# 8

# The Turaev–Viro Invariants

## 8.1 The Turaev–Viro Invariants

These invariants were first described by Turaev and Viro [126]. They possess two important properties. First, just like homology groups, they are easy to calculate. Only the limitations of the computer at hand may cause some difficulties. Second, they are very powerful, especially if used together with the first homology group.

### 8.1.1 The Construction

We divide the construction of the Turaev–Viro invariants into six steps.

Step 1. Fix an integer $N \geq 1$.

Step 2. Consider the set $\mathcal{C} = \{0, 1, \dots, N-1\}$ of integers. We will think of them as representing colors. To each integer $i = 0, 1, \dots, N-1$ assign a complex number $w_i$ called the *weight* of $i$.

Step 3. Let $E$ be a butterfly, see Fig. 1.4. Recall that it has six wings. We will color the wings by colors from the palette $\mathcal{C}$ in order to get different *colored butterflies*. The butterfly admits exactly $N^6$ different colorings.

**Definition 8.1.1.** *Two colored butterflies are called* equivalent *if there exists a color preserving homeomorphism between them.*

The number of different colored butterflies up to equivalence is significantly less than $N^6$. It is because the butterfly is very symmetric: it inherits all the 24 symmetries of the regular tetrahedron, see Fig. 1.5. It is convenient also to present a colored butterfly by coloring the edges of a regular tetrahedron $\Delta$. The body of the butterfly is the cone over the vertices of $\Delta$ while its wings are the cones over corresponding edges and have the same colors.

Step 4. To each colored butterfly, assign a complex number called the *weight* of the butterfly. There arises a problem: how to denote colored butterflies and their weights? Let us call two wings of a butterfly *opposite* if their

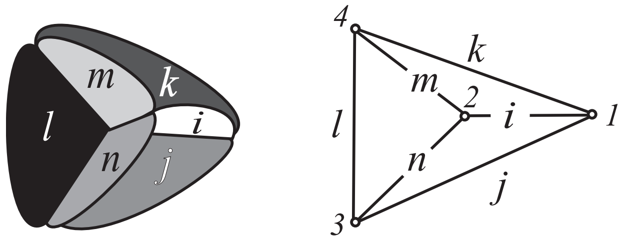



**Fig. 8.1.** The butterfly and its boundary graph

intersection is the vertex (not an edge). Note that any colored butterfly is determined (up to equivalence) by:

(a) Three pairs $(i, l), (j, m), (k, n)$ of colors that correspond to three pairs of opposite wings.
(b) A triple $(i, j, k)$ of representatives of each pair that correspond to wings having a common edge.

An example of a colored butterfly and its boundary graph are shown in Fig. 8.1.

For typographic convenience, and following some earlier conventions, such a butterfly will be denoted by the $(2 \times 3)$-matrix

$$\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix},$$

where the top row gives the colors of three adjacent wings and each column gives colors of opposite pairs of wings. The weight associated with the above butterfly is denoted by

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

and called a $(q-6j)$-symbol, for reasons we will not go into here. An interested reader is referred to [125, 126] and references therein.

The $(q-6j)$-symbol has many symmetries, corresponding to the symmetries manifest in the butterfly. The symmetry group of a butterfly presented as the cone over one-dimensional skeleton $\Delta^{(1)}$ of a regular tetrahedron is isomorphic to the symmetric group $S_4$ on four elements 1, 2, 3, 4 that correspond to the vertices of $\Delta^{(1)}$. Assume that the edges (1,2), (1,3), (1,4), (3,4), (2,4), and (2,3) have colors $i, j, k, l, m, n$, respectively. Then the following equalities correspond to generators (the transposition (2,3) and the cyclic permutation (1,2,3,4)) of $S_4$:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & i & k \\ m & l & n \end{vmatrix}, \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} n & m & i \\ k & j & l \end{vmatrix}.$$

Step 5. Let $P$ be a special polyhedron, $V(P)$ the set of its vertices, and $C(P)$ the set of its 2-cells.

**Definition 8.1.2.** *A* coloring *of $P$ is a map $\xi : C(P) \to \mathcal{C}$.*

Denote by *Col*$(P)$ the set of all possible colorings of $P$. It consists of $N^{\#C(P)}$ elements, where $N$ is the number of colors and $\#C(P)$ is the number of 2-cells in $P$. To each coloring $\xi \in$*Col*$(P)$ assign a *weight* $w(\xi)$ by the rule

$$w(\xi) = \prod_{v \in V(P)} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_v \prod_{c \in C(P)} w_{\xi(c)}. \tag{8.1}$$

Note that any coloring $\xi$ determines a coloring of a neighborhood of every vertex $v \in V(P)$. It means that in a neighborhood of $v$ we see a colored butterfly

$$\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}_v$$

with the $(q-6j)$-symbol

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_v .$$

Every 2-cell $c$ of $P$ is painted in the color $\xi(c)$ having the weight $w_{\xi(c)}$. Thus the right-hand part of the formula (8.1) is the product of all the symbols and the weights of all used colors (with multiplicity).

**Definition 8.1.3.** *Let $P$ be a special polyhedron. Then the* weight *of $P$ is given by the formula*

$$w(P) = \sum_{\xi \in Col(P)} w(\xi).$$

Step 6. Certainly, the weight of a special polyhedron $P$ depends heavily on the weights $w_i$ of colors and the values of $(q-6j)$-symbols. We will think of them as being *variables*; thus we have finitely many variables. If we fix their values, we get a well defined invariant of the topological type of $P$. Now let us try to subject the variables to constraints so that the weight of a special polyhedron will be invariant with respect to $T$-moves. In order to do that, let us write down the following system of equations:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & j & k \\ l' & m' & n' \end{vmatrix} = \sum_z w_z \begin{vmatrix} i & m & n \\ z & n' & m' \end{vmatrix} \begin{vmatrix} j & l & n \\ z & n' & l' \end{vmatrix} \begin{vmatrix} k & l & m \\ z & m' & l' \end{vmatrix}, \tag{8.2}$$

where $i, j, k, l, m, n, l', m', n'$ run over all elements of the palette $\mathcal{C}$.

The geometrical meaning of the equations is indicated in Fig. 8.2 and explained in the proof of Theorem 8.1.4. We emphasize that the system is universal, i.e., it depends neither on manifolds nor on their spines.

In order to get a feeling of the system, let us estimate the number of variables and the number of equations. If we ignore the symmetries of symbols, then the number of variables is $N + N^6$: there are $N$ weights of colors and $N^6$ symbols. The equations are parameterized by 9-tuples $(i, j, k, l, m, n, l', m', n')$

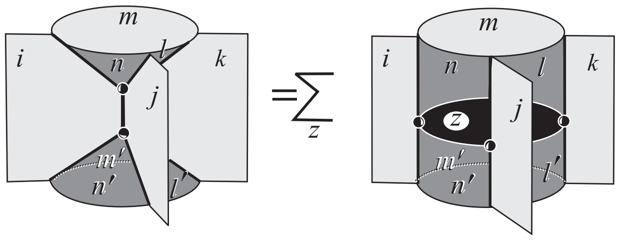



**Fig. 8.2.** Geometric presentation of equations

of colors. Therefore, we have $N^9$ equations (if we ignore symmetries of equations). In general, the system appears over-determined, but, as we shall see, solutions exist.

Let us show that every solution determines a 3-manifold invariant. Let $M$ be a 3-manifold. Construct a special spine $P$ of $M$ having $\geq 2$ vertices, and define an invariant $TV(M)$ by the formula $TV(M) = w(P)$, where $w(P)$ is the weight of $P$, see Definition 8.1.3.

**Theorem 8.1.4.** *If the $(q-6j)$-symbols and weights $w_i$ are solutions of the system (8.2), then $w(P)$ does not depend on the choice of $P$. Therefore, $TV(M)$ is a well defined 3-manifold invariant.*

*Proof.* According to Theorem 1.2.5, it is sufficient to show that $w(P)$ is invariant with respect to $T$-moves. Let a special polyhedron $P_2$ be obtained from a special polyhedron $P_1$ by exactly one $T$-move, i.e., by removing a fragment $E_T$ and inserting a fragment $E'_T$, see Definition 1.2.3. For any coloring $\xi$ of $P_1$, let $Col_\xi(P_2)$ be the set of colorings of $P_2$ that coincide with $\xi$ on $P_1 \backslash E_T = P_2 \backslash E'_T$. Since only one 2-cell of the fragment $E'_T$ (the middle disc) has no common points with $\partial E'_T$, the set $Col_\xi(P_2)$ can be parameterized by the color $z$ of this 2-cell. It follows that the set $Col_\xi(P_2)$ consists of $N$ colorings $\zeta_z, 0 \leq z \leq N-1$.

Because of distributivity, the equation of the system (8.2) that corresponds to the 9-tuple $(i,j,k,l,m,n,l',m',n')$ implies the equality $w(\xi) = \sum_z w(\zeta_z)$, see Fig. 8.2. To see this, multiply both sides of the equation by the constant factor that corresponds to the contribution made to the weights by the exteriors of the fragments. Summing up the equalities $w(\xi) = \sum_z w(\zeta_z)$ over all colorings of $P_1$, we get $w(P_1) = w(P_2)$.

**Definition 8.1.5.** *Any 3-manifold invariant obtained by the above construction will be called an* invariant of Turaev–Viro type. *The number $r = N + 1$, where $N$ is the number of colors in the palette $\mathcal{C}$, will be called the* order of the invariant.

### 8.1.2 Turaev–Viro Type Invariants of Order $r \leq 3$

There are no Turaev–Viro type invariants of order 1, since $r = N + 1$ and $N \geq 1$. If $r = 2$, then $N = 1$. Hence we have a very poor palette consisting of only one color 0, and there is only one colored butterfly

$$\begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}.$$

Denote by $w_0$ and $x$ the weight of the unique color and the symbol of the butterfly, respectively. In this case the system (8.2) consists of one equation $x^2 = w_0 x^3$. If $x = 0$, we get solutions that produce the trivial 3-manifold invariant $TV(M) \equiv 0$. Otherwise, we get a set of solutions $\{x = z^{-1}, w_0 = z\}$ parameterized by nonzero numbers $z$. Each of the solutions produces a 3-manifold invariant $TV(M) = z^{\chi(P)}$ (one should point out here that $\chi(P) = \chi(M)$ if $\partial M \neq \emptyset$, and $\chi(P) = 1$ if $M$ is closed). Indeed, by Definition 8.1.3 we have $TV(M) = w(P) = z^{-V(P)} z^{C(P)} = z^{C(P)-V(P)}$, where $V(P)$ is the number of vertices of a special spine $P \subset M$ and $C(P)$ is the number of its 2-cells. Since every vertex of $P$ is incident to exactly four edges, the number of edges of $P$ is equal to $2V(P)$. It follows that $C(P) - V(P) = \chi(P)$.

Let us investigate the case $N = 2$, when there are two colors: 0 and 1. We will call them *white* and *black*, respectively. There are 11 different colored butterflies. Their symbols together with the weights $w_0, w_1$ of the colors form a set of 13 variables. Note that the transposition $\{0, 1\} \leftrightarrow \{1, 0\}$ of the colors induces an involution $i$ on the set of variables. See Fig. 8.3, where the butterflies are presented by their boundary graphs. The lower indices of the corresponding variables show the number of black-colored wings. The weights of the black and white colors are also indicated.

It turns out that there are 74 equations. They correspond to different colorings of the boundary graph of the fragment $E_T$. See Fig. 8.4 for an example of a graphically expressed equation that corresponds to the equation

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} i & j & k \\ l' & m' & n' \end{vmatrix} = \sum_z w_z \begin{vmatrix} i & m & n \\ z & n' & m' \end{vmatrix} \begin{vmatrix} j & l & n \\ z & n' & l' \end{vmatrix} \begin{vmatrix} k & l & m \\ z & m' & l' \end{vmatrix}$$

of system (8.2) for $i = k = 1, j = l = m = n = l' = m' = n' = 0$.

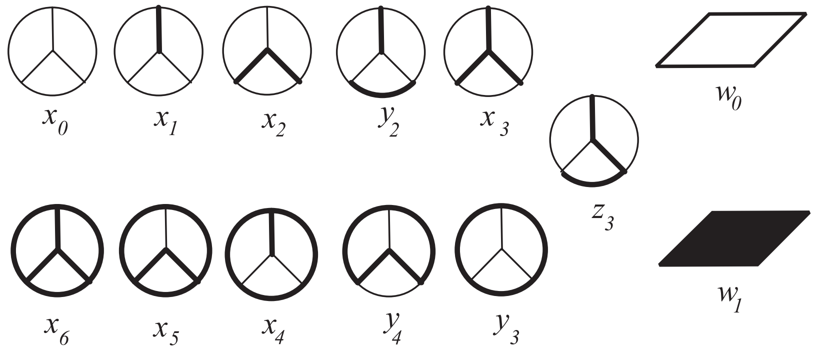
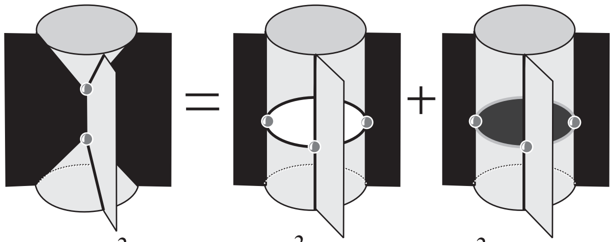



**Fig. 8.3.** Thirteen variables for $N = 2$



$$x_2^2 = w_0\, x_0\, x_1^2 + w_1\, x_1\, y^2$$

**Fig. 8.4.** An example of an equation

There are too many equations to list them all here, so we will take a shortcut. To simplify the calculations, we subject the symbols to the following constraints: if a butterfly contains one black and two white wings adjacent to the same edge, then the symbol must be zero. In other words, we assume that the symbols of the type

$$\begin{vmatrix} 0\,0\,1 \\ *\,*\,* \end{vmatrix},$$

that is, the variables $x_1, x_2, y_2, x_3, z_3$, and $x_4$ are zeros. The motivation for this restriction was *triangle inequality conditions* of Turaev–Viro, see Sect. 8.1.4. My former student Maxim Sokolov had verified that, in case $N = 2$, only restricted solutions of system (8.2) were interesting (unpublished). In other words, unrestricted solutions do not give any additional invariants. So we are left with seven variables $x_0, y_3, y_4, x_5, x_6$, and $w_0, w_1$. It is easy to see that this leaves only 14 equations:

(1) $x_0^2 = w_0 x_0^3$;
(2) $x_0 y_3 = w_1 y_3^3$;
(3) $y_3^2 = w_0 x_0 y_3^2$;
(4) $y_4^2 = w_0 y_3^2 y_4$;
(5) $y_3 y_4 = w_1 y_3 y_4^2$;
(6) $y_3^2 = w_0 y_4^3 + w_1 x_5^3$;
(7) $0 = w_0 y_4^2 x_5 + w_1 x_5^2 x_6$;
(8) $x_5^2 = w_1 y_4 x_5^2$;
(9) $y_3 x_5 = w_1 y_3 x_5^2$;
(10) $y_4 x_5 = w_1 x_5^3$;
(11) $x_5^2 = w_0 x_5 y_3^2$;
(12) $x_5^2 = w_0 y_4 x_5^2 + w_1 x_5 x_6^2$;
(13) $x_5 x_6 = w_1 x_5^2 x_6$;
(14) $x_6^2 = w_0 x_5^3 + w_1 x_6^3$.

