Applications

Recall (see vI.7.2.1) that the cohomology of the projective space $\mathbb{P}^n$ is described by $H^i(\mathbb{P}^n, \mathbb{Z}) = 0$ for odd i and $H^{2i}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}h^i$, where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^2(\mathbb{P}^n, \mathbb{Z})$.

Moreover, using the dth Veronese embedding

$$\Phi_d : \mathbb{P}^n \to \mathbb{P}^N,$$

which is given by homogeneous polynomials of degree d, and which is such that the pullback by Φ_d of a linear form on $\mathbb{P}^N$ is a polynomial of degree d on $\mathbb{P}^n$, we can consider a hypersurface $X \subset \mathbb{P}^n$ of degree d as an intersection $\Phi_d(\mathbb{P}^n) \cap \mathbb{P}^{N-1}$. Thus, the following result for hypersurfaces of projective space follows from theorem 1.23.

Corollary 1.24 *Let $X \subset \mathbb{P}^n$ be a hypersurface. Then $H^k(X, \mathbb{Z}) = 0$ for k odd, $k < \dim X$, and $H^{2k}(X, \mathbb{Z}) = \mathbb{Z}h^k$ for $2k < \dim X$.*

If moreover X is smooth, then the next result follows from Poincaré duality.

Corollary 1.25 *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface. Then $H^k(X, \mathbb{Z}) = 0$ for k odd, $k > \dim X$, and $H^{2k}(X, \mathbb{Z}) = \mathbb{Z}\alpha$ for $2k > \dim X$, where the class α has intersection with h^{n-1-k} equal to 1.*

Remark 1.26 *Let us take the case of a smooth hypersurface X in $\mathbb{P}^4$. The preceding corollary shows that $H_2(X, \mathbb{Z}) = H^4(X, \mathbb{Z})$ is generated by the unique class α such that $\langle \alpha, h \rangle = 1$. If X contains a line, the homology class of this line is thus equal to α. In general, if $d := \deg X > 5$, then X does not contain a line (at least if the equation of X is chosen generically). However, a curve $C = X \cap \mathbb{P}^2$ is of degree d and is thus of class $d\alpha$. Kollár (1990) showed that in general, for sufficiently large d, the class α is not the class of an algebraic cycle, although $d\alpha$ is. This is one of the counterexamples to the Hodge conjecture for integral cohomology.*

Corollaries 1.24 and 1.25 can be generalised immediately to complete intersections in projective space, by repeated applications of theorem 1.23.

A first application of corollary 1.24 concerns the computation of the Picard group of complete intersections. Recall (see vI.4.3, vI.11.3) that if X is a projective variety, then $\mathrm{Pic}\, X$ is the group of isomorphism classes of algebraic line bundles, or equivalently, of isomorphism classes of holomorphic line bundles, or in the smooth case, of divisors modulo rational equivalence. The second interpretation gives an identification (see vI.4.3) $\mathrm{Pic}\, X \cong H^1(X, \mathcal{O}^*_{X,\mathrm{an}})$, and the exponential exact sequence gives the long exact sequence

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}(X) \stackrel{c_1}{\to} H^2(X, \mathbb{Z}).$$

If X is now a smooth complete intersection in $\mathbb{P}^n$ such that $\dim X \geq 3$, then by corollary 1.24, we have $H^1(X, \mathbb{Z}) = 0$, so $H^1(X, \mathcal{O}_X) = 0$ by Hodge theory, which shows that $H^1(X, \mathcal{O}_X)$ is a quotient of $H^1(X, \mathbb{C})$. Moreover, we have an isomorphism

$$\operatorname{Pic} \mathbb{P}^n \overset{c_1}{\cong} H^2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}h,$$

and corollary 1.24 gives a restriction isomorphism $H^2(\mathbb{P}^n, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$. The next corollary follows immediately.

Corollary 1.27 *If X is a smooth complete intersection of dimension ≥ 3 in $\mathbb{P}^n$, then* $\operatorname{Pic} X = \mathbb{Z}\mathcal{O}_X(1)$.

In fact, using the arguments of the following section, we can show that under the hypothesis $\dim X \geq 2$, the vanishing property $H^1(X, \mathcal{O}_X) = 0$ holds even when X is not smooth, so that also corollary 1.27 holds even when X is not smooth.

1.3 Vanishing theorems and Lefschetz' theorem

As observed by Kodaira and Spencer (see Shiffman & Sommese 1985), it is possible to give a more 'algebraic' proof of the Lefschetz theorem 1.23, at least for cohomology with rational coefficients.

