Chapter 2
Linear Elastic Fracture Mechanics

2.1 Introduction

Beginning with the fabrication of stone-age axes, instinct and experience about the strength of various materials (as well as appearance, cost, availability and even divine properties) served as the basis for the design of many engineering structures. The industrial revolution of the 19th century led engineers to use iron and steel in place of traditional materials like stone and wood. Unlike stone, iron and steel had the advantage of being strong in tension, which meant that engineering structures could be made lighter and at less cost than was previously possible. In the years leading up to World War 2, engineers usually ensured that the maximum stress within a structure, as calculated using simple beam theory, was limited to a certain percentage of the "tensile strength" of the material. Tensile strength for different materials could be conveniently measured in the laboratory and the results for a variety of materials were made available in standard reference books. Unfortunately, structural design on this basis resulted in many failures because the effect of stress-raising corners and holes on the strength of a particular structure was not appreciated by engineers. These failures led to the emergence of the field of "fracture mechanics." Fracture mechanics attempts to characterize a material's resistance to fracture—its "toughness."

2.2 Stress Concentrations

Progress toward a quantitative definition of toughness began with the work of Inglis[1] in 1913. Inglis showed that the local stresses around a corner or hole in a stressed plate could be many times higher than the average applied stress. The presence of sharp corners, notches, or cracks serves to concentrate the applied stress at these points. Inglis showed, using elasticity theory, that the degree of stress magnification at the edge of the hole in a stressed plate depended on the radius of curvature of the hole.

The smaller the radius of curvature, the greater the stress concentration. Inglis found that the "stress concentration factor", κ, for an elliptical hole is equal to:

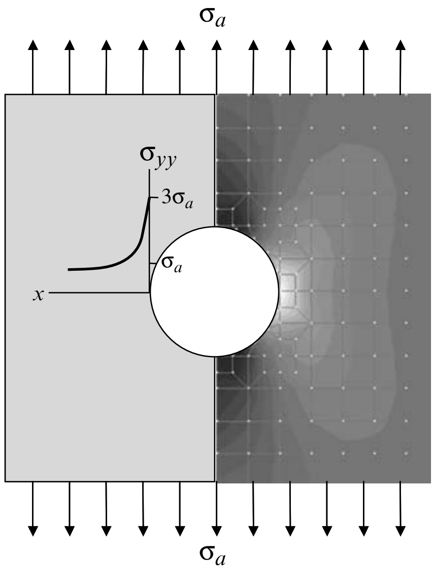

Fig. 2.2.1 Stress concentration around a hole in a uniformly stressed plate. The contours for σ_{yy} shown here were generated using the finite-element method. The stress at the edge of the hole is 3 times the applied uniform stress.

$$\kappa = 1 + 2\sqrt{\frac{c}{\rho}} \tag{2.2a}$$

where c is the hole radius and ρ is the radius of curvature of the tip of the hole.

For a very narrow elliptical hole, the stress concentration factor may be very much greater than one. For a circular hole, Eq. 2.2a gives $\kappa = 3$ (as shown in Fig. 2.2.1). It should be noted that the stress concentration factor does not depend on the absolute size or length of the hole but only on the ratio of the size to the radius of curvature.

2.3 Energy Balance Criterion

In 1920[2], A. A. Griffith of the Royal Aircraft Establishment in England became interested in the effect of scratches and surface finish on the strength of machine parts subjected to alternating loads. Although Inglis's theory showed that the stress increase at the tip of a crack or flaw depended only on the geometrical shape of the crack and not its absolute size, this seemed contrary to the well-known fact that larger cracks are propagated more easily than smaller ones. This anomaly led Griffith to a theoretical analysis of fracture based on the point of view of minimum potential energy. Griffith proposed that the reduction in strain energy due to the formation of a crack must be equal to or greater than the increase in surface energy required by the new crack faces. According to Griffith, there are two conditions necessary for crack growth:

i. The bonds at the crack tip must be stressed to the point of failure. The stress at the crack tip is a function of the stress concentration factor, which depends on the ratio of its radius of curvature to its length.
ii. For an increment of crack extension, the amount of strain energy released must be greater than or equal to that required for the surface energy of the two new crack faces.

The second condition may be expressed mathematically as:

$$\frac{dU_s}{dc} \geq \frac{dU_\gamma}{dc} \tag{2.3a}$$

where U_s is the strain energy, U_γ is the surface energy, and dc is the crack length increment. Equation 2.3a says that for a crack to extend, the rate of strain energy release per unit of crack extension must be at least equal to the rate of surface energy requirement. Griffith used Inglis's stress field calculations for a very narrow elliptical crack to show that the strain energy released by introducing a double-ended crack of length 2c in an infinite plate of unit width under a uniformly applied stress σ_a is [2]:

$$U_s = \frac{\pi\sigma_a^2 c^2}{E} \text{ Joules (per meter width)} \tag{2.3b}$$

We can obtain a semiquantitative appreciation of Eq. 2.3b by considering the strain energy released over an area of a circle of diameter 2c, as shown in Fig. 2.3.1. The strain energy is $U = (½\sigma^2/E)(\pi c^2)$. The actual strain energy computed by rigorous means is exactly twice this value as indicated by Eq. 2.3b.