They correspond to the colorings of $\partial E_T$ shown in Fig. 8.5.

Let us solve the system. It follows from (4) and (8) that if $y_3 = 0$, then $y_4 = x_5 = 0$. This leaves only two equations $x_0^2 = w_0 x_0^3, x_6^2 = w_1 x_6^3$. Just as in the case $N = 1$, one can show that then we get the sum $TV(M) = w_0^{\chi(M)} + w_1^{\chi(M)}$ of the two-order 2 invariants. Hence one may assume that $y_3 \neq 0$ and, as it follows from the third equation, $x_0 \neq 0$. Note that the system is quasihomogeneous in the following sense: if we divide all $x_i$ and $y_j$

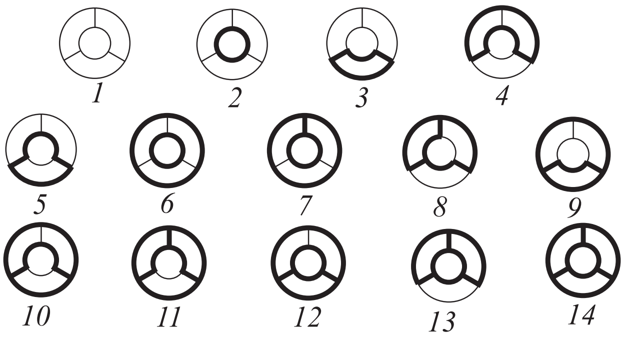


**Fig. 8.5.** Fourteen equations for $N = 2$

by $x_0$ and multiply $w_0, w_1$ by the same factor, we get an equivalent system. Hence we may assume that $x_0 = 1$ and, by the first equation, $w_0 = 1$.

The further events depend on whether or not $x_5 = 0$. Let $x_5 = 0$. Recall that $x_0 = w_0 = 1$. Set $w_1 = u$. It is easy to see that (2), (4), and (6) imply $y_3 = u^{-1/2}, y_4 = u^{-1}$, and $u^2 = 1$. All the other equations become identities except the last equation $x_6^2 = ux_6^3$. We get two solutions:

$$x_0 = w_0 = 1, w_1 = u, y_3 = u^{-1/2}, y_4 = u^{-1}, x_5 = x_6 = 0; \tag{8.3}$$

$$x_0 = w_0 = 1, w_1 = u, y_3 = u^{-1/2}, y_4 = u^{-1}, x_5 = 0, x_6 = u^{-1}, \tag{8.4}$$

where $u = \pm 1$ and for $y_3$ one may take any square root of $u^{-1}$.

Let $x_5 \neq 0$. Set $w_1 = \varepsilon$. Just as before, we get $y_3 = \varepsilon^{-1/2}$ and $y_4 = \varepsilon^{-1}$. From (9) and (7) one gets $x_5 = \varepsilon^{-1}$ and $x_6 = -\varepsilon^{-2}$. All other equations become identities except (6), (12), (14), that are equivalent to $\varepsilon^2 = 1 + \varepsilon$. We get a new solution:

$$x_0 = w_0 = 1, w_1 = \varepsilon, y_3 = \varepsilon^{-1/2}, y_4 = \varepsilon^{-1}, x_5 = \varepsilon^{-1}, x_6 = -\varepsilon^{-2}, \tag{8.5}$$

where $\varepsilon = (1 \pm \sqrt{5})/2$.

Denote by $TV_\pm(M)$ the invariants corresponding to solution (8.3) for $u = \pm 1$. Let us describe a geometric interpretation of them. Any special polyhedron contains only finitely many different closed surfaces. Denote by $n_e(P)$ and $n_o(P)$ the total number of surfaces in $P$ having *even* and, respectively, *odd* Euler characteristics.

**Lemma 8.1.6.** *For any special spine $P$ of $M$ we have $TV_\pm(M) = n_e(P) \pm n_o(P)$.*

*Proof.* There is a natural bijection between closed surfaces in $P$ and black–white colorings of $P$ with nonzero weights. Indeed, if we paint a surface $F \subset P$ in black, and the complement $P \setminus F$ in white, we get a coloring $\xi$ of $P$ such that it admits only three types of butterflies: the totally white butterfly

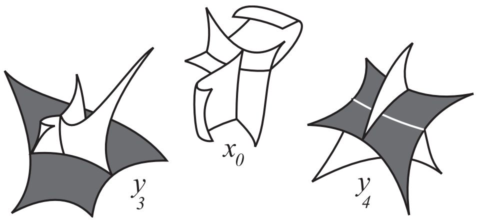



**Fig. 8.6.** Three butterflies having nonzero symbols

$$\begin{pmatrix} 0\ 0\ 0 \\ 0\ 0\ 0 \end{pmatrix},$$

and butterflies

$$\begin{pmatrix} 0\ 0\ 0 \\ 1\ 1\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 0\ 1\ 1 \\ 0\ 1\ 1 \end{pmatrix},$$

see Fig. 8.6.

Since their symbols

$$x_0 = \begin{vmatrix} 0\ 0\ 0 \\ 0\ 0\ 0 \end{vmatrix}, y_3 = \begin{vmatrix} 0\ 0\ 0 \\ 1\ 1\ 1 \end{vmatrix},$$

and

$$y_4 = \begin{vmatrix} 0\ 1\ 1 \\ 0\ 1\ 1 \end{vmatrix}$$

are nonzero, the weight $w(\xi)$ is also nonzero.

Conversely, let $\xi$ be a black–white coloring of $P$ with a nonzero weight

$$w(\xi) = \prod_{v \in V(P)} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_v \prod_{c \in C(P)} w_{\xi(c)}.$$

Denote by $F(\xi)$ the union of all black cells in $P$. Then $F(\xi)$ inherits the local structure of black parts of butterflies. Since $w(\xi) \neq 0$, the butterflies have nonzero symbols. In the case of solution (8.3), only the butterflies shown in Fig. 8.6 come into consideration. Hence $F(\xi)$ is a closed surface.

It turns out that the weight $w(\xi) \neq 0$ of a coloring $\xi$ is closely related to the Euler characteristic $\chi(F)$ of the corresponding surface $F = F(\xi)$. Let us show that $w(\xi) = u^{\chi(F)}$. Denote by $k_3$ and $k_4$ the numbers of butterflies in $P$ having the symbols $y_3$ and $y_4$, respectively. Then $F$ inherits from $P$ the cell structure with $k_3 + k_4$ vertices, $(3k_3 + 4k_4)/2$ edges, and some number of 2-cells, which we denote by $c_2(F)$. It follows that $\chi(F) = -k_3/2 - k_4 + c_2(F)$. Taking into account that $w_1 = u, y_3 = u^{-1/2}$, and $y_4 = u^{-1}$, we get that $w(\xi) = w_1^{c_2(F)} y_3^{k_3} y_4^{k_4} = u^{-k_3/2 - k_4 + c_2(F)} = u^{\chi(F)}$.

To conclude the proof, denote by $\xi_1, \dots, \xi_n$ the colorings of $P$ with nonzero weights. Then $w(P) = \sum_{i=1}^{n} w(\xi_i) = \sum_{i=1}^{n} u^{\chi(F(\xi_i))} = n_e \pm n_o$ for $u = \pm 1$.

It turns out that the invariants $TV_{\pm}(M)$ admit a very nice homological interpretation, see [126]. Note that for any compact 3-manifold $M$ the homology group $H_2(M; Z_2)$ is finite, and any homology class $\alpha \in H_2(M; Z_2)$ can be presented by an embedded closed surface. We say that $\alpha$ is *even* or *odd* if it can be presented by an embedded surface in $M$ having an even or odd Euler characteristic, respectively. Denote by $n_e(M)$ and $n_o(M)$ the number of even and, respectively, odd homology classes.

**Proposition 8.1.7.** *For any 3-manifold $M$ we have:*

*(a)* $TV_+(M) = n_e(M) + n_o(M)$ *(= the order of $H_2(M; Z_2)$);*
*(b)* $TV_-(M) = n_e(M) - n_o(M)$*;*
*(c) In case $M$ is orientable either $TV_-(M) = TV_+(M)$ or $TV_-(M) = 0$ depending on whether or not $M$ contains an odd surface.*

*Proof.* Choose a special spine $P$ of $M$. Denote by $\mathcal{F}(\mathcal{P})$ the set of all closed surfaces in $P$. Each surface $F \in \mathcal{F}(\mathcal{P})$ represents an element of $H_2(P; Z_2)$. Thus we have a map $\varphi : \mathcal{F}(P) \to H_2(P; Z_2) = H_2(M; Z_2)$. It is easy to see that, due to the nice local structure of simple polyhedra, $\varphi$ is a bijection. We may apply Lemma 8.1.6 and get $TV_{\pm}(M) = n_e(P) \pm n_o(P)$. Since $P$ is a deformation retract of $M$, we have $n_e(P) = n_e(M), n_o(P) = n_o(M)$, that implies (a) and (b). To get (c), consider the map $H_2(M; Z_2) \to Z_2$ that takes even classes to 0 and odd classes to 1. If $M$ is orientable, the map is a homomorphism. Hence even elements of $H_2(M; Z_2)$ form a subgroup that either coincides with $H_2(M; Z_2)$ or has index 2 (if there is at least one odd element). It follows that either $n_o = 0$ (and we get $TV_-(M) = TV_+(M)$) or $n_e = n_o$ (and we get $TV_-(M) = 0$).

**Examples.** The values of $TV_{\pm}$-invariants for some 3-manifolds are given in the following table. The list contains all closed orientable prime manifolds of complexity $\leq 2$ (see Chap. 2), and two nonorientable manifolds: $S^1 \times RP^2$ and $K^2 \times S^1$, where $K^2$ is the Klein bottle.

| $M$ | $TV_+$ | $TV_-$ | $M$ | $TV_+$ | $TV_-$ |
|---|---|---|---|---|---|
| $S^3$ | 1 | 1 | $L_{8,3}$ | 2 | 2 |
| $S^2 \times S^1$ | 2 | 2 | $L_{5,2}$ | 1 | 1 |
| $RP^3$ | 2 | 0 | $L_{5,1}$ | 1 | 1 |
| $L_{3,1}$ | 1 | 1 | $L_{7,2}$ | 1 | 1 |
| $L_{4,1}$ | 2 | 2 | $S^3/Q_8$ | 4 | 4 |
| $S^1 \times RP^2$ | 4 | 2 | $K^2 \times S^1$ | 8 | 4 |

Note that $TV_-$-invariant distinguishes $S^2 \times S^1$ and $RP^3$ even though the manifolds have isomorphic homology groups with coefficients in $Z_2$. Nevertheless, it does not distinguish between $L_{3,1}$ and $S^3$, or some other pairs from among the manifolds listed above.

**Remark 8.1.8.** In view of Proposition 8.1.7, the first five lines of the table are evident, since the corresponding 3-manifolds are orientable and only $RP^3$ contains an odd surface. Let us explain the last statement. The manifold $S^1 \times RP^2$ contains four homologically distinct surfaces that realize elements of $H_2(M; Z_2)$: the empty surface, projective plane $\{*\} \times RP^2$, the torus $S^1 \times RP^1$, and the Klein bottle $S^1 \tilde{\times} RP^1$. The best way to imagine $S^1 \tilde{\times} RP^1 \subset S^1 \times RP^2$ is to let a point $x \in S^1$ move around $S^1$, rotating simultaneously $\{x\} \times RP^1$ inside $\{x\} \times RP^2$ such that the total rotation angle would be 180°. Since only one of the surfaces (projective plane) has an odd Euler characteristic, $n_o = 1$ and $n_e = 3$. It follows from Proposition 8.1.7 that $TV_+(S^1 \times RP^2) = 4$ and $TV_-(S^1 \times RP^2) = 2$. The following conjecture was stated by Kauffman and Lins:

**Conjecture** [60]. *Consider an arbitrary closed 3-manifold $M$, and let $X$ be a special spine for $M$. Let $n_e$ be the number of closed surfaces contained in $X$ that have even Euler characteristic and $n_o$ the number of closed surfaces in $X$ that have odd Euler characteristic. Then either $n_e = n_o$ or $n_o = 0$.*

*Moreover, $n_e = n_o$ if and only if the same is true for all special spines of $M$, and $n_o = 0$ if and only if the values of Turaev–Viro invariants for $\theta = (2 \pm 1)\pi/4$ are integers and equal.*

As we have seen above, for orientable manifolds the first part of the conjecture is true while the manifold $S^1 \times RP^2$ disproves it for the nonorientable case. The second part of the conjecture is also wrong, see [113] and Sect. 8.1.5.

Let us turn now our attention to solution (8.4). One can easily see that it gives nothing new, since we get the sum of the $TV_\pm$-invariant and of an order two invariant $u^{\chi(P)}$. The reason is that if the weight $w(\xi)$ of a black–white coloring $\xi$ of a special polyhedron $P$ is nonzero, then the black part of $P$ is either a closed surface or coincides with $P$. Thus solution (8.4) produces invariants $TV_\pm(M) + (\pm 1)^{\chi(P)}$. On the contrary, the invariants corresponding to the solution (8.5) are very interesting since they are actually the simplest nontrivial invariants of Turaev–Viro type. We consider them in Sect. 8.1.3.

### 8.1.3 Construction and Properties of the $\varepsilon$-Invariant

We start with an alternative description of the new invariant. Let $P$ be a simple polyhedron. Denote by $\mathcal{F}(\mathcal{P})$ the set of all simple subpolyhedra of $P$ including $P$ and the empty set.

**Lemma 8.1.9.** *$\mathcal{F}(\mathcal{P})$ is finite.*

*Proof.* It is easy to see that if a simple subpolyhedron $F \subset P$ contains at least one point of a 2-component $\alpha$ of $P$, then $\alpha \subset F$. It follows that for describing $F$ it is sufficient to specify which 2-components of $P$ are contained in $F$. Thus the total number of simple subpolyhedra of $P$ is no greater than $2^n$, where $n$ is the number of 2-components in $P$.

