Preface

It is well known that many phenomena in biology, chemistry, engineering, physics can be described by boundary value problems associated with various types of partial differential equations or systems. When we associate a mathematical model with a phenomenon, we generally try to capture what is essential, retaining the important quantities and omitting the negligible ones which involve small parameters. The model that would be obtained by maintaining the small parameters is called the perturbed model, whereas the simplified model (the one that does not include the small parameters) is called unperturbed (or reduced model). Of course, the unperturbed model is to be preferred, because it is simpler. What matters is that it should describe faithfully enough the respective phenomenon, which means that its solution must be "close enough" to the solution of the corresponding perturbed model. This fact holds in the case of regular perturbations (which are defined later). On the other hand, in the case of singular perturbations, things get more complicated. If we refer to an initial-boundary value problem, the solution of the unperturbed problem does not satisfy in general all the original boundary conditions and/or initial conditions (because some of the derivatives may disappear by neglecting the small parameters). Thus, some discrepancy may appear between the solution of the perturbed model and that of the corresponding reduced model. Therefore, to fill in this gap, in the asymptotic expansion of the solution of the perturbed problem with respect to the small parameter (considering, for the sake of simplicity, that we have a single parameter), we must introduce corrections (or boundary layer functions).

More than half a century ago, A.N. Tikhonov [43]–[45] began to systematically study singular perturbations, although there had been some previous attempts in this direction. In 1957, in a fundamental paper [50], M.I. Vishik and L.A. Lyusternik studied linear partial differential equations with singular perturbations, introducing the famous method which is today called the Vishik-Lyusternik method. From that moment on, an entire literature has been devoted to this subject.

> This book offers a detailed asymptotic analysis of some important classes of singularly perturbed boundary value problems which are mathematical models for various phenomena in biology, chemistry, engineering.

> We are particularly interested in nonlinear problems, which have hardly been examined so far in the literature dedicated to singular perturbations. This book proposes to fill in this gap, since most applications are described by nonlinear models. Their asymptotic analysis is very interesting, but requires special methods and tools. Our treatment combines some of the most successful results from different parts of mathematics, including functional analysis, singular perturbation theory, partial differential equations, evolution equations. So we are able to offer the reader a complete justification for the replacement of various perturbed models with corresponding reduced models, which are simpler but in general have a different character. From a mathematical point of view, a change of character modifies dramatically the model, so a deep analysis is required.
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> Although we address specific applications, our methods are applicable to other mathematical models.

We continue with a few words about the structure of the book. The material is divided into four parts. Each part is divided into chapters, which, in turn, are subdivided into sections (see the Contents). The main definitions, theorems, propositions, lemmas, corollaries, remarks are labelled by three digits: the first digit indicates the chapter, the second the corresponding section, and the third the respective item in the chapter.

Now, let us briefly describe the material covered by the book.

The first part, titled *Preliminaries*, has an introductory character. In Chapter 1 we recall the definitions of the regular and singular perturbations and present the Vishik-Lyusternik method. In Chapter 2, some results concerning existence, uniqueness and regularity of the solutions for evolution equations in Hilbert spaces are brought to attention.

In Part II, some nonlinear boundary value problems associated with the telegraph system are investigated. In Chapter 3 (which is the first chapter of Part II) we present the classes of problems we intend to study and indicate the main fields of their applications. In Chapters 4 and 5 we discuss in detail the case of algebraic boundary conditions and that of dynamic boundary conditions, respectively. We determine formally some asymptotic expansions of the solutions of the problems under discussion and find out the corresponding boundary layer functions. Also, we establish results of existence, uniqueness and high regularity for the other terms of our asymptotic expansions. Moreover, we establish estimates for the components of the remainders in the asymptotic expansions previously deducted in a formal way, with respect to the uniform convergence topology, or with respect to some weaker topologies. Thus, the asymptotic expansions are validated.

Part III, titled *Singularly perturbed coupled problems*, is concerned with the coupling of some boundary value problems, considered in two subdomains of a given domain, with transmission conditions at the interface.

In the first chapter of Part III (Chapter 6) we introduce the problems we are going to investigate in the next chapters of this part. They are mathematical models for diffusion-convection-reaction processes in which a small parameter is present. We consider both the stationary case (see Chapter 7) and the evolutionary one (see Chapter 8). We develop an asymptotic analysis which in particular allows us to determine appropriate transmission conditions for the reduced models.

What we do in Part III may also be considered as a first step towards the study of more complex coupled problems in Fluid Mechanics.

While in Parts II and III the possibility to replace singular perturbation problems with the corresponding reduced models is discussed, in Part IV we aim at reversing the process in the sense that we replace given parabolic problems with singularly perturbed, higher order (with respect to t) problems, admitting solutions which are more regular and approximate the solutions of the original problems. More precisely, we consider the classical heat equation with homogeneous Dirichlet boundary conditions and initial conditions. We add to the heat equation the term $\pm\varepsilon u_{tt}$, thus obtaining either an elliptic equation or a hyperbolic one. If we associate with each of the resulting equations the original boundary and initial conditions we obtain new problems, which are incomplete, since the new equations are of a higher order with respect to t. For each problem we need to add one additional condition to get a complete problem. We prefer to add a condition at $t = T$ for the elliptic equation, either for u or for u_t, and an initial condition at $t = 0$ for u_t for the hyperbolic equation. So, depending on the case, we obtain an elliptic or hyperbolic regularization of the original problem. In fact, we have to do with singularly perturbed problems, which can be treated in an abstract setting. In the final chapter of the book (Chapter 11), elliptic and hyperbolic regularizations associated with the nonlinear heat equation are investigated.

Note that, with the exception of Part I, the book includes original material mainly due to the authors, as considerably revised or expanded versions of previous works, including in particular the 2000 authors' Romanian book [6].

The present book is designed for researchers and graduate students and can be used as a two-semester text.

The authors December 2006