As mentioned in Chapter 1, for cases of plane strain, where the thickness of the specimen is significant, E should be replaced by $E/(1-\nu^2)$. In this chapter, we omit the $(1-\nu^2)$ factor for brevity, although it should be noted that in most practical applications it should be included.

The total surface energy for *two* surfaces of unit width and length 2c is:

$$U_\gamma = 4\gamma c \text{ Joules (per meter width)} \tag{2.3c}$$

The factor 4 in Eq. 2.3c arises because of there being two crack surfaces of length 2c. γ is the *fracture* surface energy of the solid. This is usually larger than the surface free energy since the process of fracture involves atoms located a small distance into the solid away from the surface. The fracture surface energy may additionally involve energy dissipative mechanisms such as microcracking, phase transformations, and plastic deformation.

Thus, taking the derivative with respect to c in Eq. 2.3b and 2.3c, this gives us the strain energy release rate (J/m per unit width) and the surface energy creation rate (J/m per unit width). The critical condition for crack growth is:

$$\frac{\pi\sigma_a^2 c}{E} \geq 2\gamma \tag{2.3d}$$

The left-hand side of Eq. 2.3d is the rate of strain energy release per crack tip and applies to a double-ended crack in an infinite solid loaded with a uniformly applied tensile stress. Equation 2.3d shows that strain energy release rate per increment of crack length is a linear function of crack length and that the required rate of surface energy per increment of crack length is a constant. Equation 2.3d is the Griffith energy balance criterion for crack growth, and the relationships between surface energy, strain energy, and crack length are shown in Fig. 2.3.2.

A crack will not extend until the strain energy release rate becomes equal to the surface energy requirement. Beyond this point, more energy becomes available by the released strain energy than is required by the newly created crack surfaces which leads to unstable crack growth and fracture of the specimen.

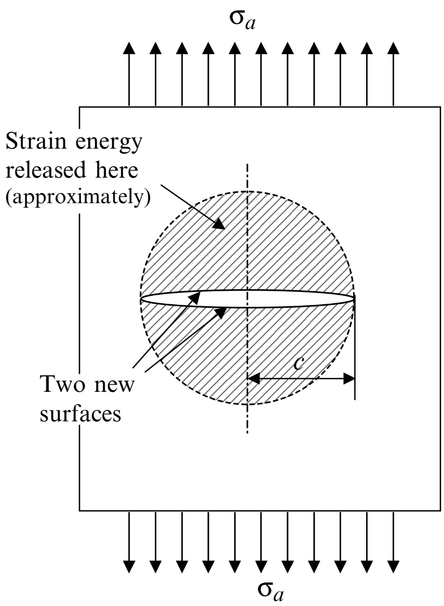

Fig. 2.3.1 The geometry of a straight, double-ended crack of unit width and total length $2c$ under a uniformly applied stress σ_a. Stress concentration exists at the crack tip. Strain energy is released over an approximately circular area of radius c. Growth of crack creates new surfaces.

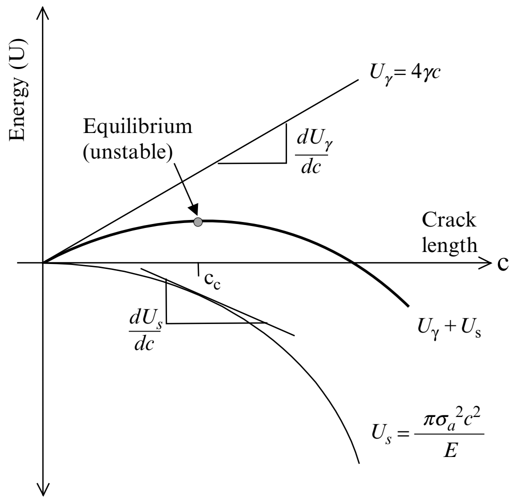

Fig. 2.3.2 Energy versus crack length showing strain energy released and surface energy required as crack length increases for a uniformly applied stress as shown in Fig. 2.3.1. Cracks with length below c_c will not extend spontaneously. Maximum in the total crack energy denotes an unstable equilibrium condition.

The equilibrium condition shown in Fig. 2.3.2 is unstable, and fracture of the specimen will occur at the equilibrium condition. The presence of instability is given by the second derivative of Eq. 2.3b. For $\mathrm{d}^2U_s/\mathrm{d}c^2 < 0$, the equilibrium condition is unstable. For $\mathrm{d}^2U_s/\mathrm{d}c^2 > 0$, the equilibrium condition is stable. Figure 2.3.3 shows a configuration for which the equilibrium condition is stable. In this case, crack growth occurs at the equilibrium condition, but the crack only extends into the material at the same rate as the wedge.