Let us associate to each simple polyhedron $F$ its *$\varepsilon$-weight*

$$w_\varepsilon(F) = (-1)^{V(F)} \varepsilon^{\chi(F)-V(F)},$$

where $V(F)$ is the number of vertices of $F$, $\chi(F)$ is its Euler characteristic, and $\varepsilon$ is a solution of the equation $\varepsilon^2 = \varepsilon + 1$. One may take $\varepsilon = (1+\sqrt{5})/2$ as well as $\varepsilon = (1-\sqrt{5})/2$.

**Definition 8.1.10.** *The $\varepsilon$-invariant $t(P)$ of a simple polyhedron $P$ is given by the formula $t(P) = \sum_{F \in \mathcal{F}(\mathcal{P})} w_\varepsilon(F)$.*

Below we will prove that the $\varepsilon$-invariant of a special polyhedron $P$ coincides with the weight $w(P)$ that corresponds to solution (8.5), see Sect. 8.1.2. Hence it is invariant under $T$-moves. Nevertheless, we prefer to give an independent proof since it reveals better the geometric nature of the invariance.

**Theorem 8.1.11.** *$t(P)$ is invariant under $T$-moves.*

*Proof.* Let a simple polyhedron $P_2$ be obtained from a simple polyhedron $P_1$ by the move $T$. Denote by $E_T$ the fragment of $P_1$ which is cut out and replaced by a fragment $E'_T$ of $P_2$. It is convenient to assume that the complement $P_1 \backslash E_T$ of $E_T$ and the complement $P_2 \backslash E'_T$ of the fragment $E'_T$ do coincide. Let us analyze the structures of $E_T$ and $E'_T$.

The fragment $E_T$ consists of two cones and three sheets called *wings*. Each cone consists of three-curved triangles. $E'_T$ consists of six-curved rectangles, the middle disc, and three wings. Let us divide the set $\mathcal{F}(\mathcal{P}_\infty)$ of all simple subpolyhedra of $P_1$ into two subsets. A simple subpolyhedron $F \in \mathcal{F}(P_1)$ is called *rich* (with respect to $E_T$), if $F \cap E_T$ contains all six triangles of $E_T$, and *poor* otherwise.

We wish to arrange a finite-to-(one or zero) correspondence between simple subpolyhedra of $P_2$ and those of $P_1$ such that the correspondence respects $\varepsilon$-weights.

(a) Let $F_1$ be a poor subpolyhedron of $P_1$. Since $E_T$ without a triangle is homeomorphic to $E'_T$ without the corresponding rectangle, there exists exactly one simple subpolyhedron $F_2$ of $P_2$ such that $F_1 \cap (P_1 - E_T) = F_2 \cap (P_2 - E'_T)$. Moreover, $F_2$ is homeomorphic to $F_1$ and hence has the same $\varepsilon$-weight.

(b) Let $F_1$ be a rich subpolyhedron of $P_1$. Then there exist exactly two simple subpolyhedra $F'_2$ and $F''_2$ of $P_2$ such that $F_1 \cap (P_1 \setminus E_T) = F'_2 \cap (P_2 \setminus E'_T) = F''_2 \cap (P_2 \setminus E'_T)$, namely, the one that contains the middle disc, and the other that does not. The intersection $F_1 \cap E_T$ can contain 0, 2, or 3 wings. It is easy to verify that in all three cases the equation $w_\varepsilon(F_1) = w_\varepsilon(F'_2) + w_\varepsilon(F''_2)$ is equivalent to $\varepsilon^2 = \varepsilon + 1$. See Fig. 8.7 for the case of 0 wings: $C$ denotes the product of weights and symbols that correspond to 2-cells and vertices outside $E_T$ and $E'_T$.

(c) We have not considered simple subpolyhedra of $P_2$ the intersections of which with $E'_T$ contain six rectangles and exactly one wing. The set of such polyhedra can be decomposed onto pairs $F'_2, F''_2$ such that

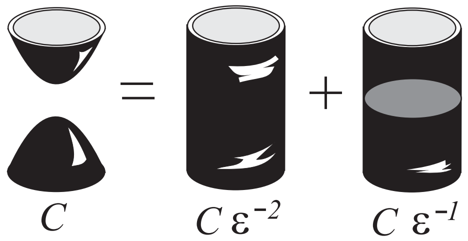



**Fig. 8.7.** The behavior of a rich surface that does not contain wings

$F_2' \cap (P_2 \setminus E_T') = F_2'' \cap (P_2 \setminus E_T')$, and exactly one of the subpolyhedra $F_2', F_2''$ contains the middle disc. For each such pair we have $w_\varepsilon(F_2') + w_\varepsilon(F_2'') = 0$. We can conclude that $t(P_1) = t(P_2)$.

Theorems 8.1.11 and 1.2.5 show that the following definition makes sense.

**Definition 8.1.12.** *Let $M$ be a compact 3-manifold. Then the $\varepsilon$-invariant $t(M)$ of $M$ is given by the formula $t(M) = t(P)$, where $P$ is a special spine of $M$.*

Now we relate the $\varepsilon$-invariant to the $TV$-invariant that corresponds to solution (8.5) in Sect. 8.1.11.

**Proposition 8.1.13.** *The $\varepsilon$-invariant coincides with the $TV$-invariant corresponding to solution (8.5).*

*Proof.* We use the same ideas as in the proof of Lemma 8.1.6. Assign to any black–white coloring $\xi$ of a special spine $P$ the union $F(\xi)$ of all black cells of $P$. Note that the black part of any butterfly has a singularity allowed for simple polyhedra if and only if the corresponding symbol (with respect to solution (8.5)) is nonzero. Hence the assignment $\xi \to F(\xi)$ induces a bijection between colorings with nonzero weights and simple subpolyhedra of $P$.

Now, let us verify that the $\varepsilon$-weight $w_\varepsilon(F)$ of a simple subpolyhedron $F = F(\xi) \subset P$ coincides with the weight $w(\xi)$ of the coloring $\xi$. Denote by $k_3, k_4, k_5, k_6$ the numbers of butterflies in $P$ with symbols $y_3, y_4, x_5, x_6$, respectively. Clearly, $k_6$ is the number of totally black butterflies in $P$ and thus coincides with the number $V(F)$ of true vertices of $F$. Note that the first butterfly has three black edges while the other three have four black edges each. $F(\xi)$ inherits from $P$ the cell structure with $k_3 + k_4 + k_5 + k_6$ vertices, $(3k_3 + 4k_4 + 4k_5 + 4k_6)/2$ edges, and some number of 2-cells that we denote by $c_2(F)$. It follows that $\chi(F) = -k_3/2 - k_4 - k_5 - k_6 + c_2(F)$. Taking into account that $w_1 = \varepsilon, y_3 = \varepsilon^{-1/2}$, $y_4 = x_5 = \varepsilon^{-1}$, and $x_6 = -\varepsilon^{-2}$, we have $w(\xi) = w_1^{c_2(F)} y_3^{k_3/2} y_4^{k_4} x_5^{k_5} x_6^{k_6} = (-1)^{V(P)} \varepsilon^{\chi(F)-V(P)}$. Taking sums of the weights, we get the conclusion.

**Example.** Let us compute "by hand" the $\varepsilon$-invariant of $S^3$. One should take a special spine of $S^3$, no matter which one. Let us take the Abalone $A$, see Fig. 1.1.4. It contains two 2-cells; one of them is the meridional disc $D$ of the tube. There are three simple subpolyhedra of $A$:

1. The empty subpolyhedron with the $\varepsilon$-weight 1.
2. The whole Abalone with the $\varepsilon$-weight $(-1)^V \varepsilon^{\chi(A)-V(A)} = -1$, since $V(A) = 1$ and $\chi(A) = 1$.
3. The subpolyhedron $A \backslash \mathrm{Int}\, D$, i.e., the subpolyhedron covered by the remaining 2-cell. It contains no vertices and has zero Euler characteristic. Hence it has $\varepsilon$-weight 1.

Summing up, we get $t(S^3) = 1$.

One can see from this example that the calculation of the $\varepsilon$-invariant is theoretically simple, but may be cumbersome in practice, especially when the manifold is complicated. Sokolov wrote a computer program that, given a special spine, calculates the $\varepsilon$-invariant of the corresponding 3-manifold. The results are presented in Table 8.1.

### 8.1.4 Turaev–Viro Invariants of Order $r \geq 3$

As we have seen in Sect. 8.1.1, the number of variables of the system (8.2) grows as $N^6$, where $N = r - 1$ is the number of colors in the palette $\mathcal{C} = \{0, 1, \ldots, N-1\}$. One may decrease the number of variables by imposing some constraints on butterflies with nonzero symbols.

**Definition 8.1.14.** *An unordered triple $i, j, k$ of colors taken from the palette $\mathcal{C}$ is called* admissible *if*

1. $i + j \geq k, j + k \geq i, k + i \geq j$ *( triangle inequalities).*
2. $i + j + k$ *is even.*
3. $i + j + k \leq 2r - 4$.

**Remark 8.1.15.** In the original paper [126], where this definition is taken from, Turaev and Viro used the half-integer palette $\{0, 1/2, \ldots, (r-2)/2\}$. There are some deep reasons behind this choice but we prefer to consider the integer palette $\mathcal{C}$. In any case, this is only a problem of notation.

Let us give a geometric interpretation of admissibility. Consider a disc $D$ with three adjacent strips that contain $i, j$, and $k$ strings, respectively. Then the triple $(i, j, k)$ satisfies conditions 1, 2 if and only if the strings can be joined together in a nonsingular way as shown in Fig. 8.8.

To be more precise, the united strings should be disjoint and no string should return to the strip it is coming out of. The third condition $i + j + k \leq 2r - 4$ is of technical nature and can be avoided.

**Definition 8.1.16.** *A coloring $\xi$ of a special polyhedron $P$ is called* admissible *if the colors of any three wings adjacent to the same edge form an admissible triple. The set of all admissible colorings will be denoted by* $\mathrm{Adm}(P)$.

**Table 8.1.** $\varepsilon$-Invariants of closed irreducible orientable 3-manifolds up to complexity 5

| $c_i$ | $M$ | $t(M)$ | $c_i$ | $M$ | $t(M)$ |
|---|---|---|---|---|---|
| $0_1$ | $S^3$ | $1$ | $5_2$ | $L_{13,2}$ | $\varepsilon+1$ |
| $0_2$ | $RP^3$ | $\varepsilon+1$ | $5_3$ | $L_{16,3}$ | $1$ |
| $0_3$ | $L_{3,1}$ | $\varepsilon+1$ | $5_4$ | $L_{17,3}$ | $\varepsilon+1$ |
| $1_1$ | $L_{4,1}$ | $1$ | $5_5$ | $L_{17,4}$ | $\varepsilon+1$ |
| $1_2$ | $L_{5,2}$ | $0$ | $5_6$ | $L_{19,4}$ | $1$ |
| $2_1$ | $L_{5,1}$ | $\varepsilon+2$ | $5_7$ | $L_{20,9}$ | $\varepsilon+2$ |
| $2_2$ | $L_{7,2}$ | $\varepsilon+1$ | $5_8$ | $L_{22,5}$ | $\varepsilon+1$ |
| $2_3$ | $L_{8,3}$ | $\varepsilon+1$ | $5_9$ | $L_{23,5}$ | $\varepsilon+1$ |
| $2_4$ | $S^3/Q_8$ | $\varepsilon+3$ | $5_{10}$ | $L_{23,7}$ | $\varepsilon+1$ |
| $3_1$ | $L_{6,1}$ | $1$ | $5_{11}$ | $L_{24,7}$ | $1$ |
| $3_2$ | $L_{9,2}$ | $1$ | $5_{12}$ | $L_{25,7}$ | $0$ |
| $3_3$ | $L_{10,3}$ | $0$ | $5_{13}$ | $L_{25,9}$ | $\varepsilon+2$ |
| $3_4$ | $L_{11,3}$ | $1$ | $5_{14}$ | $L_{26,7}$ | $1$ |
| $3_5$ | $L_{12,5}$ | $\varepsilon+1$ | $5_{15}$ | $L_{27,8}$ | $\varepsilon+1$ |
| $3_6$ | $L_{13,5}$ | $\varepsilon+1$ | $5_{16}$ | $L_{29,8}$ | $1$ |
| $3_7$ | $S^3/Q_{12}$ | $\varepsilon+3$ | $5_{17}$ | $L_{29,12}$ | $1$ |
| $4_1$ | $L_{7,1}$ | $\varepsilon+1$ | $5_{18}$ | $L_{30,11}$ | $\varepsilon+2$ |
| $4_2$ | $L_{11,2}$ | $1$ | $5_{19}$ | $L_{31,12}$ | $1$ |
| $4_3$ | $L_{13,3}$ | $\varepsilon+1$ | $5_{20}$ | $L_{34,13}$ | $1$ |
| $4_4$ | $L_{14,3}$ | $\varepsilon+1$ | $5_{21}$ | $S^3/Q_8\times Z_5$ | $\varepsilon+2$ |
| $4_5$ | $L_{15,4}$ | $\varepsilon+2$ | $5_{22}$ | $S^3/Q_{12}\times Z_5$ | $\varepsilon+2$ |
| $4_6$ | $L_{16,7}$ | $1$ | $5_{23}$ | $S^3/Q_{16}\times Z_3$ | $\varepsilon+1$ |
| $4_7$ | $L_{17,5}$ | $\varepsilon+1$ | $5_{24}$ | $S^3/Q_{20}$ | $3\varepsilon+2$ |
| $4_8$ | $L_{18,5}$ | $\varepsilon+1$ | $5_{25}$ | $S^3/Q_{20}\times Z_3$ | $-\varepsilon+2$ |
| $4_9$ | $L_{19,7}$ | $1$ | $5_{26}$ | $S^3/D_{40}$ | $-\varepsilon+2$ |
| $4_{10}$ | $L_{21,8}$ | $1$ | $5_{27}$ | $S^3/D_{48}$ | $\varepsilon+3$ |
| $4_{11}$ | $S^3/Q_8\times Z_3$ | $2\varepsilon+3$ | $5_{28}$ | $S^3/P_{24}\times Z_5$ | $\varepsilon+2$ |
| $4_{12}$ | $S^3/Q_{16}$ | $1$ | $5_{29}$ | $S^3/P_{48}$ | $\varepsilon+1$ |
| $4_{13}$ | $S^3/D_{24}$ | $2\varepsilon+3$ | $5_{30}$ | $S^3/P'_{72}$ | $\varepsilon+3$ |
| $4_{14}$ | $S^3/P_{24}$ | $2\varepsilon+3$ | $5_{31}$ | $S^3/P_{120}$ | $3\varepsilon+2$ |
| $5_1$ | $L_{8,1}$ | $\varepsilon+1$ | | | |

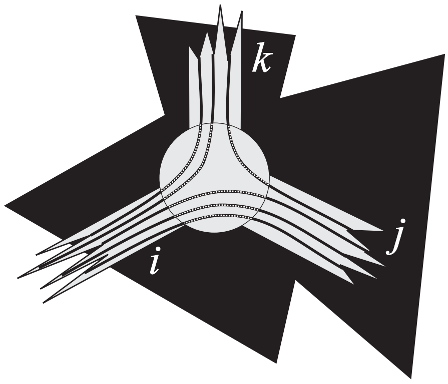



**Fig. 8.8.** Disjoint strings without returns

Similarly, one may speak about admissible butterflies: a colored butterfly

$$\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$$

is *admissible* if all the triples $(i, j, k)$, $(k, l, m)$, $(m, n, i)$, $(j, l, n)$ are admissible (they represent the wings that meet together along the four edges of the butterfly).