For this, we use the following vanishing theorem due to Akizuki, Kodaira, and Nakano (see Demailly 1996; Griffiths & Harris 1978). Let X be a complex variety, and let L be a holomorphic line bundle on X. Recall that L is said to be positive if L can be equipped with a Hermitian metric whose associated Chern form is positive (i.e. is a Kähler form). By the Kodaira embedding theorem (see vI.7.1.3), this is equivalent to the fact that L is ample, i.e. that the holomorphic sections of $L^{\otimes N}$ for sufficiently large N give an embedding

$$\Phi_{NL} : X \hookrightarrow \mathbb{P}^r.$$

Theorem 1.28 *Let $L \to X$ be a positive line bundle, where X is compact. Then for $p + q > n :=$* $\dim X$*, we have*

$$H^q\big(X, \Omega_X^p(L)\big) = 0.$$

Applying Serre duality (see vI.5.3.2) and noting that by the exterior product,

$$\big(\Omega_X^p\big)^* \otimes K_X \cong \Omega_X^{n-p},$$

we also obtain the following equivalent statement: *under the same hypotheses, we have*

$$H^q\big(X, \Omega_X^p(-L)\big) = 0 \quad \text{for} \quad p+q<n.$$

Here, the notation $\mathcal{F}(-L)$ means $\mathcal{F} \otimes (L^{-1})$, where $\mathcal{F}$ denotes a coherent sheaf on X.

This theorem yields the following version of the Lefschetz theorem.

Theorem 1.29 *Let X be an n-dimensional compact complex variety, and let $Y \stackrel{j}{\hookrightarrow} X$ be a smooth hypersurface such that the line bundle $\mathcal{O}_X(Y) = (\mathcal{I}_Y)^*$ is positive. Then for $k < n-1$, the restriction*

$$j^* : H^k(X, \mathbb{Q}) \to H^k(Y, \mathbb{Q})$$

is an isomorphism, and for $k = n-1$, it is injective.

Remark 1.30 *This statement is weaker than theorem 1.23. Indeed, note that by definition Y is the zero locus of a section σ_Y of $\mathcal{O}_X(Y)$, so NY is the divisor of the section σ_Y^N of $\mathcal{O}_X(NY)$. Under the preceding hypotheses, there exists an embedding Φ_{NY} of X into $\mathbb{P}^r$ such that the pullback of the linear forms gives exactly the sections of $\mathcal{O}_X(NY)$. Thus, under the embedding Φ_{NY}, $Y \subset X$ is the set-theoretic intersection of X with a hyperplane in $\mathbb{P}^r$. Thus, theorem 1.23 implies theorem 1.29. But the latter is strictly weaker, insofar as it deals only with rational cohomology and smooth hypersurfaces.*

Proof of theorem 1.29 By the change of coefficients theorem, we have $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$, and a similar statement for Y; thus it suffices to prove this theorem for the cohomology with complex coefficients instead of the rational cohomology. Now, under the hypotheses of the theorem, X and Y are Kähler (and even projective), and thus, from the Hodge decomposition for X and Y (see vI.6.1), we have

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q\big(X, \Omega_X^p\big)$$

and

$$H^k(Y, \mathbb{C}) = \bigoplus_{p+q=k} H^q\big(Y, \Omega_Y^p\big)$$

with $j^* = \bigoplus_{p+q=k} j^*_{p,q}$, where $j^*_{p,q} : H^q(X, \Omega_X^p) \to H^q(Y, \Omega_Y^p)$ is the restriction morphism induced by the morphism

$$j_p^* : \Omega_X^p \to j_*\Omega_Y^p$$

of coherent sheaves. Thus, it suffices to show that $j^*_{p,q}$ is an isomorphism for $p+q < \dim Y$ and is injective for $p+q = \dim Y$. Now, the morphism j^*_p is the composition of the natural morphisms

$$\Omega^p_X \to \Omega^p_X \otimes \mathcal{O}_Y = j_*\big(\Omega^p_X|_Y\big) \tag{1.10}$$

and

$$\Omega^p_X|_Y \to \Omega^p_Y \tag{1.11}$$

(the latter should in fact be composed with j_*, which induces an isomorphism in cohomology, and is thus usually omitted). It thus suffices to show that each of the morphisms (1.10) and (1.11) induces an isomorphism on the cohomology of degree q for $p+q < n-1 = \dim Y$, and an injection for $p+q = n-1$.

For this, we apply theorem 1.28 to X and to Y. Let us first consider the case of (1.10). We have the exact sequence

$$0 \to \Omega^p_X(-Y) \to \Omega^p_X \to \Omega^p_X|_Y \to 0.$$

The associated long exact sequence and the vanishing property $H^q(X, \Omega^p_X(-Y)) = 0$ for $p+q < \dim X$ immediately imply that the arrow

$$H^q\big(X, \Omega^p_X\big) \to H^q\big(Y, \Omega^p_X|_Y\big)$$

induced by (1.10) is an isomorphism for $p+q < \dim Y = \dim X - 1$ and is injective for $p+q = \dim Y$.