The energy balance criterion indicates whether crack growth is possible, but whether it will actually occur depends on the state of stress at the crack tip. A crack will not extend until the bonds at the crack tip are loaded to their tensile strength, even if there is sufficient strain energy stored to permit crack growth. For example, if the crack tip is blunted or rounded, then the crack may not extend because of an insufficient stress concentration. The energy balance criterion is a necessary, but not a sufficient condition for fracture. Fracture only occurs when the stress at the crack tip is sufficient to break the bonds there. It is customary to assume the presence of an infinitely sharp crack tip to approximate the worst-case condition. This does not mean, however, that all solids fail upon the immediate application of a load. In practice, stress singularities that arise due to an “infinitely sharp” crack tip are avoided by plastic deformation of the material. However, if such an infinitely sharp crack tip could be obtained, then the crack would not extend unless there was sufficient energy for it to do so.

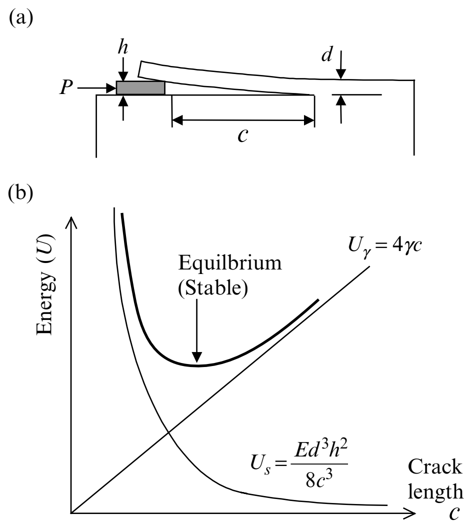

Fig. 2.3.3 (a) Example of stable equilibrium (Obreimoff's experiment). (b) Energy versus crack length showing stable equilibrium as indicated by the minimum in the total crack energy.

For a given stress, there is a minimum crack length that is not self-propagating and is therefore "safe." A crack will not extend if its length is less than the critical crack length, which, for a given uniform stress, is:

$$c_c = \frac{2\gamma E}{\pi\sigma_a^2} \tag{2.3e}$$

In the analyses above, Eq. 2.3b implicitly assumes that the material is linearly elastic and γ in Eq. 2.3d is the fracture surface energy, which is usually greater than the intrinsic surface energy due to energy dissipative mechanisms in the vicinity of the crack tip.

The discussion above refers to a decrease in strain potential energy with increasing crack length. This type of loading would occur in a "fixed-grips" apparatus, where the load is applied, and the apparatus clamped into position. It can be shown that exactly the same arguments apply for a "dead-weight" loading, where the fracture surface energy corresponds to a decrease in potential energy of the loading system. The term "mechanical energy release rate," may be more appropriate than "strain energy release rate" but the latter term is more commonly used.

2.4 Linear Elastic Fracture Mechanics

2.4.1 Stress intensity factor

During the Second World War, George R. Irwin[3] became interested in the fracture of steel armor plating during penetration by ammunition. His experimental work at the U.S. Naval Research Laboratory in Washington, D.C. led, in 1957[4], to a theoretical formulation of fracture that continues to find wide application. Irwin showed that the stress field $\sigma(r,\theta)$ in the vicinity of an infinitely sharp crack tip could be described mathematically by:

$$\sigma_{yy} = \frac{K_1}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \quad (2.4.1a)$$

The first term on the right hand side of Eq. 2.4.1a describes the magnitude of the stress whereas the terms involving θ describe its distribution. K_1 is defined as*:

$$K_1 = \sigma_a Y\sqrt{\pi c} \quad (2.4.1b)$$

The coordinate system for Eqs. 2.4.1a and 2.4.1b is shown in Fig. 2.4.1. In Eqn. 2.4.1b, σ_a is the externally applied stress, Y is a geometry factor, and c is the crack half-length. K_1 is called the "stress intensity factor." There is an important reason for the stress intensity factor to be defined in this way. For a particular crack system, π and Y are constants so the stress intensity factor tells us that the magnitude of the stress at position (r,θ) depends only on the external stress applied and the square root of the crack length. For example, doubling the externally applied stress σ_a will double the magnitude of the stress in the vicinity of the crack tip at coordinates (r,θ) for a given crack size. Increasing the crack length by four times will double the stress at (r,θ) for the same value of applied stress. The stress intensity factor K_1, which includes both applied stress and crack length, is a combined "scale factor," which characterizes the *magnitude* of the stress at some coordinates (r,θ) near the crack tip. The shape of the stress *distribution* around the crack tip is exactly the same for cracks of all lengths.