The constraint on butterflies we have mentioned above is the following:

*The symbols of nonadmissible butterflies must be zeros.*

Another way of saying this is that we define the weight (see Definition 8.1.3) of a special polyhedron $P$ by taking the sum over admissible colorings only:

$$w(P) = \sum_{\xi \in Adm(P)} w(\xi)$$

The following nonrigorous considerations show that the system (8.2) (subjected to the admissibility restrictions from Definition 8.1.14) should not have too many solutions (if any). For simplicity assume that the symbols of all admissible butterflies are nonzero. Since all equations are quasihomogeneous, we may assume $w_0 = 1$ (see Sect. 8.1.2). Denote by $s_k$ the symbol

$$\begin{vmatrix} 0 & 0 & 0 \\ k & k & k \end{vmatrix} = \begin{vmatrix} k & 0 & k \\ 0 & k & 0 \end{vmatrix}.$$

(a) Write down an equation of system (8.2) for the case $l = 0$. If $j \neq n$ or $k \neq m$, all terms in both sides of the equation contain symbols of nonadmissible butterflies (this is because a triple of the type $(0, x, y)$ is admissible if and only if $x = y$). This annihilates the equation. We may assume therefore that $n = j$ and $m = k$. Similarly, $l' = z$, and we get the equation

$$\begin{vmatrix} i & j & k \\ 0 & k & j \end{vmatrix} \begin{vmatrix} i & j & k \\ z & m' & n' \end{vmatrix} = w_z \begin{vmatrix} i & k & j \\ z & n' & m' \end{vmatrix} \begin{vmatrix} j & 0 & j \\ z & n' & z \end{vmatrix} \begin{vmatrix} k & 0 & k \\ z & m' & z \end{vmatrix},$$

that after dividing both sides by

$$\begin{vmatrix} i & j & k \\ z & m' & n' \end{vmatrix} = \begin{vmatrix} i & k & j \\ z & n' & m' \end{vmatrix}$$

gives

$$\begin{vmatrix} i & j & k \\ 0 & k & j \end{vmatrix} = w_z \begin{vmatrix} j & 0 & j \\ z & n' & z \end{vmatrix} \begin{vmatrix} k & 0 & k \\ z & m' & z \end{vmatrix}.$$

(b) Taking $z = 0$, we get $n' = n = j, m' = m = k$, and

$$\begin{vmatrix} i & j & k \\ 0 & k & j \end{vmatrix} = s_j s_k.$$

This converts the preceding equation to $s_j s_k = w_z s_j s_z s_k s_z$ or, equivalently, to $w_z = s_z^{-2}$.

(c) Next, let us write down the equation of system (8.2) for $i = j = k = 0$. The admissibility implies that $l' = m' = n'$ and $l = m = n$. Taking into account that

$$\begin{vmatrix} 0 & l & l \\ z & l' & l' \end{vmatrix} = \begin{vmatrix} z & l' & l \\ 0 & l & l' \end{vmatrix} = s_l s_{l'},$$

we get the equation $s_l s_{l'} = \sum_z w_z (s_l s_{l'})^3$, which is equivalent to $w_l w_{l'} = \sum_z w_z$ (both sums are taken over all $z \leq r-2$ such that the triple $(l, l', z)$ is admissible). In particular, for $l' = 1$ and $1 \leq l \leq r-2$ we get the system

$$\begin{aligned} &w_1 w_1 = w_0 + w_2 \\ &w_2 w_1 = w_1 + w_3 \\ &\dots \\ &w_{r-3} w_1 = w_{r-4} + w_{r-2} \\ &w_{r-2} w_1 = w_{r-3} \end{aligned}$$

To solve it, present $w_1$ in the form

$$w_1 = -(q + q^{-1}) = -\frac{q^2 - q^{-2}}{q - q^{-1}},$$

where $q$ is a new variable. Since

$$\frac{q^{i+1} - q^{-i-1}}{q - q^{-1}} = \frac{q^i - q^{-i}}{q - q^{-1}}(q + q^{-1}) - \frac{q^{i-1} - q^{-i+1}}{q - q^{-1}},$$

we get inductively

$$w_i = (-1)^i \frac{q^{i+1} - q^{-i-1}}{q - q^{-1}} \text{ for } 1 \leq i \leq r-2 \text{ and } \frac{q^r - q^{-r}}{q - q^{-1}} = 0.$$

We can conclude that the solutions of the system have the form

$$w_i = (-1)^i \frac{q^{i+1} - q^{-i-1}}{q - q^{-1}}, 1 \leq i \leq r-2,$$

where $q$ runs over all roots of unity of degree $2r$.

It follows from the above considerations that system (8.2) is very restrictive because a very small part of it allows us to find the weights $w_i$ of all colors and some symbols. It is surprising that any solutions exist! Below we present solutions found by Turaev and Viro [126]. To adjust our notation to the original one (see Remark 8.1.15), we adopt the following notational convention:

$$\hat{k} = k/2$$

for any integer $k$. Let $q$ be a $2r$-th root of unity such that $q^2$ is a primitive root of unity of degree $r$. Note that $q$ itself may not be primitive.

For an integer $n$ set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{8.6}$$

Note that all $[n]$ are real numbers and $[n] = 0$ if and only if $n = 0 \bmod r$.

Define the quantum factorial $[n]!$ by setting

$$[n]! = [n][n-1]\dots[2][1].$$

In particular, $[1]! = [1] = 1$. Just as for the usual factorial, set by definition, $[0]! = 1$.

For an admissible triple $(i, j, k)$ put

$$\Delta(i,j,k) = \left( \frac{[\hat{i} + \hat{j} - \hat{k}]!\, [\hat{j} + \hat{k} - \hat{i}]!\, [\hat{k} + \hat{i} - \hat{j}]!}{[\hat{i} + \hat{j} + \hat{k} + 1]!} \right)^{1/2}.$$

**Remark 8.1.17.** As we will see later, it does not matter, which square root of the expression in the round brackets is taken for $\Delta(i,j,k)$. The resulting 3-manifold invariant will be the same.

Now we are ready to present the solution. The weights of colors from the palette $\mathcal{C} = \{0, 1, \dots, r-2\}$ are given by

$$w_i = (-1)^i [i+1]. \tag{8.7}$$

The symbol

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$$

of the butterfly

$$\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$$

is given by

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sum_z \frac{(-1)^z [z+1]!\, A(i,j,k,l,m,n)}{B\left(z, \begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}\right) C\left(z, \begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}\right)}, \tag{8.8}$$

where

$$A(i,j,k,l,m,n) = \boldsymbol{i}^{(i+j+k+l+m+n)} \Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m),$$

$$B\left(z, \begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}\right) = [\hat{z} - \hat{i} - \hat{j} - \hat{k}]!\, [z - \hat{i} - \hat{m} - \hat{n}]!\, [z - \hat{j} - \hat{l} - \hat{n}]!\, [z - \hat{k} - \hat{l} - \hat{m}]!,$$

$$C\left(z, \begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}\right) = [\hat{i} + \hat{j} + \hat{l} + \hat{m} - z]!\, [\hat{i} + \hat{k} + \hat{l} + \hat{n} - z]!\, [\hat{j} + \hat{k} + \hat{m} + \hat{n} - z]!,$$

and the sum is taken over all integer $z$ such that all expressions in the square brackets are nonnegative. In other words, one should have $\alpha \le z \le \beta$, where

$$\alpha = \max(\hat{i} + \hat{j} + \hat{k}, \hat{i} + \hat{m} + \hat{n}, \hat{j} + \hat{l} + \hat{n}, \hat{k} + \hat{l} + \hat{m}),$$
$$\beta = \min(\hat{i} + \hat{j} + \hat{l} + \hat{m}, \hat{i} + \hat{k} + \hat{l} + \hat{n}, \hat{j} + \hat{k} + \hat{m} + \hat{n})$$

(it follows from the triangle inequalities that $\alpha \le \beta$).

**Remark 8.1.18.** The bold letter $\boldsymbol{i}$ in the above expression for $A(i,j,k,l,m,n)$ is the imaginary unit (do not confuse with the symbol $i$ denoting a color from the palette $\mathcal{C}$). If we replace $\boldsymbol{i}$ by $-\boldsymbol{i}$, we get a different solution producing the same 3-manifold invariant. It is because for any coloring with a nonzero weight the number of butterflies

$$\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$$

with an odd number $(i+j+k+l+m+n)$ is even.

Why do the presented values of variables form a solution to the system (8.2)? Turaev and Viro proved this by a reference to a paper of Kirillov and Reshetikhin [64], who had used the so-called Biederharn-Elliot identity [10,27] to obtain a solution to a similar system. Meanwhile there appeared many different ways to prove the existence of the invariants. Probably, one of the simplest approaches is based on remarkable results of Kauffman, Lickorish and others, and belongs to Roberts, see [108] and references therein. An exhaustive exposition of the subject along with deep connections to quantum groups, motivating ideas in physics and to other areas of mathematics can be found in the fundamental monograph [125].

**Definition 8.1.19.** *The 3-manifold invariant corresponding to the above solution will be called the* order $r$ Turaev–Viro invariant *and denoted by* $TV_q(M)$.

**Remark 8.1.20 (On the Terminology).** We distinguish between *invariants of Turaev–Viro type* and *Turaev–Viro invariants* $TV_q(M)$. The former correspond to arbitrary solutions (that potentially would be found in future), the latter are related to the particular solutions given by (8.7) and (8.8). For example, the $\varepsilon$-invariant is of Turaev–Viro type but it is not a Turaev–Viro invariant.

**Remark 8.1.21.** One should point out that our exposition of results in [126] differs from the original approach. In the first place, to simplify the construction, we do not pay any attention to the relative case, which is very important from the point of view of category theory. In particular, we do not reveal the functorial nature of the invariants, nor how they fit into the conception of Topological Quantum Field Theory (TQFT) [7]. On the other hand, it is sometimes convenient to consider (as we do) absolute invariants of not necessarily closed 3-manifolds

Secondly, our version of the invariants is $S^3$-normalized, i.e., $TV_q(S^3) = 1$ for all $q$. The invariant $\mid M \mid$ presented in [126] for a degree $2r$ root of unity $q = q_0$ is related to $TV_q(M)$ by the formula

$$\mid M \mid = -\frac{(q-q^{-1})^2}{2r} TV_q(M).$$

Thirdly, the solution given by (8.7) and (8.8) satisfies additional equations of the type

$$\sum_z w_z \begin{vmatrix} i & l & m \\ z & m' & l' \end{vmatrix} \begin{vmatrix} j & l & m \\ z & m' & l' \end{vmatrix} = \delta^i_j,$$

where $i, j, l, m, l', m'$ run over all elements of the palette $\mathcal{C}$ and $\delta^i_j$ is the Kronecker symbol. These equations guarantee that the weight of a simple polyhedron is invariant under lune moves, see Fig. 1.16. Moreover, one can calculate the invariants starting from any simple (not necessarily special) spine of a manifold. The only difference is that one should take into account the Euler characteristics of 2-components by defining the weight of a coloring $\xi$ by

$$w(\xi) = \prod_{v \in V(P)} \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_v \prod_{c \in C(P)} w_{\xi(c)}^{\chi(c)},$$

instead of corresponding formula (8.1) for the case of disc 2-components.

Finally, for the solution given by (8.7) and (8.8), the following holds: there exists a number $w$ (it is equal to $-2r/(q-q^{-1})^2$) such that for all $j$

$$w_j = w^{-1} \sum_{(k,l)} w_k w_l,$$

where the sum is taken over all $k, l$ such that the triple $(j, k, l)$ is admissible. This condition guarantees that performing a bubble move on a simple polyhedron is equivalent to multiplying its weight by $w$.

### 8.1.5 Computing Turaev–Viro Invariants

Just after discovering the invariants, Turaev and Viro calculated all of them for the sphere $S^3$, real projective space $RP^3$, for $S^2 \times S^1$, and the lens space $L_{3,1}$. The calculation was facilitated by the fact that these manifolds have simple spines without vertices. For example, $L_{3,1}$ admits a spine consisting of one 2-cell and having one triple circle. It can be presented as the identification space of a disc by a free action of the group $Z_3$ on the boundary. Sometimes it is called *triple hat*. Note that the three wings adjacent to any segment of the triple circle belong to the same 2-cell, and hence have the same color $i \in \mathcal{C}$ (for any coloring). The admissibility conditions (see Definition 8.1.14) imply that $i$ must be even and no greater than $(2r-4)/3$. It follows that

$$TV_q(L_{3,1}) = \sum_i w_i, \tag{8.9}$$

where the sum is taken over all even $i$ such that $0 \leq i \leq (2r-4)/3$.

If every simple spine of a given 3-manifold contains vertices, obtaining explicit expressions for order $r$ Turaev–Viro invariants (as functions on $q$) for all $r$ is difficult, see [137] for the case of lens spaces. On the other hand, if $r$ is fixed, the problem has a purely combinatorial nature and can be solved by means of a computer program. One should construct a special spine of the manifold and, using formulas (8.6)–(8.8), calculate the weights and symbols. Then one can enumerate all colorings and find out the value of the invariant by taking the sum of their weights. Extensive numerical tables of that kind can be found in [60, 113].

