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An Introduction to Noncommutative Noetherian Rings

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Deceased

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Contents

1. A Few Noetherian Rings

After a review of the definition and basic properties of noetherian modules and rings, we introduce a few classes of examples of noetherian rings, which will serve to illustrate and support the later theory. We concentrate particularly on some of the "surrogate" examples outlined in the Prologue, namely, module-finite algebras over commutative rings, skew-Laurent rings, and the corresponding skew polynomial rings twisted by automorphisms. The general theory of skew polynomial rings will be addressed in the following chapter, where we study the Weyl algebras, formal differential operator rings, and other examples from the Prologue.

• THE NOETHERIAN CONDITION •

We begin with several basic equivalent conditions which are abbreviated by the adjective "noetherian," honoring E. Noether, who first demonstrated the importance and usefulness of these conditions. Recall that a collection $\mathcal{A}$ of subsets of a set A satisfies the *ascending chain condition* (or *ACC*) if there does not exist a properly ascending infinite chain $A_1 \subset A_2 \subset \cdots$ of subsets from $\mathcal{A}$. Recall also that a subset $B \in \mathcal{A}$ is a *maximal element* of $\mathcal{A}$ if there does not exist a subset in $\mathcal{A}$ that properly contains B. To emplasize the order-theoretic nature of these considerations, we often use the notation of inequalities ($\leq$, $<$, $\nleq$, etc.) for inclusions among submodules and/or ideals. In particular, if A is a module, the notation $B \leq A$ means that B is a submodule of A, and the notation $B < A$ (or $A > B$) means that B is a proper submodule of A.

Proposition 1.1. *For a module A, the following conditions are equivalent:*

(a) *A has the ACC on submodules.*

(b) *Every nonempty family of submodules of A has a maximal element.*

(c) *Every submodule of A is finitely generated.*

Proof. (a) $\Longrightarrow$ (b): Suppose that $\mathcal{A}$ is a nonempty family of submodules of A without a maximal element. Choose $A_1 \in \mathcal{A}$. Since A_1 is not maximal, there exists $A_2 \in \mathcal{A}$ such that $A_2 > A_1$. Continuing in this manner, we obtain a properly ascending infinite chain $A_1 < A_2 < A_3 < \cdots$ of submodules of A, contradicting the ACC.

(b) $\Longrightarrow$ (c): Let B be a submodule of A, and let $\mathcal{B}$ be the family of all finitely generated submodules of B. Note that $\mathcal{B}$ contains 0 and so is nonempty. By (b), there exists a maximal element $C \in \mathcal{B}$. If $C \neq B$, choose an element $x \in B \setminus C$, and let C' be the submodule of B generated by C and x. Then $C' \in \mathcal{B}$ and $C' > C$, contradicting the maximality of C. Thus $C = B$, whence B is finitely generated.

(c) $\Longrightarrow$ (a): Let $B_1 \leq B_2 \leq \cdots$ be an ascending chain of submodules of A. Let B be the union of the B_n. By (c), there exists a finite set X of generators for B. Since X is finite, it is contained in some B_n, whence $B_n = B$. Thus $B_m = B_n$ for all $m \geq n$, establishing the ACC for submodules of A. $\square$

Definition. A module A is *noetherian* if and only if the equivalent conditions of Proposition 1.1 are satisfied. As follows from the proof of (b) $\Longrightarrow$ (c), a further equivalent condition is that A have the ACC on *finitely generated* submodules.

For example, any finite dimensional vector space V over a field k is a noetherian k-module, since a properly ascending chain of submodules (subspaces) of V cannot contain more than $\dim_k(V) + 1$ terms.

Definition. A ring R is *right (left) noetherian* if and only if the right module R_R (left module ${}_RR$) is noetherian. If both conditions hold, R is called a *noetherian ring*.

Rephrasing Proposition 1.1 for the ring itself, we see that a ring R is right (left) noetherian if and only if R has the ACC on right (left) ideals, if and only if all right (left) ideals of R are finitely generated. For example, $\mathbb{Z}$ is a noetherian ring because all its ideals are principal (singly generated). The same is true of a polynomial ring $k[x]$ in one indeterminate over a field k.

Exercise 1A. (a) Show that the 2×2 matrices over $\mathbb{Q}$ of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a \in \mathbb{Z}$ and $b, c \in \mathbb{Q}$ make a ring which is right noetherian but not left noetherian.

(b) Show that any finite direct product of right (left) noetherian rings is right (left) noetherian. $\square$

Proposition 1.2. *Let B be a submodule of a module A. Then A is noetherian if and only if B and A/B are both noetherian.*

Proof. First assume that A is noetherian. Since any ascending chain of submodules of B is also an ascending chain of submodules of A, it is immediate that B is noetherian. If $C_1 \leq C_2 \leq \cdots$ is an ascending chain of submodules of A/B, each C_i is of the form A_i/B for some submodule A_i of A that contains B, and $A_1 \leq A_2 \leq \cdots$. Since A is noetherian, there is some n such that $A_i = A_n$ for all $i \geq n$, and then $C_i = C_n$ for all $i \geq n$. Thus A/B is noetherian.