Equation 2.4.1a shows that, for all sizes of cracks, the stresses at the crack tip are infinite. Despite this, the Griffith energy balance criterion must be satisfied for such a crack to extend in the presence of an applied stress σ_a. The stress intensity factor K_1 thus provides a numerical "value," which quantifies the magnitude *of the effect* of the stress singularity at the crack tip. We shall see later that there is a critical value for K_1 for different materials which corresponds to the energy balance criterion being met. In this way, this critical value of K_1 characterizes the fracture strength of different materials.

* Some authors prefer to define K_1 without $\pi^{1/2}$ in Eq. 2.4.1b. In this case, $\pi^{-1/2}$ does not appear in Eq. 2.4.1a.

In Eq. 2.4.1b, Y is a function whose value depends on the geometry of the specimen, and σ_a is the applied stress. For a straight double-ended crack in an infinite solid, $Y = 1$. For a *small* single-ended surface crack (i.e., a semi-infinite solid), $Y = 1.12$[5,6]. This 12% correction arises due to the additional release in strain potential energy (compared with a completely embedded crack) caused by the presence of the free surface† as indicated by the shaded portion in Fig. 2.4.1. This correction has a diminished effect as the crack extends deeper into the material. For embedded penny-shaped cracks, $Y = 2/\pi$. For half-penny-shaped surface flaws in a semi-infinite solid, the appropriate value is $Y = 0.713$. Values of Y for common crack geometries and loading conditions can be found in standard engineering texts.

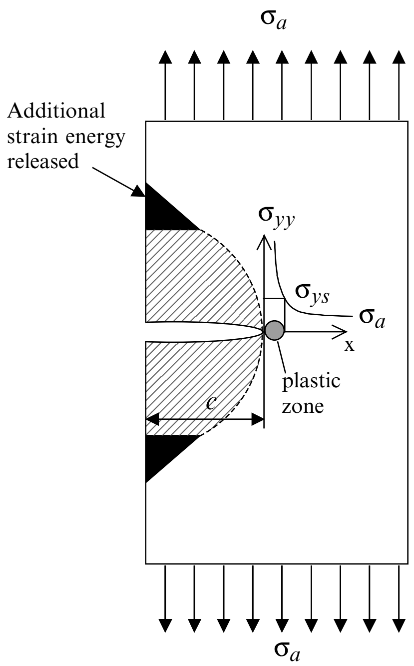

Fig. 2.4.1 Semi-infinite plate under a uniformly applied stress with single-ended surface crack of half-length c. Dark shaded area indicates additional release in strain energy due to the presence of the surface compared to a fully embedded crack in an infinite solid.

† A further correction can be made for the effect of a free surface in front of the crack (i.e., the surface to which the crack is approaching). This correction factor is very close to 1 for cracks with a length less than one-tenth the width of the specimen.

Equation 2.4.1a arises from Westergaard's solution[7] for the Airy stress function, which fulfills the equilibrium equations of stresses subject to the boundary conditions associated with a sharp crack, $\rho = 0$, in an infinite, biaxially loaded plate. Equation 2.4.1a applies only to the material in the vicinity of the crack tip. A cursory examination of Eq. 2.4.1a shows that σ_{yy} approaches zero for large values of r rather than the applied stress σ_a. To obtain values for stresses further from the crack tip, additional terms in the series solution must be included. However, near the crack tip, the localized stresses are usually very much greater than the applied uniform stress that may exist elsewhere, and the error is thus negligible.

The subscript 1 in K_1 is associated with tensile loading, as shown in Fig. 2.4.2. Stress intensity factors exist for other types of loading, as also shown in this figure, but our interest centers mainly on type 1 loading—the most common type that leads to brittle failure.

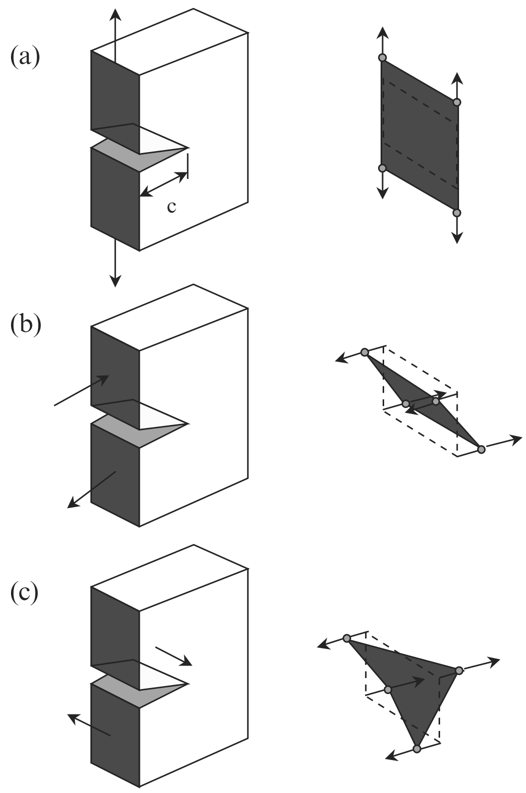

Fig. 2.4.2 Three modes of fracture. (a) Mode I, (b) Mode II, and (c) Mode III. Type I is the most common. The figures on the right indicate displacements of atoms on a plane normal to the crack near the crack tip.