Let us make a digression. Soon after the discovery of the invariants, many mathematicians (and certainly Turaev and Viro) noticed that the invariants of $M$ were actually sums of invariants of pairs $(M, h)$, where $M$ is a 3-manifold and $h \in H_2(M; Z_2)$. We describe this observation in detail. Let $\xi$ be an admissible coloring of a special spine $P$ of a 3-manifold $M$ by colors taken from the palette $\mathcal{C} = \{0, 1, \ldots, N-1\}$. Reducing all colors mod 2, we get a black–white coloring $\xi$ mod 2 that happens to be also admissible owing to condition 2 of Definition 8.1.14. As we know from the proof of Lemma 8.1.6, admissible black–white colorings correspond to black (i.e., colored by the color 1) surfaces in $P$ or, what is just the same, to elements of $H_2(M; Z_2)$. This decomposes the set of all admissible colorings of $P$ into classes corresponding to elements of $H_2(M; Z_2)$: two colorings belong to the same class $Adm(P, h)$ if their mod 2 reductions determine the same homology class $h \in H_2(M; Z_2)$ (and hence the same surface in $P$).

Assume now that the pair $(M, h)$ and a special spine $P$ of $M$ are given. Define an invariant $TV_q(M, h)$ by setting $TV_q(M, h) = w(P, h)$, where the

$h$-weight $w(P,h)$ is given by

$$w(P,h) = \sum_{\xi \in Adm(P,h)} w(\xi).$$

The same proof as given for Theorem 8.1.4 shows that this definition is correct, i.e., $TV_q(M,h)$ does not depend on the choice of $P$. We need only one additional observation. Let a special spine $P_2$ of $M$ be obtained from a special spine $P_1$ by exactly one $T$-move, i.e., by removing a fragment $E_T$ and inserting a fragment $E'_T$, see Definition 1.2.3. For any admissible coloring $\xi$ of $P_1$, let $Col_\xi(P_2)$ be the set of admissible colorings of $P_2$ that coincide with $\xi$ on $P_1 \backslash E_T = P_2 \backslash E'_T$. Then all the colorings in $Col_\xi(P_2)$ determine the same homology class $h \in H_2(M; Z_2)$ as the coloring $\xi$.

It follows from the definition of $TV_q(M,h)$ that the Turaev–Viro invariant $TV_q(M)$ is the sum of $TV_q(M,h)$ taken over all $h \in H_2(M; Z_2)$. Especially important is the *homologically trivial part* $TV_q(M)_0$ of the Turaev–Viro invariant that corresponds to the zero element of $H_2(M; Z_2)$. Recall that $h \in H_2(M; Z_2)$ is even or odd, if it can be realized by a closed surface in $M$ having the Euler characteristic of the same parity. We follow [113] and denote by $TV_q(M)_1$ the *odd* part of $TV_q(M)$, that is equal to the sum of $TV_q(M,h)$ over all odd elements $h \in H_2(M; Z_2)$. Similarly, by $TV_q(M)_2$ we denote the sum of $TV_q(M,h)$ taken over all even elements $h \in H_2(M; Z_2)$ different from 0. Clearly, $TV_q(M) = TV_q(M)_0 + TV_q(M)_1 + TV_q(M)_2$.

**Remark 8.1.22.** Note that since any special spine $P \subset M$ is two-dimensional, the 2-cycle group $C_2(P, Z_2)$ coincides with $H_2(P; Z_2)$. Therefore, the mod 2 reduction of an admissible coloring $\xi \in Adm(P)$ determines the trivial element of $H_2(P; Z_2)$ if and only if just even colors $0, 2, \ldots$ have been used. Thus, the only difference between $TV_q(M)$ and its homologically trivial part $TV_q(M)_0$ is that we consider all admissible colorings in the first case and only even ones in the second.

At the end of the book we reproduce from [116] (with notational modifications) tables of Turaev–Viro invariants of order $\leq 7$ and their summands for all closed orientable irreducible 3-manifolds up to complexity 6 (Table A.1; see Chap. 2 for the definition of complexity). We subject $q$ to the following constraint: $q$ must be a primitive root of unity of degree $2r$. This constraint is slightly stronger than the one in the definition of Turaev–Viro invariants, see Sect. 8.1.4. Nevertheless, we do not lose any information because of the following relation proved in [116]: $TV_q(M)_\nu = (-1)^\nu TV_{-q}(M)_\nu$, where $\nu \in \{0, 1, 2\}$.

The invariants are presented by polynomials of $q$. This presentation is much better than the numerical form since we simultaneously encode the invariants evaluated at all degree $2r$ primitive roots of unity, and avoid problems with the precision of calculations. For the sake of compactness of notation, we write $\sigma_k$ instead of $q^k + q^{-k}$. For instance, we set $\sigma_1 = q + q^{-1}, \sigma_2 = q^2 + q^{-2}$,

**Table 8.2.** Turaev–Viro invariants of order $r \le 7$ and their summands for closed orientable irreducible 3-manifolds of complexity $\le 2$

| $M$ | $\nu\backslash r$ | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $S^3$ | 0 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 0 | 0 | 0 | 0 | 0 |
| | $\sum$ | 1 | 1 | 1 | 1 | 1 |
| $RP^3$ | 0 | 1 | 2 | $\sigma_2+2$ | 4 | $-\sigma_3+2\sigma_2+3$ |
| | 1 | $-1$ | $-\sigma_1$ | $-\sigma_2-2$ | $-2\sigma_1$ | $\sigma_3-2\sigma_2-3$ |
| | 2 | 0 | 0 | 0 | 0 | 0 |
| | $\sum$ | 0 | $-\sigma_1+2$ | 0 | $-2\sigma_1+4$ | 0 |
| $L_{3,1}$ | 0 | 1 | 1 | $\sigma_2+2$ | 3 | $\sigma_2+2$ |
| | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 0 | 0 | 0 | 0 | 0 |
| | $\sum$ | 1 | 1 | $\sigma_2+2$ | 3 | $\sigma_2+2$ |
| $L_{4,1}$ | 0 | 1 | 2 | 1 | 4 | $\sigma_2+2$ |
| | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 1 | 0 | 1 | 0 | $\sigma_2+2$ |
| | $\sum$ | 2 | 2 | 2 | 4 | $2\sigma_2+4$ |
| $L_{5,2}$ | 0 | 1 | 1 | 0 | 1 | $-\sigma_3+2\sigma_2+3$ |
| | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 0 | 0 | 0 | 0 | 0 |
| | $\sum$ | 1 | 1 | 0 | 1 | $-\sigma_3+2\sigma_2+3$ |
| $L_{5,1}$ | 0 | 1 | 1 | $\sigma_2+3$ | 1 | $-\sigma_3+2\sigma_2+3$ |
| | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 0 | 0 | 0 | 0 | 0 |
| | $\sum$ | 1 | 1 | $\sigma_2+3$ | 1 | $-\sigma_3+2\sigma_2+3$ |
| $L_{7,2}$ | 0 | 1 | 1 | $\sigma_2+2$ | 1 | 0 |
| | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 0 | 0 | 0 | 0 | 0 |
| | $\sum$ | 1 | 1 | $\sigma_2+2$ | 1 | 0 |
| $L_{8,3}$ | 0 | 1 | 2 | $\sigma_2+2$ | 4 | 1 |
| | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 1 | 2 | $\sigma_2+2$ | 0 | 1 |
| | $\sum$ | 2 | 4 | $2\sigma_2+4$ | 4 | 2 |
| $S^3/Q_8$ | 0 | 1 | 4 | $\sigma_2+4$ | 10 | $2\sigma_2+7$ |
| | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 3 | 6 | $3\sigma_2+12$ | 18 | $6\sigma_2+21$ |
| | $\sum$ | 4 | 10 | $4\sigma_2+16$ | 28 | $8\sigma_2+28$ |

For each $M$ the first three lines present $TV_q(M)_\nu$; the fourth line contains the values of $TV_q(M)$. For brevity, we write $\sigma_k$ instead of $q^k+q^{-k}$

and so on. For your convenience, a small part of the table (for manifolds of complexity $\leq 2$) is given.

Items (1–4) below are devoted to analysis of the table and commentaries.

(1) Selected testing has shown that the table agrees with the ones presented in [60, 61, 63], as well as with the above-mentioned calculations made by the authors of the invariants.
(2) The manifold $4_{12} = S^3/Q_{16}$ disproves the second part of the Kauffman–Lins Conjecture (see Remark 8.1.8). We see from Table A.9 that $TV_q$ $(S^3/Q_{16})$ is equal to 6 for every primitive root of unity $q$ of degree 8, including $q = exp((2 \pm 1)\pi/4)$. Nevertheless, since $TV_q(S^3/Q_{16})_1 \neq 0$, there is at least one surface with odd Euler characteristic. Therefore, $n_o \neq 0$.
(3) Let us call 3-manifolds *twins* if their Turaev–Viro invariants of order $\leq 7$ have the same triples of summands. The distribution of twins is shown in Fig. 8.9. Each line of the table consists of twin manifolds. Cells painted in gray contain *genuine twins*, i.e., manifolds having the same TV-invariants of all orders. They cannot be distinguished by Turaev–Viro invariants.

Let us comment on the table. There are no twins up to complexity 3. First two pairs of twins appear on the level of complexity $\leq 4$: manifold $3_4$ $(= L_{11,3})$ is a twin of $4_2$ $(= L_{11,2})$, and $3_6$ $(= L_{13,5})$ is a twin of $4_3$ $(= L_{13,3})$. At the level of complexity $\leq 5$ there appear new twin pairs and twin triples, and at the level $\leq 6$ we can find even a 7-tuple of twins.

Note that $TV_q(M)_1$ and $TV_q(M)_2$ are *not* invariants of Turaev–Viro type (see Definition 8.1.5) since the constraints on mod 2 reduction of colorings

| 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $0_1$ | | | | | $5_{16}$ $5_{17}$ | $6_{11}$ $6_{26}$ $6_{27}$ $6_{28}$ |
| | | | $3_2$ | | | $6_1$ |
| | | | $3_4$ | $4_2$ | $5_{19}$ | $6_{13}$ $6_{14}$ |
| | | | $3_6$ | $4_3$ | $5_2$ | $6_{29}$ |
| | | | | $4_7$ | $5_4$ $5_5$ | |
| | | | | $4_9$ | $5_6$ | $6_3$ |
| | | | | | $5_{15}$ | $6_9$ |
| | | | | | $5_{20}$ | $6_{18}$ |
| | | | | | $5_9$ $5_{10}$ | $6_6$ $6_{21}$ $6_{22}$ $6_{33}$ |
| | | | | | | $6_{16}$ $6_{17}$ |
| | | | | | | $6_{23}$ $6_{24}$ |
| | | | | | | $6_{65}$ $6_{67}$ |
| | | | | | | $6_{68}$ $6_{69}$ |
| | | | | | | $6_{70}$ $6_{71}$ |

**Fig. 8.9.** Each line of the table contains twin manifolds. Cells painted in *gray* contain *genuine twins* (manifolds having the same TV-invariants of all orders)

have global nature. On the other hand, one can extract from Table A.9 that $TV_q(M)_1$ and $TV_q(M)_2$ add actually nothing to information given by $TV_q(M)_0$ and $TV_q(M)$. For that reason, for manifolds of complexity 6 we include only values of $TV_q(M)$ and $TV_q(M)_0$.

(4) Analyzing the tables, we can notice that $TV_3(M)TV_q(M)_0 = TV_q(M)$ for odd $r$ the following holds: In fact, this equality is always true and follows from the similar formula of Kirby and Melvin for Reshetikhin–Turaev invariants [63], and the Turaev–Walker theorem saying that each Turaev–Viro invariant is equal to the square of the absolute value of the corresponding Reshetikhin–Turaev invariant (see, for instance, [108, 125]).

(5) It is not difficult to observe that all coefficients in the polynomials presenting Turaev–Viro invariants in our tables are integers. This is not an accident. Robers and Masbaum gave in [75] an elegant proof that values of Turaev–Viro invariants are algebraic integers.

(6) As we have mentioned above, there exist explicit expressions for Turaev–Viro invariants for lens spaces. Using Yamada's formulas [137], Sokolov found a simple solution to the following interesting problem: *Which lens spaces can be distinguished by Turaev–Viro invariants?*

For any integer $v$ define a *characteristic* function $h_v : Z \to Z_2$ by setting $h_v(k) = 1$ if $k = \pm 1 \mod v$, and $h_v(k) = 0$, otherwise.

**Theorem 8.1.23.** *[117] Lens spaces $L_{p_1,q_1}$ and $L_{p_2,q_2}$ have the same Turaev–Viro invariants of all orders if and only if $p_1 = p_2$ and for any divisor $v > 2$ of $p_1$ we have $h_v(q_1) = h_v(q_2)$.*

Sokolov noticed also that if $p_1 \neq p_2$, then $L_{p_1,q_1}$ and $L_{p_2,q_2}$ can be distinguished by Turaev–Viro invariant $TV_q$ of some order $r \leq 2R$, where $R$ is the minimal natural number such that $R$ is coprime with $p_1, p_2$, and $p_1 \neq p_2 \mod R$. In case $p_1 = p_2$ it is sufficient to consider only invariants of order $r \leq p_1$. If $p$ is prime, then the criterion is especially simple.

**Corollary 8.1.24.** *If $p$ is prime, then $L_{p,q_1}$ and $L_{p,q_2}$ have the same Turaev–Viro invariants if and only if for $i = 1, 2$ either $q_i = \pm 1 \mod p$ or $q_i \neq \pm 1 \mod p$.*

Note that all lines of the table in Fig. 8.9 except the last three contain only lens spaces. Thus Theorem 8.1.23 and the above corollary are sufficient for selecting genuine twins among them. See Remark 8.2.15 in Sect. 8.2 for an explanation why the last three lines contain genuine twin pairs.

**Remark 8.1.25.** It is interesting to note that if $p$ is prime, then Reshetikhin–Turaev invariants can distinguish any two nonhomeomorphic lens spaces $L_{p,q_1}$ and $L_{p,q_2}$, see [51].

### 8.1.6 More on $\varepsilon$-Invariant

Comparing Tables 8.1 and A.9, one can see that $\varepsilon$-invariant coincides with the homologically trivial part $TV_q(M)_0$ of order 5 Turaev–Viro invariant. Let us prove that.