Conversely, assume that B and A/B are noetherian, and let $A_1 \leq A_2 \leq \cdots$ be an ascending chain of submodules of A. There are ascending chains of submodules

$$A_1 \cap B \leq A_2 \cap B \leq \cdots$$
$$(A_1 + B)/B \leq (A_2 + B)/B) \leq \cdots$$

in B and in A/B. Hence, there is some n such that $A_i \cap B = A_n \cap B$ and $(A_i + B)/B = (A_n + B)/B$ for all $i \geq n$, and the latter equation yields $A_i + B = A_n + B$. For all $i \geq n$, we conclude that

$$A_i = A_i \cap (A_i + B) = A_i \cap (A_n + B) = A_n + (A_i \cap B) = A_n + (A_n \cap B) = A_n$$

(using the modular law for the third equality). Therefore A is noetherian. □

In particular, Proposition 1.2 shows that any factor ring of a right noetherian ring is right noetherian. (Note that if I is an ideal of a ring R, then the right ideals of R/I are the same as the right R-submodules.)

Corollary 1.3. *Any finite direct sum of noetherian modules is noetherian.*

Proof. It suffices to prove that the direct sum of any two noetherian modules A_1 and A_2 is noetherian. The module $A = A_1 \oplus A_2$ has a submodule $B = A_1 \oplus 0$ such that $B \cong A_1$ and $A/B \cong A_2$. Then B and A/B are noetherian, whence A is noetherian by Proposition 1.2. □

Corollary 1.4. *If R is a right noetherian ring, all finitely generated right R-modules are noetherian.*

Proof. If A is a finitely generated right R-module, then $A \cong F/K$ for some finitely generated free right R-module F and some submodule $K \leq F$. Since F is isomorphic to a finite direct sum of copies of the noetherian module R_R, it is noetherian by Corollary 1.3. Then, by Proposition 1.2, A must be noetherian. □

Corollary 1.5. *Let S be a subring of a ring R. If S is right noetherian and R is finitely generated as a right S-module, then R is right noetherian.*

Proof. By Corollary 1.4, R is noetherian as a right S-module. Since all right ideals of R are also right S-submodules, the ACC on right ideals follows. □

Using Corollary 1.5, we obtain some easy examples of noncommutative noetherian rings.

Proposition 1.6. *If R is a module-finite algebra over a commutative noetherian ring S, then R is a noetherian ring.*

Proof. The image of S in R is a noetherian subring S' of the center of R such that R is a finitely generated (right or left) S'-module. Apply Corollary 1.5. □

For instance, let $S = \mathbb{Z} + \mathbb{Z}\mathbf{i} + \mathbb{Z}\mathbf{j} + \mathbb{Z}\mathbf{k}$, a subring of the division ring $\mathbb{H}$. Since S is a finitely generated module over the noetherian ring $\mathbb{Z}$, Proposition 1.6 shows that S is a noetherian ring. For another example, Proposition 1.6 shows that, for any positive integer n, the ring of all $n \times n$ matrices over a commutative noetherian ring is noetherian. This also holds for matrix rings over noncommutative noetherian rings, as follows.

Definition. Given a ring R and a positive integer n, we use $M_n(R)$ to denote the ring of all $n \times n$ matrices over R. The *standard $n \times n$ matrix units* in $M_n(R)$ are the matrices e_{ij} (for $i, j = 1, \ldots, n$) such that e_{ij} has 1 for the i, j-entry and 0 for all other entries.

Proposition 1.7. *Let R be a right noetherian ring and S a subring of a matrix ring $M_n(R)$. If S contains the subring*

$$R' = \left\{ \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix} \middle|\, r \in R \right\}$$

of all "scalar matrices," then S is right noetherian. In particular, $M_n(R)$ is a right noetherian ring.

Proof. Clearly $R' \cong R$, whence R' is a right noetherian ring. Observe that $M_n(R)$ is generated as a right R'-module by the standard $n \times n$ matrix units. Hence, Corollary 1.4 implies that $M_n(R)$ is a noetherian right R'-module. As all right ideals of S are also right R'-submodules of $M_n(R)$, we conclude that S is right noetherian. $\square$

• FORMAL TRIANGULAR MATRIX RINGS •

One way to construct rings to which Corollary 1.5 and Proposition 1.7 apply is to take an upper (or lower) triangular matrix ring over a known ring, or to take a subring of a triangular matrix ring. For instance, if S and T are subrings of a ring B, the set R of all matrices of the form $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix}$ (for $s \in S$, $b \in B$, $t \in T$) is a subring of $M_2(B)$. (If S and T are right noetherian, and B_T is finitely generated, it follows easily from Corollary 1.5 that R is right noetherian.) Note that B need not be a ring itself in order for R to be a ring – rather, B must be closed under addition, left multiplication by elements of S, and right multiplication by elements of T. More formally, the symbols $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix}$ will form a ring under matrix addition and multiplication provided only that B is simultaneously a left S-module and a right T-module satisfying an associative law connecting its left and right module structures. We focus on this ring construction because it provides a convenient source for any number of interesting examples. Later, we shall see such left/right modules as B appearing for their own sake in noetherian ring theory.