An important property of the stress intensity factors is that they are additive for the same type of loading. This means that the stress intensity factor for a complicated system of loads may be derived from the addition of the stress intensity factors determined for each load considered individually. It shall be later shown how the additive property of K_1 permits the stress field in the vicinity of a crack can be calculated on the basis of the stress field that existed in the solid prior to the introduction of the crack.

The power of Eq. 2.4.1b cannot be overestimated. It provides information about events at the crack tip in terms of easily measured macroscopic variables. It implies that the magnitude and distribution of stress in the vicinity of the crack tip can be considered separately and that a criterion for failure need only be concerned with the "magnitude" or "intensity" of stress at the crack tip. Although the stress at an infinitely sharp crack tip may be "infinite" due to the singularity that occurs there, the stress intensity factor is a measure of the "strength" of the singularity.

2.4.2 Crack tip plastic zone

Equation 2.4.1a implies that at $r = 0$ (i.e., at the crack tip) σ_{yy} approaches infinity. However, in practice, the stress at the crack tip is limited to at least the yield strength of the material, and hence linear elasticity cannot be assumed within a certain distance of the crack tip (see Fig. 2.4.1). This nonlinear region is sometimes called the "crack tip plastic zone[8]." Outside the plastic zone, displacements under the externally applied stress mostly follow Hooke's law, and the equations of linear elasticity apply. The elastic material outside the plastic zone transmits stress to the material inside the zone, where nonlinear events occur that may preclude the stress field from being determined exactly. Equation 2.4.1a shows that the stress is proportional to $1/r^{1/2}$. The strain energy release rate is not influenced much by events within the plastic zone *if the plastic zone is relatively small*. It can be shown that an approximate size of the plastic zone is given by:

$$r_p = \frac{K_1^2}{2\pi\sigma_{ys}^2} \tag{2.4.2a}$$

where σ_{ys} is the yield strength (or yield stress) of the material.

The concept of a plastic zone in the vicinity of the crack tip is one favored by many engineers and materials scientists and has useful implications for fracture in metals. However, the existence of a crack tip plastic zone in brittle solids appears to be objectionable on physical grounds. The stress singularity predicted by Eq. 2.4.1a may be avoided in brittle solids by nonlinear, but elastic, deformations. In Chapter 1, we saw how linear elasticity applies between two atoms for *small displacements* around the equilibrium position. At the crack tip, the displacements are not small on an atomic scale, and nonlinear behavior is to be expected. In brittle solids, strain energy is absorbed by the nonlinear stretching of atomic bonds, not plastic events, such as dislocation movements, that may be expected in a ductile metal. Hence, brittle materials do not fall to pieces under the application of even the smallest of loads even though an infinitely large stress appears to exist at the tip of any surface flaws or cracks within it. The energy balance criterion must be satisfied for such flaws to extend.

2.4.3 Crack resistance

The assumption that all the strain energy is available for surface energy of new crack faces does not apply to ductile solids where other energy dissipative mechanisms exist. For example, in crystalline solids, considerable energy is consumed in the movement of dislocations in the crystal lattice and this may happen at applied stresses well below the ultimate strength of the material. Dislocation *movement* in a ductile material is an indication of yield or plastic deformation, or plastic flow.

Irwin and Orowan[9] modified Griffith's equation to take into account the nonreversible energy mechanisms associated with the plastic zone by simply including this term in the original Griffith equation:

$$\frac{dU_s}{dc} = \frac{dU_\gamma}{dc} + \frac{dU_p}{dc} \qquad (2.4.3a)$$

The right-hand side of Eq. 2.4.3a is given the symbol R and is called the crack resistance. At the point where the Griffith criterion is met, the crack resistance indicates the minimum amount of energy required for crack extension in J/m^2 (i.e., J/m per unit crack width). This energy is called the "work of fracture" (units J/m^2) which is a measure of toughness.

Ductile materials are tougher than brittle materials because they can absorb energy in the plastic zone, as what we might call "plastic strain energy," which is no longer available for surface (i.e., crack) creation. By contrast, brittle materials can only dissipate stored elastic strain energy by surface area creation. The work of fracture is difficult to measure experimentally.

2.4.4 K_{1C}, the critical value of K_1

The stress intensity factor K_1 is a "scale factor" which characterizes the magnitude of the stress at some coordinates (r,θ) near the crack tip. If each of two cracks in two different specimens are loaded so that K_1 is the same in each specimen, then the magnitude of the stresses in the vicinity of each crack is precisely the same. Now, if the applied stresses are increased, keeping the same value of K_1 in each specimen, then eventually the energy balance criterion will be satisfied and the crack in each will extend. The stresses at the crack tip are exactly the same at this point although unknown (theoretically infinite for a perfectly elastic material but limited in practice by inelastic deformations). The value of K_1 at the point of crack extension is called the critical value: K_{1C}.