**Theorem 8.1.26.** *[89] Let $M$ be a closed 3-manifold. Then $\varepsilon(M) = TV_q(M)_0$, where $\varepsilon = (1+\sqrt{5})/2$ for $q = exp(\pm\frac{\pi}{5}\boldsymbol{i})$ and $\varepsilon = (1-\sqrt{5})/2$ for $q = exp(\pm\frac{3\pi}{5}\boldsymbol{i})$.*

*Proof.* The values of $q$ and $\varepsilon$ presented above are related by the equality $\varepsilon = q^2 + 1 + q^{-2}$. Indeed, $q^{10} = 1$ implies

$$\frac{q^5 - q^{-5}}{q + q^{-1}} = q^4 + q^2 + 1 + q^{-2} + q^{-4} = 0,$$

which is equivalent to $(q^2 + 1 + q^{-2})^2 = (q^2 + 1 + q^{-2}) + 1$. Recall that for calculating $\varepsilon$-invariant we use two colors: 0 and 1. For calculating $TV_q(M)_0$ in the case $r = 5$ we use *even* colors from the palette $\{0, 1, 2, 3\}$, i.e., two colors 0 and 2 (see Remark 8.1.22). It remains to verify that the correspondence $(0, 1) \to (0, 2)$ transforms weights of colors and symbols for $\varepsilon$-invariant (see solution (8.5) on page (389)) to those for $TV_q(M)_0$ (see formulas (8.7) and (8.8) on page 399). For instance, the weight of the color 1, in the case of $\varepsilon$-invariant, equals $\varepsilon$ while the weight of the color 2, in the case of Turaev–Viro invariant, equals $q^2 + 1 + q^{-2}$.

Let us discuss briefly the relation between $\varepsilon$-invariant and TQFT. A three-dimensional TQFT is a functor $\mathcal{F}$ from the three-dimensional cobordism category to the category of vector spaces. The functor should satisfy some axioms, see [7]. In particular, "quantum" means that $\mathcal{F}$ takes the disjoint union of surfaces to the tensor product of vector spaces. In our case the base field corresponding to the empty surface is the field $R$ of real numbers. It follows that to every closed 3-manifold there corresponds a linear map from $R$ to $R$, that is, the multiplication by a number. This number is an invariant of the manifold.

As explained in [126], Turaev–Viro invariants fit into conception of TQFT. The only difference is that instead of the cobordism category one should consider a category whose objects have the form $(F, \Gamma)$, where $F$ is a surface and $\Gamma$ is a fixed one-dimensional special spine of $F$. Here a one-dimensional special polyhedron is a regular graph of valence 3. Morphisms between objects $(F_-, \Gamma_-), (F_+, \Gamma_+)$ have the form $(M, i_-, i_+)$, where $M$ is a 3-manifold with boundary $\partial M$ presented as the union of two disjoint surfaces $\partial_- M$, $\partial_+ M+$, and $i_\pm : F_\pm \to \partial_\pm M$ are homeomorphisms.

The $\varepsilon$-invariant, as any other Turaev–Viro type invariant, admits a similar interpretation. From general categorical considerations (see [126, Sect. 2.4]) it follows that there arises a homomorphism $\Phi$ from the mapping class group of the two-dimensional torus $T^2 = S^1 \times S^1$ to the matrix group $GL_5(R)$. Given a homomorphism $h : T^2 \to T^2$, we construct the cobordism $(T^2 \times I, i_-, i_+(h))$,

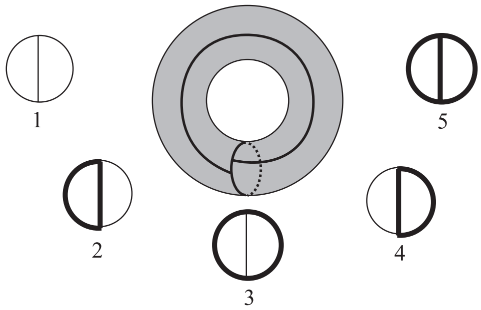



**Fig. 8.10.** Five *black–white* colorings for a spine of a torus

where $i_- : T^2 \to T^2 \times \{0\} \subset T^2 \times I$ is the standard inclusion and $i_+(h) : T^2 \to T^2 \times \{1\} \subset T^2 \times I$ is the inclusion induced by $h$. We assume that $T^2$ is equipped with a fixed special spine $\Theta$. By definition, put $\Phi(h) = \mathcal{F}(T^2 \times I, i_-, i_+(h))$, where $\mathcal{F}$ is the functor corresponding to the $\varepsilon$-invariant. Then $\Phi(h)$ is a linear map $R^5 \to R^5$. The dimension is 5, since $\Theta$ admits exactly five black–white colorings (see Fig. 8.10) that are admissible in the following sense: one black and two white edges never meet at the same vertex.

The mapping class group of the torus is generated by twists $\tau_m$ and $\tau_l$ along a meridian and a longitude, respectively. The twists satisfy the relations $\tau_m\tau_l\tau_m = \tau_l\tau_m\tau_l$ and $(\tau_m\tau_l\tau_m)^4 = 1$. Denote by $a$ and $b$ the matrices of the corresponding linear maps $R^5 \to R^5$. One can verify that $a^5 = 1$, $(aba)^2 = 1$ and that the group $\langle a, b \mid aba = bab, (aba)^2 = 1, a^5 = 1\rangle$ is finite. Actually, the presentation coincides with the standard presentation $\langle a, b \mid aba = bab, (a^2b)^2 = a^5 = 1\rangle$ of the alternating group $A_5$.

The following theorem is a direct consequence of this observation [89]:

**Theorem 8.1.27.** *Let $F$ be a closed surface and $n$ a nonnegative integer. Denote by $\mathcal{M}(F, n)$ the set of all Seifert manifolds over $F$ with $n$ exceptional fibers. Then the set $\{t(M), M \in \mathcal{M}(F, n)\}$ of the values of the $\varepsilon$-invariant is finite.*

The number $60^n$ (60 is the order of the alternating symmetric group $A_5$) serves as an upper estimate for the number of values of $t(M)$. Certainly, the estimate is very rough. More detailed considerations show that for lens spaces the number of values of $\varepsilon$-invariant is equal to 4. We give without proof an exact expression for the $\varepsilon$-invariant of the lens space $L(p, q)$.

**Theorem 8.1.28.**

$$t(L_{p,q}) = \begin{cases} 1, & \textit{if } p \equiv \pm 1 \bmod 5; \\ \varepsilon + 1, & \textit{if } p \equiv \pm 2 \bmod 5; \\ \varepsilon + 2, & \textit{if } p \equiv 0 \bmod 5 \textit{ and } q \equiv \pm 1 \bmod 5; \\ 0, & \textit{if } p \equiv 0 \bmod 5 \textit{ and } q \equiv \pm 2 \bmod 5. \end{cases}$$

## 8.2 3-Manifolds Having the Same Invariants of Turaev–Viro Type

This section is based on the following observation of Lickorish [69]: if two 3-manifolds $M_1, M_2$ have special spines with the same incidence relation between 2-cells and vertices (in a certain strong sense), then their Turaev–Viro invariants of all orders coincide. Manifolds having spines as above are called *similar*. We construct a simple example of similar 3-manifolds with different homology groups, and present a result of Nowik and the author [88] stating that under certain conditions similar manifolds are homeomorphic.

Let $P$ be a special spine of a 3-manifold $M$ and $V = V(P)$ the set of its vertices. Denote by $N(V, P)$ a regular neighborhood of $V$ in $P$. It consists of some number of disjoint copies of the butterfly $E$. The intersection of the union of all open 2-cells in $P$ with each butterfly consists of exactly six wings.

**Definition 8.2.1.** *Two special polyhedra $P_1$ and $P_2$ are called* similar *if there exists a homeomorphism $\varphi : N(V(P_1), P_1) \to N(V(P_2), P_2)$ such that for any two wings $w_1$ and $w_2$ of $P_1$ the following condition holds: $w_1$ and $w_2$ belong to the same 2-cell of $P_1$ if and only if $\varphi(w_1)$ and $\varphi(w_2)$ belong to the same 2-cell of $P_2$. The homeomorphism $\varphi$ is called a* similarity homeomorphism.

A good way to think of it is the following: let us paint the 2-cells of $P_1$ in different colors and the corresponding 2-cells of $P_2$ in the same colors. Then the similarity homeomorphism $\varphi$ is required to preserve the colors of wings. In other words, $P_2$ must contain exactly the same colored butterflies as $P_2$.

**Definition 8.2.2.** *Two 3-manifolds $M_1$ and $M_2$ are said to be* similar *if a special spine of $M_1$ is similar to a special spine of $M_2$.*

Examples of similar but nonhomeomorphic 3-manifolds will be presented later. The following proposition is based on an idea of Lickorish [69]. It is related to all invariants of Turaev–Viro type, not only to Turaev–Viro ones (see Definition 8.1.5 and Remark 8.1.20).

**Proposition 8.2.3.** *Similar manifolds have the same invariants of Turaev–Viro type.*

*Proof.* Let us look carefully through the construction of Turaev–Viro type invariants (Sect. 8.1.1). We come to the conclusion that all what we need to know to calculate the invariants is just the number of vertices and 2-cells, and the incidence relation between vertices and 2-cells, see Definition 8.2.1. For similar spines these data coincide and hence produce the same invariants.

Below we describe moves on special polyhedra and moves on manifolds that transform them into similar ones. We start with moves on manifolds.

Let $M$ be a (not necessarily orientable) 3-manifold and $F \subset \operatorname{Int} M$ a closed connected surface such that $F$ is two-sided in $M$ and $\chi(F) \geq 0$. The last condition means that $F$ is homeomorphic to $S^2$, $RP^2$, $T^2 = S^1 \times S^1$, or to the Klein bottle $K^2$. Choose a homeomorphism $r : F \to F$ such that:

(1) If $F = S^2$, then $r$ reverses the orientation.
(2) If $F = RP^2$, then $r$ is the identity.
(3) If $F = T^2$ or $F = K^2$, then $r$ induces multiplication by $-1$ in $H_1(F; Z)$.

It is clear that $r$ is unique up to isotopy. In case (3) one can explicitly describe it as follows: present the torus or the Klein bottle as a square with identified opposite edges. Then $r$ is induced by the symmetry of the square with respect to the center.

Now cut $M$ along $F$ and repaste the two copies of $F$ thus obtained according to the homeomorphism $r$. We get a new 3-manifold $M_1$.

**Definition 8.2.4.** *We say the new 3-manifold $M_1$ arising in such a way is obtained from $M$ by the* manifold move *along $F$.*

**Remark 8.2.5.** The manifold move along $RP^2$ does not change the manifold, and neither does the move along any trivial (i.e., bounding a ball) 2-sphere. Suppose $F = T^2$ and $F$ bounds a solid torus in $M$. Since $r : T^2 \to T^2$ can be extended to the interior of the solid torus, we have $M_1 = M$. The same is true for any Klein bottle that bounds in $M$ a solid Klein bottle $S^1 \tilde{\times} D^2$.

Now let us turn our attention to moves on special polyhedra. Let $G$ be a connected graph with two vertices of valence 3. There exist two such graphs: a theta-curve (a circle with a diameter) and an *eyeglass curve* (two circles joined by a segment). Choose a homeomorphism $\varrho : G \to G$ such that:

(1) If $G$ is a theta-curve, then $\varrho = \varrho_1$, where $\varrho_1 : G \to G$ permutes the vertices and takes each of the three edges into itself.
(2) If $G$ is an eyeglass curve, then $\varrho = \varrho_2$, where $\varrho_2 : G \to G$ leaves the joining segment fixed and inverses both loops, see Fig. 8.11.

**Definition 8.2.6.** *An one-dimensional subpolyhedron $G$ of a special polyhedron $P$ is called* proper *if a regular neighborhood $N(G, P)$ of $G$ in $P$ is a twisted or untwisted $I$-bundle over $G$. If $N(G, P) \approx G \times I$, then $G$ is called* two-sided.

Let $G \subset P$ be a two-sided theta-curve or an eyeglass curve in a special polyhedron $P$. Cut $P$ along $G$ and repaste the two copies of $G$ thus obtained according to the homeomorphism $\varrho$. We get a new special polyhedron $P_1$.

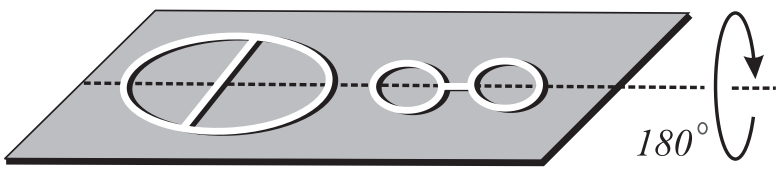



**Fig. 8.11.** Involution $\varrho$ on the theta-curve and eyeglass curve

**Definition 8.2.7.** *We say the new special polyhedron $P_1$ is obtained from $P$ by a* spine move $\sigma_i$ *along $G$, where $i = 1$ if $\varrho = \varrho_1$ and $i = 2$ if $\varrho = \varrho_2$.*

**Proposition 8.2.8.** *Spine moves transform special polyhedra to similar ones.*

*Proof.* Let $G \subset P$ be a two-sided theta-curve or a two-sided eyeglass curve in a special polyhedron $P$. The edges of $G$ decompose some 2-cells of $P$ into smaller parts that are glued together to new 2-cells. Since $\varrho$ takes each edge of $G$ to itself, the boundary curves of the new 2-cells run along the same edges as before, although may pass along them in a different order. Nothing happens near vertices. It follows that the new special polyhedron is similar to $P$.

Recall that we have two types of spine moves: theta-move $\sigma_1$ and glasses-move $\sigma_2$. It is convenient to introduce the third move $\sigma_3$.

Let $G \subset P$ be a proper theta-curve with edges $l_1, l_2, l_3$ such that

(1) $G$ separates $P$.
(2) $l_1$ and $l_2$ belong to the same 2-cell $C$ of $P$.

Choose a homeomorphism $\varrho_3 : G \to G$ such that $\varrho_3$ leaves $l_3$ fixed and permutes $l_1$ and $l_2$. Cut $P$ along $G$ and repaste the two copies of $G$ thus obtained according to $\varrho_3$. We say that the new special polyhedron $P_1$ arising in such a way is obtained from $P$ by the move $\sigma_3$.

**Lemma 8.2.9.** *$\sigma_3$ can be expressed through $\sigma_1$ and $\sigma_2$.*

*Proof.* Let $l_1, l_2$ be the two edges of $G$ which are contained in the same 2-cell $C$ such that $\sigma_3$ transposes them. Then there exists a simple arc $l \subset C$ such that $l \cap G = \partial l$ and $l$ connects $l_1$ with $l_2$. Consider a regular neighborhood $N = N(G \cup l)$ of $G \cup l$ in $P$. Since $G$ separates $P$, it has a neighborhood homeomorphic to $G \times [0, 1]$. Hence $N$ can be presented as $G \times [0, 1]$ with a twisted or an untwisted band $B$ attached to $G \times \{1\}$, see Fig. 8.12.