Moreover, we have the conormal exact sequence

$$0 \to \mathcal{O}_Y(-Y) \to \Omega_X|_Y \to \Omega_Y \to 0$$

on Y (see vI.3.3.3), where the identification of $\mathcal{O}_Y(-Y) = \mathcal{I}_Y \otimes \mathcal{O}_Y$ with the conormal bundle $N^*_{Y/X}$ is induced by the differential $d : \mathcal{I}_Y \to \Omega_X$. Passing to the pth exterior power, this exact sequence induces the exact sequence

$$0 \to \Omega^{p-1}_Y(-Y) \to \Omega^p_X|_Y \to \Omega^p_Y \to 0.$$

The associated long exact sequence of cohomology and theorem 1.28 applied to Y thus show that the morphism

$$H^q\big(Y, \Omega^p_X|_Y\big) \to H^q\big(Y, \Omega^p_Y\big)$$

induced by (1.11) is an isomorphism for $p+q < \dim Y$, and is injective for $p+q = \dim Y$. □

Exercises

1. *Morse theory and the Euler–Poincaré characteristic.* Let $X_1 \subset X$ be varieties with compact boundaries.
   (a) Using the long exact sequence of relative cohomology, show that
   $$\chi_{\text{top}}(X) = \chi_{\text{top}}(X_1) + \chi_{\text{top}}(X, X_1),$$
   where $\chi_{\text{top}}(X) := \sum_i (-1)^i b_i(X)$ and $\chi_{\text{top}}(X, X_1) := \sum_i (-1)^i \dim H^i(X, X_1)$.

   Now let X be a compact differentiable variety, and let $f : X \to \mathbb{R}$ be a Morse function.
   (b) Let $x \in X$ be a critical point of f of index i. Show that for $\epsilon > 0$, the level sets $X_{\leq f(x)\pm\epsilon}$ satisfy
   $$\chi_{\text{top}}\big(X_{\leq f(x)+\epsilon}\big) = \chi_{\text{top}}\big(X_{\leq f(x)-\epsilon}\big) + (-1)^i.$$
   (c) Deduce the formula
   $$\chi_{\text{top}}(X) = \sum_i (-1)^i N_i,$$
   where N_i is the number of critical points of index i.
2. *Subvarieties with ample normal bundle and Lefschetz theorems.* Let X be an n-dimensional smooth projective variety and E a holomorphic vector bundle of rank r on X. We say that E is ample if the invertible bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample on the projective bundle $\mathbb{P}(E^*) \xrightarrow{\pi} X$. Thus, $h := c_1(\mathcal{O}_{\mathbb{P}(E^*)}(1))$ is a Kähler class on $\mathbb{P}(E^*)$. Recall (see vI.11.2) that the Chern classes $c_i(E^*)$ are characterised by the relation
   $$h^r + \sum_{0<i\leq r} \pi^* c_i(E^*) h^{r-i} = 0 \text{ in } H^{2r}(\mathbb{P}(E^*), \mathbb{Z}).$$
   (a) Deduce from the hard Lefschetz theorem applied to $\mathbb{P}(E^*)$ that if E is ample, then the map
   $$\cup h : H^{k+r-2}(\mathbb{P}(E^*), \mathbb{Q}) \to H^{k+r}(\mathbb{P}(E^*), \mathbb{Q})$$
   is injective for $k \leq n$.
   (b) Under the same hypothesis, deduce from the decomposition of $H^*(\mathbb{P}(E^*), \mathbb{Q})$ (see vI.7.3.3) that the map
   $$\cup c_r(E) : H^k(X, \mathbb{Q}) \to H^{k+2r}(X, \mathbb{Q})$$
   is injective for $k \leq n - r$.

(c) Show that the conclusion of (b) still holds if E^* is assumed to be ample.

Let X be an $(n+r)$-dimensional complex variety, and let $Y \overset{j}{\hookrightarrow} X$ be an n-dimensional compact complex subvariety of X.

(d) Show that the map

$$j^* \circ j_* : H^k(Y, \mathbb{Z}) \to H^{k+2r}(Y, \mathbb{Z})$$

is equal to $\cup c_r(N_{Y/X})$ (see vI.1.2.2, and vI, chapter 11, exercise 3). Deduce that if the normal bundle $N_{Y/X}$ or its dual is ample, then the map

$$j_* : H^k(Y, \mathbb{Q}) \to H^{k+2r}(X, \mathbb{Q})$$

is injective for $k \leq n - r$.