K_{1C} then defines the onset of crack extension. It does not necessarily indicate fracture of the specimen—this depends on the crack stability. It is usually regarded as a material property and can be used to characterize toughness. In contrast to the work of fracture, its determination does not depend on exact knowledge of events within the plastic zone. Consistent and reproducible values of K_{1C} can only be obtained when specimens are tested in plane strain. In plane stress, the critical value of K_1 for fracture depends on the thickness of the plate. Hence, K_{1C} is often called the "plane strain fracture toughness" and has units MPa $m^{1/2}$. Low values of K_{1C} mean that, for a given stress, a material can only withstand a small length of crack before a crack extends.

The condition $K_1 = K_{1C}$ does not necessarily correspond to fracture, or failure, of the specimen. K_{1C} describes the onset of crack extension. Whether this is a stable or unstable condition depends upon the crack system. Catastrophic fracture occurs when the equilibrium condition is unstable. For cracks in brittle materials initiated by contact stresses, the crack may be initially unstable and then become stable due to the sharply diminishing stress field. For example, in Chapter 7, we find that the variation in strain energy release rate (directly related to K_1), the quantity dG/dc, is initially positive and then becomes negative as the crack becomes longer. In terms of stress intensity factor, the crack is stable when $dK_1/dc < 0$ and unstable when $dK_1/dc > 0$. The condition $K_1 = K_{1C}$ for the stable configuration means that the crack is on the point of extension but will not extend unless the applied stress is increased. If this happens, a new stable equilibrium crack length will result. Under these conditions, each increment of crack extension is sufficient to account for the attendant release in strain potential energy. For the unstable configuration, the crack will immediately extend rapidly throughout the specimen and lead to failure. Under these conditions, for each increment of crack extension there is insufficient surface energy to account for the release in strain potential energy.

2.4.5 Equivalence of *G* and *K*

Let G be defined as being equal to the strain energy release rate *per crack tip* and given by the left-hand side of Eq. 2.3d, that is, for a double-ended crack within an infinite solid, the rate of release in strain energy per crack tip is:

$$G = \frac{\pi\sigma^2 c}{E} \tag{2.4.5a}$$

Thus, substituting Eq. 2.4.1b into Eq. 2.4.5a, we have:

$$G = \frac{K_1^2}{E} \tag{2.4.5b}$$

When $K_1 = K_{1C}$, then G_c becomes the critical value of the rate of release in strain energy for the material which leads to crack extension and possibly fracture of the specimen. The relationship between K_1 and G is significant because it means that the K_{1C} condition is a necessary and sufficient criterion for crack growth since it embodies both the stress and energy balance criteria. The value of K_{1C} describes the stresses (indirectly) at the crack tip *as well as* the strain energy release rate at the onset of crack extension.

It should be remembered that various corrections to K, and hence G, are required for cracks in bodies of finite dimensions. Whatever the correction, the correspondence between G and K is given in Eq. 2.4.5b.

A factor of π sometimes appears in Eq. 2.4.5b depending on the particular definition of K_1 used. Consistent use of π in all these formulae is essential, especially when comparing equations from different sources. Again, we should recognize that Eq. 2.4.5b applies to plane stress conditions. In practice, a condition of plane strain is more usual, in which case one must include the factor $(1-\nu^2)$ in the numerator.

2.5 Determining Stress Intensity Factors

2.5.1 Measuring stress intensity factors experimentally

Direct application of Griffith's energy balance criterion is seldom practical because of difficulties in determining work of fracture γ. Furthermore, the Griffith criterion is a necessary but not sufficient condition for crack growth. However, stress intensity factors are more easily determined and represent a necessary *and* sufficient condition for crack growth, but in determining the stress intensity factor, Eq. 2.4.1b cannot be used directly because the shape factor Y is not generally known.

As mentioned previously, $Y = 2/\pi$ applies for an embedded penny shaped circular crack of radius c in an infinite plate. Expressions such as this for other types of cracks and loading geometries are available in standard texts. To find the critical value of K_1, it is necessary simply to apply an increasing load P to a prepared specimen, which has a crack of known length c already introduced, and record the load at which the specimen fractures.

Figure 2.5.1 shows a beam specimen loaded so that the side in which a crack has been introduced is placed in tension. Equation 2.5.1 allows the fracture toughness to be calculated from the crack length c and load P at which fracture of the specimen occurs. Note that in practice the length of the beam specimen is made approximately 4 times its height to avoid edge effects.