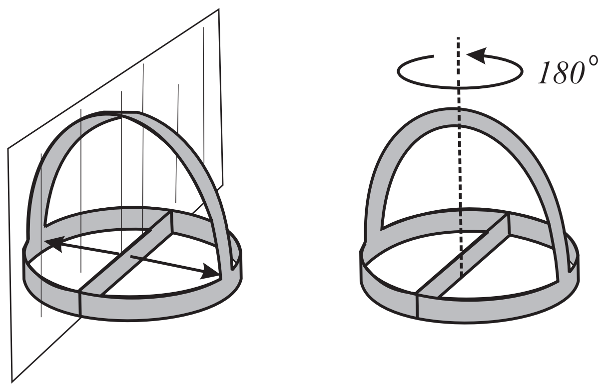



**Fig. 8.12.** Two types of $N = (G \times [0, 1]) \cup B$; the rotation by 180° determines a homeomorphism of $N$

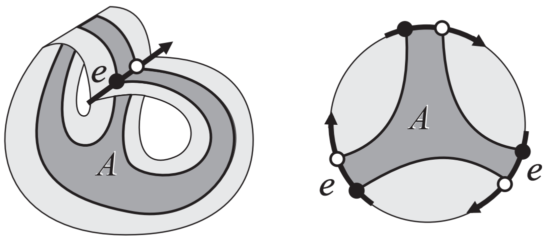



**Fig. 8.13.** Spine move across $e$: we cut out the region $A$ and paste it back by a homeomorphism that permutes the *white* and *black* vertices and is invariant on edges

If the band is untwisted, then $N$ is bounded by $G_1$ and $G_2$, where $G_1$ is a theta-curve isotopic to $G$ and $G_2$ is an eyeglass curve. There exists a homeomorphism $h : N \to N$ such that $h|_{G_1} = \varrho_3$ and $h|_{G_2} = \varrho_2$ ($h$ can be visualized as the symmetry in the vertical plane shown on Fig. 8.12). It follows that the move $\sigma_3$ along $G_1$ (and along $G$) is equivalent to the move $\sigma_2$ along $G_2$.

Let the band be twisted. Then $N$ is bounded by two theta-curves $G_1$ and $G_2$, where $G_1$ is isotopic to $G$. There exists a homeomorphism $h : N \to N$ (this time the rotation by 180° around the vertical axis) such that $h\big|_{G_1} = \varrho_1\varrho_3$ and $h\big|_{G_2} = \varrho_1$. Hence, the superposition of the moves $\sigma_1$ and $\sigma_3$ along $G_1$ is equivalent to the move $\sigma_1$ along $G_2$. Taking into account that $\varrho_1^2 = 1$, we can conclude that the move $\sigma_3$ along $G$ is equivalent to the superposition of the move $\sigma_1$ along $G_1$ and the move $\sigma_1$ along $G_2$.

Suppose the boundary curve of a 2-cell $C$ of a special spine $P$ passes along an edge $e$ of $P$ three times. Choose two points on $e$ and join them by three arcs in $C$ as it is shown on Fig. 8.13. The union $G$ of the arcs is a proper two-sided theta or an eyeglass curve in $P$. One can consider the spine move along $G$. To distinguish this type of spine move we supply it with a special name.

**Definition 8.2.10.** *Let $G$ be a proper two-sided theta-curve or an eyeglass curve in a special polyhedron $P$ such that both vertices of $G$ lie in the same edge. Then the spine move along $G$ is called a* spine move across $e$.

Our next goal is to prove that spine moves induce moves on manifolds, and vice versa, manifold moves can be realized by spine moves.

**Lemma 8.2.11.** *Let $G$ be a proper theta or an eyeglass curve in a special spine $P$ of a closed 3-manifold $M$. Then there exists a closed connected surface $F \subset M$ such that $\chi(F) \geq 0$, $F \cap P = G$, and $F$ is transversal to the singular graph $SP$ of $P$.*

*Proof.* Let $N = N(P, M)$ be a regular neighborhood of $P$ in $M$. Present $N$ as the mapping cylinder $C_f = P \cup (\partial N \times [0,1])/\sim$ of an appropriate locally homeomorphic map $f : \partial N \to P$. Then $F_1 = (f^{-1}(G) \times [0,1])/\sim$ is a surface in $N$ such that

(1) $F_1 \cap \partial N = \partial F_1$, $F_1 \cap P = G$, and $F_1$ is transversal to $SP$.
(2) $G$ is a spine of $F_1$.

To obtain $F$, attach disjoint 2-cells contained in the 3-cell $M \setminus N$ to the boundary components of $F_1$. Since $\chi(F_1) = \chi(G) = -1$, we have $\chi(F) \geq 0$.

**Proposition 8.2.12.** *Let $P$ be a special spine of a closed 3-manifold $M$, and let $G \subset P$ be a two-sided theta-curve or a two-sided eyeglass curve. Denote by $P_1$ the special polyhedron obtained from $P$ by the spine move along $G$. Then*

*(1) $P_1$ is a spine of a closed 3-manifold $M_1$.*
*(2) $M_1$ can be obtained from $M$ by a manifold move.*

*Proof.* Let $F \subset M$ be the surface constructed in Lemma 8.2.11. Since $G$ is two-sided, $F$ is also two-sided. The homeomorphism $\varrho : G \to G$ can be extended to a homeomorphism $r : F \to F$. It is clear that $r$ satisfies conditions (1)–(3) preceding Definition 8.2.4 of a manifold move. Denote by $M_1$ the 3-manifold obtained from $M$ by the manifold move along $F$. Since $r\big|_G = \varrho$, $P_1$ is a spine of $M_1$.

**Proposition 8.2.13.** *Let a closed 3-manifold $M_1$ be obtained from a closed 3-manifold $M$ by a manifold move along a surface $F \subset M$. Then $M$ and $M_1$ are similar.*

*Proof.* Let us construct a special spine $P$ of $M$ such that $G = P \cap F$ is a proper two-sided theta-curve. To do it, remove an open ball $D^3$ from $M$ such that $D = D^3 \cap F$ consists of one open disc if $F = T^2, K^2$, and of three open discs if $F = S^2$. We do not take $F = RP^2$ since in this case the manifold move is trivial. Denote by $F_1$ the surface $F \setminus D$. Starting from $F_1 \times \partial I$, collapse a regular neighborhood $N = F_1 \times I$ in $M \setminus D^3$ onto $G \times I$, where $G$ is a theta-graph in $F_1$. The collapsing can be easily extended to a collapsing of $M \setminus D^3$ onto a special spine $P \supset G \times I$.

Apply to $P$ the spine move along $G$. It follows from Proposition 8.2.12 that the special polyhedron $P_1$ thus obtained is a spine of $M_1$. Since $P$ and $P_1$ are similar, the same is true for $M$ and $M_1$.

**Example 8.2.14.** We are ready now to construct two similar manifolds with different homology groups. Take $M_1 = S^1 \times S^1 \times S^1$ and consider the torus $T^2 = S^1 \times S^1 \times \{*\} \subset M$. To construct $M_2$, perform the manifold move on $M_1$ along $T^2$. By Proposition 8.2.13, $M_2$ is similar to $M_1$. A simple calculation shows that $H_1(M_1; Z) = Z \oplus Z \oplus Z$, and $H_1(M_2; Z) = Z_2 \oplus Z_2 \oplus Z$.

**Remark 8.2.15.** According to Preposition 8.2.3, manifolds $M_1$ and $M_2$ above have the same Turaev–Viro invariants. For instance, if $q = exp(\frac{\pi}{7}i)$, then $TV_q(M_1) = TV_q(M_2) = -63q^3 + 189q^2 + 378 + 189q^{-2} - 63q^{-3}$, see Table A.1, where $M_1$, $M_2$ have names $6_{70}$, $6_{71}$, and Fig. 8.9, where they are shown as genuine twins. Manifolds $6_{65}$, $6_{67}$ as well as $6_{68}$, $6_{69}$ occupying the neighboring lines, also form genuine twin pairs since they are related by the same manifold move.

It is interesting to recall here that Turaev–Viro invariants of order 2 determine the order of the second homology group with coefficients $Z_2$, see Sect. 8.1.7. This agrees with the observation that $H_2(M_1; Z_2) = H_2(M_2; Z_2) = Z_2 \oplus Z_2 \oplus Z_2$.

As we have claimed at the beginning of this section, under certain conditions similarity of 3-manifolds implies homeomorphism. The idea of the proof is to transform a special spine $P_1$ of the first manifold into a similar special spine $P_2$ of the second one step by step. Our first goal is to define *graph moves* for transforming the singular graph of $P_1$ to the one of $P_2$.

Let $\Gamma$ be a finite (multi)graph. Fix a finite set $A$. By a *coloring* of $\Gamma$ by $A$ we mean a map $c : E(\Gamma) \to A$, where $E(\Gamma)$ is the set of all open edges of $\Gamma$. Denote by $V(\Gamma)$ the set of vertices of $\Gamma$ and by $N(V, \Gamma)$ a regular neighborhood of $V$ in $\Gamma$. The intersection of open edges with $N(V, \Gamma)$ consists of half-open 1-cells, which are called *thorns*.

**Definition 8.2.16.** *Two colored graphs $\Gamma_1$ and $\Gamma_2$ are called* similar*, if there exists a homeomorphism $\varphi : N(V(\Gamma_1), \Gamma_1) \to N(V(\Gamma_2), \Gamma_2)$ preserving the colors of thorns. The homeomorphism $\varphi$ is called a* similarity homeomorphism.

Let $\Gamma$ be a colored graph. Choose two edges $e_1$ and $e_2$ of the same color and cut each of them in the middle. Repaste the four "half edges" thus obtained into two new edges which do not coincide with the initial ones.

**Definition 8.2.17.** *We say the new colored graph $\Gamma_1$ arising in such a way is obtained from $\Gamma$ by a* graph move *along $e_1$ and $e_2$. The graph move is called* admissible*, if $\Gamma$ and $\Gamma_1$ are connected.*

**Remark 8.2.18.** For any given $e_1$ and $e_2$ there exist two different graph moves along $e_1$ and $e_2$. Suppose $\Gamma$ is connected and $\Gamma \setminus \text{Int}\,(e_1 \cup e_2)$ consists of two connected components such that each of them contains one vertex of each edge. Then precisely one of the moves is admissible, see Fig. 8.14. If $\Gamma \setminus \text{Int}\,(e_1 \cup e_2)$ is connected, then both moves are admissible.

**Lemma 8.2.19.** *Let $\Gamma_1$ and $\Gamma_2$ be similar colored graphs. If they are connected, then one can pass from $\Gamma_1$ to $\Gamma_2$ by a sequence of admissible graph moves.*

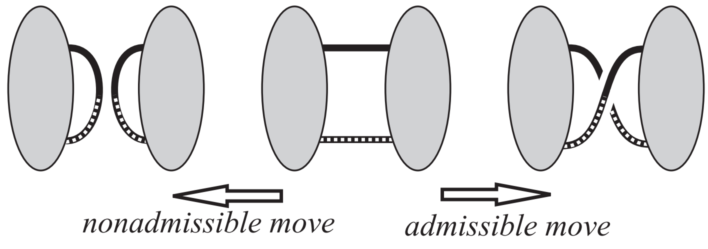



**Fig. 8.14.** Admissible and nonadmissible graph moves

*Proof.* It follows from Definition 8.2.16 that there exists a homeomorphism $\varphi : N(V(\Gamma_1), \Gamma_1) \to N(V(\Gamma_2), \Gamma_2)$ preserving the colors of thorns. We call an edge $e$ in $\Gamma_1$ *correct* if $\varphi$ maps the two thorns $t_1$ and $t_2$ contained in it into the same edge $f$ in $\Gamma_2$. The thorns $t_1$ and $t_2$, the edge $f$ and the thorns $\varphi(t_1), \varphi(t_2) \subset f$ are also called *correct*. The homeomorphism $\varphi$ can be extended to an edge $e$ if and only if $e$ is correct, so to prove Lemma 8.2.19 it is sufficient to show that the number of correct edges can be increased by admissible graph moves on $\Gamma_1$ and $\Gamma_2$.

Let $t_1$ be an incorrect thorn in $\Gamma_1$ and let $t_2, t_3, \ldots, t_{2n}$ be all other incorrect thorns of the same color (say, red). We shall say that a thorn $t_i$, $2 \le i \le 2n$, is *good* (with respect to $t_1$), if $t_1$ and $t_i$ belong to the same edge or if they can be transferred to the same edge by an admissible graph move on $\Gamma_1$. Denote by $T$ the set $\{\tau_i = \varphi(t_i),\ 1 \le i \le 2n\}$ of all red incorrect thorns in $\Gamma_2$. We shall say that a thorn $\tau_i$, $2 \le i \le 2n$, is *good*, if $\tau_1$ and $\tau_i$ belong to the same edge or if they can be transferred to the same edge by an admissible graph move on $\Gamma_2$.

Consider two subsets $A_1$ and $A_2$ of the set $T$. The subset $A_1 \subset T$ consists of the images of good thorns in $\Gamma_1$, the subset $A_2 \subset T$ is the set of all good thorns in $\Gamma_2$. Let $\#X$ denote the number of elements in $X$. Since any red incorrect edge in $\Gamma_1$ and $\Gamma_2$ contains at least one good thorn, we have $\#A_1 \ge n$ and $\#A_2 \ge n$. Note that $\#T = 2n$ and, because $t_1$ and $\tau_1 = \varphi(t_1)$ are not good, $\tau_1$ does not belong to $A_1 \cup A_2$. Hence, $\#(A_1 \cup A_2) < 2n$, and $A_1 \cap A_2 \neq \emptyset$. We can conclude that there exist $i$ and $j$, $2 \le i, j \le 2n$, such that $t_i$ and $\tau_j$ are good. By definition of good edges, we can perform admissible graph moves such that after these moves $t_1$ and $t_2$ belong to the same edge and $\tau_1$ and $\tau_2$ also belong to the same edge. The moves are performed along incorrect edges. Hence, all correct edges are preserved, but now a new correct edge has appeared (just the one containing $t_1$ and $t_2$).

Our next step is to prove Proposition 8.2.22 below stating that under certain conditions any similarity homeomorphism between neighborhoods of vertices of special spines can be extended to the union of edges. We need two lemmas.

**Lemma 8.2.20.** *Let $P$ be a special spine of a closed 3-manifold $M$. Suppose that every surface $F \subset M$ with $\chi(F) \geq 0$ separates $M$. Then each proper theta or eyeglass curve $G \subset P$ separates $P$.*

*Proof.* Let $F \subset M$ be the surfaces constructed in Lemma 8.2.11. Since $F$ separates $M$ and $P$ is a spine of $M$, $\Gamma = F \cap P$ separates $P$.