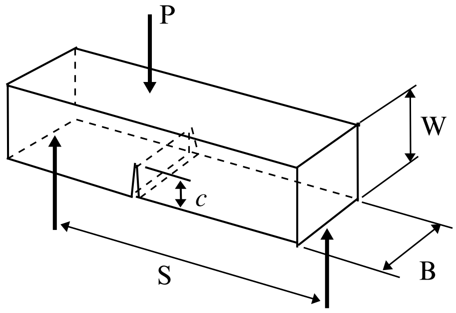

Fig. 2.5.1 Single edge notched beam (SENB)

$$K_1 = \frac{PS}{BW^{3/2}}\left[\begin{array}{l} 2.9\left(\frac{c}{W}\right)^{1/2} - 4.6\left(\frac{c}{W}\right)^{3/2} + ... \\ ...21.8\left(\frac{c}{W}\right)^{5/2} - 37.6\left(\frac{c}{W}\right)^{7/2} + 38.7\left(\frac{c}{W}\right)^{9/2} \end{array}\right] \quad (2.5.1)$$

Consistent and reproducible results for fracture toughness can only be obtained under conditions of plane strain. In plane stress, the values of K_1 at fracture depend on the thickness of the specimen. For this reason, values of K_{1C} are measured in plane strain, hence the term "plane strain fracture toughness."

2.5.2 Calculating stress intensity factors from prior stresses

Under some circumstances, it is possible[10] to calculate the stress intensity factor for a given crack path using the stress field in the solid *before the crack actually exists*. The procedure makes use of the property of superposition of stress intensity factors.

Consider an internal crack of length $2c$ within an infinite solid, loaded by a uniform externally applied stress σ_a, as shown in Fig. 2.5.2a. The presence of the crack intensifies the stress in the vicinity of the crack tip, and the stress intensity factor K_1 is readily determined from Eq. 2.4.1b. Now, imagine a series of surface tractions in the direction opposite the stress and applied to the crack faces so as to close the crack completely, as shown in Fig. 2.5.2b. At this point, the stress distribution within the solid, uniform or otherwise, is precisely equal to what would have existed in the absence of the crack because the crack is now completely closed. The stress intensity factor thus drops to zero, since there is no longer a concentration of stress at the crack tip. Thus, in one case, the presence of the crack causes the applied stress to be intensified in the vicinity of the crack, and in the other, application of the surface tractions causes this intensification to be reduced to zero.

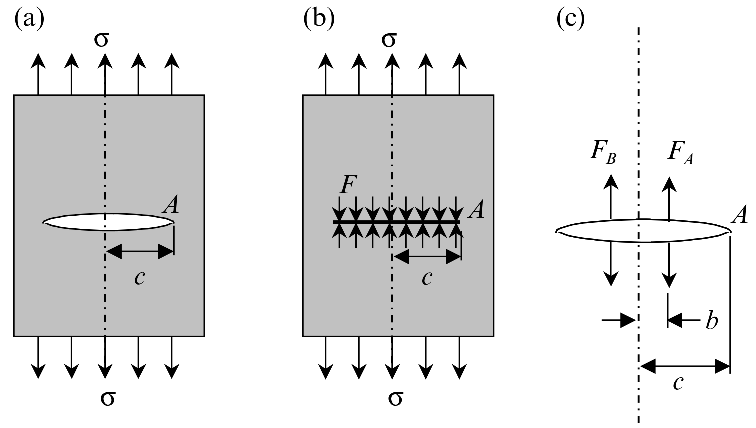

Fig 2.5.2 (a) Internal crack in a solid loaded with an external stress σ. (b) Crack closed by the application of a distribution of surface tractions *F*. (c) Internal crack loaded with surface tractions F_A and F_B.

Consider now the situation illustrated in Fig. 2.5.2c. Wells[11] determined the stress intensity factor K_1 at one of the crack tips A for a symmetric internal crack of total length 2*c* being loaded by forces F_A applied on the crack faces at a distance *b* from the center. The value for K_1 for this condition is:

$$K_{1A} = \frac{F_A}{(\pi c)^{1/2}} \left(\frac{c+b}{c-b} \right)^{1/2} \tag{2.5.2a}$$

Forces F_B also contribute to the stress field at *A*, and the stress intensity factor due to those forces is:

$$K_{1B} = \frac{F_B}{(\pi c)^{1/2}} \left(\frac{c-b}{c+b} \right)^{1/2} \tag{2.5.2b}$$

Due to the additive nature of stress intensity factors, the total stress intensity factor at crack tip *A* shown in Fig. 2.5.2c due to forces F_A and F_B, where $F_A = F_B = F$, is‡:

$$K_1 = K_{1A} + K_{1B} = \frac{2F}{\pi^{1/2}} \left(\frac{c}{c^2 - b^2} \right)^{1/2} \tag{2.5.2c}$$

‡ It is important to note that the Green's weighting functions here apply to a double-ended crack in an infinite solid. For example, Eq. 2.5.2a applies to a force F_A applied to a double-ended symmetric crack and not F_A applied to a single crack tip alone.