Suppose $P$ is a special spine of a closed 3-manifold $M$. Let us color the 2-cells of $P$ in different colors. At each edge of $P$ three 2-cells meet, and so to each edge there corresponds some unordered triplet of colors (possibly with multiplicity). We call this triplet the *tricolor* of the edge. Thus, we may consider $SP$ as a colored graph. Note that each spine move on $P$ induces an admissible graph move on $SP$. It turns out that under certain conditions all admissible graph moves can be obtained in this way.

**Lemma 8.2.21.** *Let $P$ be a special spine of a 3-manifold $M$. If every surface $F \subset M$ with $\chi(F) \geq 0$ is separating, then each admissible graph move $\gamma$ on $SP$ is induced by a spine move on $P$.*

*Proof.* Let $\gamma$ be performed along edges $e_1$ and $e_2$. Then $e_1$ and $e_2$ have the same tricolor. Connect the middle points of $e_1$ and $e_2$ by three disjoint arcs $l_j \subset P$ $(j = 1, 2, 3)$ in such a way that $G = l_1 \cup l_2 \cup l_3$ is a proper theta-curve. If the tricolor has multiplicity, this is also possible. By Lemma 8.2.20, $G$ separates $P$ into two parts such that each part contains one vertex of $e_1$ and one vertex of $e_2$. Denote by $\sigma_1$ the spine move along $G$. Then $\sigma_1$ induces an admissible graph move along $e_1$ and $e_2$. Since such a move is unique (see Remark 8.2.18), it coincides with $\gamma$.

Recall that if a closed 3-manifold $M$ is irreducible, then any compressible torus or Klein bottle in $M$ bounds a solid torus or Klein bottle, respectively. There exist no compressible projective planes at all. It follows that if an irreducible $M$ contains no closed incompressible surfaces with nonnegative Euler characteristic, then the following holds:

(1) Every surface $F \subset M$ with $\chi(F) \geq 0$ separates $M$.
(2) Every manifold move on $M$ produces a homeomorphic manifold, see Remark 8.2.5.

**Proposition 8.2.22.** *Let $M_1$ and $M_2$ be similar closed 3-manifolds. Suppose $M_1$ is irreducible and does not contain closed incompressible surfaces with nonnegative Euler characteristic. Then there exist special spines $P_i$ of $M_i$ $(i = 1, 2)$ and a homeomorphism $\psi : N_1 \cup SP_1 \to N_2 \cup SP_2$ such that $\psi|_{N_1} : N_1 \to N_2$ is a similarity homeomorphism, where $N_i = N(V(P_i), P_i)$.*

*Proof.* Let $\varphi : N_1 \to N_2$ be a similarity homeomorphism, where $P_1$ and $P_2$ are special spines of $M_1$ and $M_2$, respectively. We imagine the 2-cells of $P_1$ and $P_2$ as being painted in different colors such that $\varphi$ preserves the colors of wings. As above, we paint also each edge in the corresponding tricolor. Then $\varphi$ induces a similarity homeomorphism between $SP_1$ and $SP_2$. If all edges of $SP_1$ are correct, then $\varphi$ can be extended to a homeomorphism $\psi$ satisfying the conclusion of the proposition. If not, we use Lemma 8.2.19 to correct them by a sequence of graph moves. By Lemma 8.2.21 this sequence can be realized by a sequence of spine moves. It remains to note that each move on a spine of $M_1$ produces a spine of the same manifold, so we do not violate the assumption on $M$.

Let $P_1$ and $P_2$ be special spines of $M_1$ and $M_2$, and let a homeomorphism $\psi : N_1 \cup SP_1 \to N_2 \cup SP_2$ induce a similarity homeomorphism $\psi'$ between $N_1 = N(V(P_1), P_1)$ and $N_2 = N(V(P_2), P_2)$. Identify $N_1 \cup SP_1$ and $N_2 \cup SP_2$ via $\psi$. We obtain two special spines $P_1$ and $P_2$ such that their singular graphs and wings coincide.

Let $e$ be an edge of $P_1$. It contains two thorns $t_1, t_2$. Let $\omega_1^{(i)}, \omega_2^{(i)}, \omega_3^{(i)}$ be the wings adjacent to $t_i$, $i = 1, 2$. A regular neighborhood $N(e \setminus \text{Int } (t_1 \cup t_2), P_1)$ of a middle part of $e$ in $P_1$ is homeomorphic to $Y \times I$, where $Y$ is a wedge of three segments. Hence, we have a natural bijection $a_{1e} : \{\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}\} \to \{\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}\}$. In the same way a direct product structure on $N(e \setminus \text{Int } (t_1 \cup t_2), P_2)$ determines a natural bijection $a_{2e} : \{\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}\} \to \{\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}\}$. Denote by $\beta_e$ the permutation $a_{2e}^{-1} a_{1e}$.

**Definition 8.2.23.** *An edge $e$ is called* even *(*odd*) if $\beta_e$ is an even (odd) permutation.*

Let $C$ be a 2-cell of $P_1$. Denote by $E_C$ the collection of edges incident to $C$. We allow multiplicity, so if the boundary curve of $C$ passes along an edge $e$ two (three) times, then $e$ is included in $E_C$ two (three) times. Note that $E_C$ coincides with the set of edges incident to the 2-cell of $P_2$ having the same color.

**Lemma 8.2.24.** *For any 2-cell $C$ of $P_1$ the collection $E_C$ contains an even number of odd edges.*

*Proof.* Regular neighborhoods $N(V(P_i), M_i)$ $(i = 1, 2)$ consist of 3-balls. Choose orientations of the 3-balls such that the similarity homeomorphism $\psi' : N(V(P_1), P_1) \to N(V(P_2), P_2)$ is extendible to an orientation preserving homeomorphism between $N(V(P_1), M_1)$ and $N(V(P_2), M_2)$. The orientations induce a cyclic order on the set $\{\omega_1^{(j)}, \omega_2^{(j)}, \omega_3^{(j)}\}$ of wings adjacent to each thorn of $P_1$ or $P_2$. We shall say that an edge $e$ is *orientation reversing* with respect to $P_i$, if the corresponding bijection $a_{ie} : \{\omega_1^{(1)}, \omega_2^{(1)}, \omega_3^{(1)}\} \to \{\omega_1^{(2)}, \omega_2^{(2)}, \omega_3^{(2)}\}$ preserves the cyclic order, $i = 1, 2$. Since the boundary curve of each 2-cell in a 3-manifold is orientation preserving, $E_C$ contains an even number of orientation reversing edges with respect to $P_1$ and an even number of orientation reversing edges with respect to $P_2$. It remains to note that $e$ is odd if and only if $e$ is orientation reversing with respect to one of spines $P_1$, $P_2$, and orientation preserving with respect to the other.

**Theorem 8.2.25.** *Let $M_1$ and $M_2$ be similar closed 3-manifolds. Suppose $M_1$ is irreducible and does not contain closed incompressible surfaces with non-negative Euler characteristics. Then $M_1$ and $M_2$ are homeomorphic.*

*Proof.* According to Proposition 8.2.22, there exist special spines $P_i$ of $M_i$ $(i = 1, 2)$ and a homeomorphism $\psi : N_1 \cup SP_1 \to N_2 \cup SP_2$ such that $\psi\big|_{N_1} : N_1 \to N_2$ is a similarity homeomorphism, where $N_i = N(V(P_i), P_i)$. As above, identify $N_1 \cup SP_1$ with $N_2 \cup SP_2$ via $\psi$. We define an edge $e$ of $P_1$ to be *strongly correct* (SC) if the corresponding permutation $\beta_e$ is trivial. In other words, $e$ is SC if and only if the identification $\psi$ can be extended to a neighborhood of $e$ in $P_1$. Note that if all edges are SC, then $\psi$ can be extended to a homeomorphism between $P_1$ and $P_2$ and to a homeomorphism between $M_1$ and $M_2$. We claim that one can perform spine moves on $P_1$ until all edges become SC. This will prove Theorem 8.2.25, because each spine move can be extended to a manifold move on $M_1$ that does not change its homeomorphism type.

As above, we paint the 2-cells of $P_1$ and $P_2$ in different colors and the edges in tricolors. Note that if the tricolor of an edge $e$ consists of three different colors, then $e$ is obviously SC. Assume that the tricolor of $e$ is bichromatic (i.e., it has the form $(x, y, y)$, $x \neq y$), and that $e$ is not SC. Then $e$ is odd. It follows from Lemma 8.2.24 that there is another non-SC edge $e'$ of tricolor $(x, z, z)$ (possibly $z = x$ or $z = y$). Assuming first that $z \neq y$, we construct a proper eyeglass curve $G$ with the vertices on $e$ and $e'$ (this is also possible when $z = x$). By Lemma 8.2.20, $G$ is two-sided, and the spine move $\sigma_2$ along $G$ can be performed. The edge $e$ will now be SC. If $z = y$, there are two possibilities for the relative displacement of $e$ and $e'$ along the boundary curve of $y$-colored 2-cell: the displacement $(e, e, e', e')$ and the displacement $(e, e', e, e')$. In the first case we can still construct an eyeglass curve with vertices on $e$ and $e'$ and perform $\sigma_2$. In the second case we construct a proper theta-curve $G$ with the vertices on $e$ and $e'$. The move $\sigma_3$ along $G$ makes $e$ strongly correct.

Assume now $e$ is a monochromatic non-SC edge of tricolor $(x, x, x)$, and assume that there is another edge $e'$ with the same tricolor. Denote by $C_x$ the $x$-colored 2-cell of $P_1$. We shall say that $e$ and $e'$ are *linked* if the boundary curve of $C_x$ cannot be decomposed into two arcs $d$ and $d'$, such that $d$ passes three times along $l$ and $d'$ passes three times along $l'$. Suppose that $l$ and $l'$ are linked. In order to make $l$ strongly correct, we use spine moves $\sigma_3$ along theta-curves with vertices on $l$ and $l'$. Each such move changes $\beta_e$ by some permutation. It is sufficient to show that each transposition $\tau$ of wings can be achieved. In essence, there are two possibilities for the relative displacement of $e$ and $e'$ on the boundary curve of $C_x$. It is clear that in both cases $\tau$ can be realized by a move $\sigma_3$ along the theta-curve $G = l_1 \cup l_2 \cup l_3$, see Fig. 8.15.

Suppose now that each two non-SC edges of tricolor $(x, x, x)$ are unlinked. If $e$ is an odd edge with tricolor $(x, x, x)$, then there is another odd edge $e'$

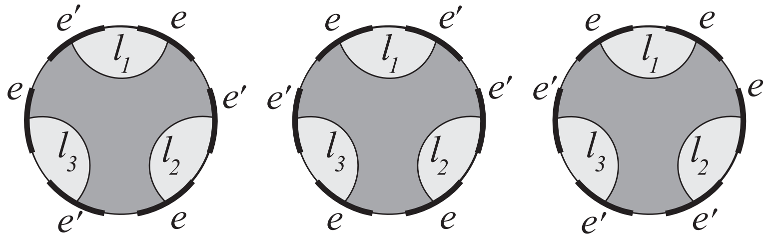



**Fig. 8.15.** Two linked and one unlinked positions of edges $e, e'$ in the boundary of a 2-cell

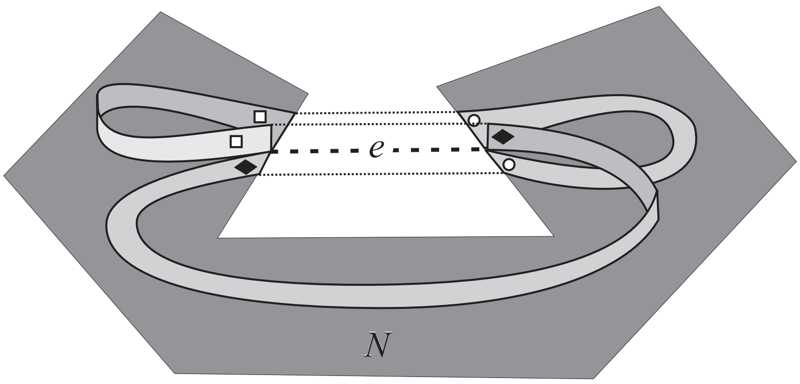



**Fig. 8.16.** Decomposition of wings into pairs

with the same tricolor. We use the manifold move along $G = l_1 \cup l_2 \cup l_3$ (see Fig. 8.15) to make $e$ and $e'$ even.

It remains to consider the following situation: all non-SC edges are monochromatic and even, and there are no linked edges among them. Let $e$ be a non-SC edge with tricolor $(x, x, x)$. Denote by $P_3$ the spine obtained from $P_1$ by the spine move across $e$, see Definition 8.2.10. Let $t_1$ and $t_2$ be the thorns in $e$ and let $w_1^{(i)}, w_2^{(i)}, w_3^{(i)}$ be the wings adjacent to $t_i$, $i = 1, 2$. The direct product structures on regular neighborhoods of $e \setminus \text{Int } (t_1 \cup t_2)$ in $P_i$ determine natural bijections $a_{ie} : \{w_1^{(1)}, w_2^{(1)}, w_3^{(1)}\} \to \{w_1^{(2)}, w_2^{(2)}, w_3^{(2)}\}$, $i = 1, 2, 3$. It is sufficient to prove that $a_{2e}$ coincides with $a_{3e}$, because this means that the spine move across $e$ makes $e$ strongly correct.

Consider a regular neighborhood $N$ of $SP_1 \setminus e$ in $P_1$. The difference $N \setminus SP_1$ consists of some number of half-open annuli and precisely three $x$-colored half-open discs. Each of the discs contains two wings from the set $W = \{w_j^{(i)},\ 1 \leq j \leq 3,\ i = 1, 2\}$. Thus, we have a decomposition of the set $W$ into three pairs. In Fig. 8.16 the wings forming each pair are marked with similar signs. Taking $P_2$ or $P_3$ instead of $P_1$, we obtain two other decompositions. A very important observation: since all non-monochromatic edges are SC and $e$ is not linked with any other edge, these three decompositions coincide.

At least one pair of the decomposition contains a wing adjacent to the pair, $1 \leq j,\ k \leq 3$. Since each of the spines $P_1, P_2, P_3$ contains only one $x$-colored 2-cell, we have $a_{ie}(w_j^{(1)}) \neq w_k^{(2)}$, $1 \leq i \leq 3$. Hence, among $a_{1e}(w_j^{(1)}), a_{2e}(w_j^{(1)})$, $a_{3e}(w_j^{(1)})$ at least two wings coincide. Taking into account that any two different bijections $a_{1e}, a_{2e}, a_{3e}$ differ on an even permutation, we can conclude that at least two of them do coincide. Since $e$ is not SC and since the spine move across $e$ changes the corresponding bijection, we have $a_{1e} \not\equiv a_{2e}$ and $a_{1e} \not\equiv a_{3e}$. It follows that $a_{2e} \equiv a_{3e}$.