Now, if the tractions F are continuous along the length of the crack, then the force per unit length may be associated with a stress applied $\sigma(b)$ normal to the crack. The total stress intensity factor is given by integrating Eq. 2.5.2c with F replaced by $dF = \sigma(b)db$.

$$K_1 = \frac{2}{\pi^{1/2}}\int_0^c c^{1/2}\frac{\sigma(b)}{\sqrt{c^2-b^2}}db \tag{2.5.2d}$$

However, if the forces F are reversed in sign such that they *close the crack completely*, then the associated stress distribution $\sigma(b)$ must be that which existed *prior* to the introduction of the crack. The stress intensity factor, as calculated by Eq. 2.5.2d, for continuous surface tractions applied *so as to close the crack*, is precisely the same as that (except for a reversal in sign) calculated for the crack using the macroscopic stress σ_a in the absence of such tractions. For example, for the uniform stress case, where $\sigma(b) = \sigma_a$, Eq. 2.5.2d reduces to Eq. 2.4.1b§.

As long as the prior stress field within the solid is known, the stress intensity factor for any proposed crack path can be determined using Eq. 2.5.2d. The strain energy release rate G can be calculated from Eq. 2.4.5b. Of course, one cannot always immediately determine whether a crack will follow any particular path within the solid. It may be necessary to calculate strain energy release rates for a number of proposed paths to determine the maximum value for G. The crack extension that results in the maximum value for G is that which an actual crack will follow.

In brittle materials, cracks usually initiate from surface flaws. The strain energy release rate as calculated from the prior stress field (i.e., prior to there being any flaws) applies to the complete growth of the subsequent crack. The conditions determining subsequent crack growth depend on the *prior* stress field. The strain energy release rate, G, can be used to describe the crack growth for all flaws that exist in the prior stress field *but can only be considered applicable for the subsequent growth of the flaw that actually first extends*. Assuming there is a large number of cracks or surface flaws to consider, the one that first extends is that giving the highest value for G (as calculated using the prior stress field) for an increment of crack growth. Subsequent growth of that flaw depends upon the Griffith energy balance criterion (i.e., $G \geq 2\gamma$) being met as calculated along the crack path still using the prior stress field, even though the actual stress field is now different due to the presence of the extending crack.

§ To show this, one must make use of the standard integral:

$$\int \frac{1}{\left(a^2-x^2\right)^{1/2}}dx = \sin^{-1}\frac{x}{a} + C$$

2.5.3 Determining stress intensity factors using the finite-element method

Stress intensity factors may also be calculated using the finite-element method. The finite-element method is useful for determining the state of stress within a solid where the geometry and loading is such that a simple analytical solution for the stress field is not available. The finite-element solution consists of values for local stresses and displacements at predetermined node coordinates. A value for the local stress σ_{yy} at a judicious choice of coordinates (r,θ) can be used to determine the stress intensity factor K_1. For example, at θ = 0, Eq. 2.4.1a becomes:

$$K_1 = \sigma_{yy}(2\pi r)^{1/2} \tag{2.5.3a}$$

where σ_{yy} is the magnitude of the local stress at r. It should be noted that the stress at the node that corresponds to the location of the crack tip (r = 0) cannot be used because of the stress singularity there. Stress intensity factors determined for points away from the crack tip, outside the plastic zone, or more correctly the "nonlinear" zone, may only be used. However, one cannot use values that are too far away from the crack tip since Eq. 2.4.1a applies only for small values of r. At large r, σ_{yy} as given by Eq. 2.4.1a approaches zero, and not as is actually the case, σ_a.

Values of K_1 determined from finite-element results and using Eq. 2.5.3a should be the same no matter which node is used for the calculation, subject to the conditions regarding the choice of r mentioned previously. However, it is not always easy to choose which value of r and the associated value of σ_{yy} to use. In a finite-element model, the specimen geometry, density of nodes in the vicinity of the crack tip, and the types of elements used are just some of the things that affect the accuracy of the resultant stress field. One method of estimation is to determine values for K_1 at different values of r along a line ahead of the crack tip at θ = 0. These values for K_1 are then fitted to a smooth curve and extrapolated to r = 0, as shown in Fig. 2.5.3.

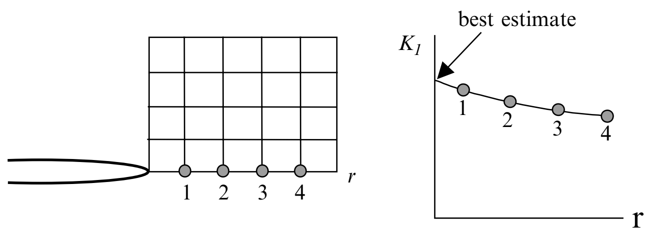

Fig. 2.5.3 Estimating K_1 from finite-element results. For elements near the crack tip, Eq. 2.4.1a is valid and K_1 can be determined from the stresses at any of the nodes near the crack tip. In practice, one needs to determine a range of K_1 for a fixed θ (e.g., θ = 0) for a range of r and extrapolate back to r = 0.

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