Definition. Let S and T be rings. An (S,T)*-bimodule* is an abelian group B equipped with a left S-module structure and a right T-module structure (both utilizing the given addition) such that $s(bt) = (sb)t$ for all $s \in S$, $b \in B$, $t \in T$. The symbol ${}_SB_T$ is used to denote this situation. An (S,T)*-sub-bimodule* of B (or just a *sub-bimodule*, if S and T are clear from the context) is any subgroup of B which is both a left S-submodule and a right T-submodule. Note that if C is a sub-bimodule of B, the factor group B/C is a bimodule in the obvious manner.

For instance, if S is a ring and T is a subring, then S itself (or an ideal of S) can be regarded as an (S,T)-bimodule (or as a (T,S)-bimodule). For another example, if B is a right module over a ring T and S is a subring of $\mathrm{End}_T(B)$, then B is an (S,T)-bimodule. Perhaps most importantly, if $I \subseteq J$ are ideals in a ring S, then J/I is an (S,S)-bimodule. The next exercise shows that in a sense every bimodule appears this way, as an ideal of a formal triangular matrix ring.

Exercise 1B. Let ${}_SB_T$ be a bimodule, and write $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ for the abelian group $S \oplus B \oplus T$, where triples (s,b,t) from $S \oplus B \oplus T$ are written as formal 2×2 matrices $\left(\begin{smallmatrix} s & b \\ 0 & t \end{smallmatrix}\right)$.

(a) Show that formal matrix addition and multiplication make sense in $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$, and that by using those operations $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ becomes a ring.

(b) Show that there is also a ring $\left(\begin{smallmatrix} T & 0 \\ B & S \end{smallmatrix}\right)$ of formal lower triangular matrices, and that $\left(\begin{smallmatrix} T & 0 \\ B & S \end{smallmatrix}\right) \cong \left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$.

(c) Observe that the set $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ of matrices $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$ is an ideal of $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$, and that, under the obvious abelian group isomorphism of B onto $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$, left S-submodules (right T-submodules, (S,T)-sub-bimodules) of B correspond precisely to left ideals (right ideals, two-sided ideals) of $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ contained in $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$. $\square$

Definition. A *formal triangular matrix ring* is any ring of the form $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} T & 0 \\ B & S \end{smallmatrix}\right)$ as described in Exercise 1B. By way of abbreviation, we write "let $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ be a formal triangular matrix ring" in place of "let S and T be rings, let B be an (S,T)-bimodule, and let $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ be the corresponding formal triangular matrix ring."

Observe that if S and T are subrings of a ring U, and B is an (S,T)-sub-bimodule of U, the formal triangular matrix ring $\left(\begin{smallmatrix} S & B \\ 0 & T \end{smallmatrix}\right)$ is isomorphic to the subring of $M_2(U)$ consisting of all honest matrices of the form $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix}$ with $s \in S$, $b \in B$, $t \in T$.

Proposition 1.8. *Let $R = \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be a formal triangular matrix ring. Then R is right noetherian if and only if S and T are right noetherian and B_T is finitely generated. Similarly, R is left noetherian if and only if S and T are left noetherian and ${}_SB$ is finitely generated.*

Proof. Assume first that S and T are right noetherian and B_T is finitely generated. Observe that the diagonal subring $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$ is isomorphic to $S \times T$ and so is right noetherian. Observe also that if elements $b_1, \dots, b_n$ generate B as a right T-module, then the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix}$$

generate R as a right $\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$-module. Consequently, Corollary 1.5 shows that R is right noetherian.

Conversely, assume that R is right noetherian. Observing that the projection maps $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix} \mapsto s$ and $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix} \mapsto t$ are ring homomorphisms of R onto S and of R onto T, we see that S and T must be right noetherian. Moreover, $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ is a right ideal of R and must have a finite list of generators

$$\begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix},$$

from which we infer that the elements $b_1, \dots, b_n$ generate B_T.

The left noetherian analog is proved in the same manner. □

For example, it is immediate from Proposition 1.8 that the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right noetherian but not left noetherian (Exercise 1A(a)).

Exercise 1C. Let $R = \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$ be a formal triangular matrix ring. The purpose of this exercise is to give a description of all right R-modules in terms of right S-modules and T-modules.

(a) Let A be a right S-module, C a right T-module, and f a homomorphism in $\mathrm{Hom}_T(A \otimes_S B, C)$. For $(a, c) \in A \oplus C$ and $\begin{pmatrix} s & b \\ 0 & t \end{pmatrix} \in R$, define

$$(a, c) \begin{pmatrix} s & b \\ 0 & t \end{pmatrix} = (as,\ f(a \otimes b) + ct).$$

Show that, using this multiplication rule, $A \oplus C$ is a right R-module.

(b) Show that the R-module $A \oplus C$ in (a) is finitely generated if and only if A is a finitely generated S-module and $C/f(A \otimes_S B)$ is a finitely generated T-module.

(c) Show that every right R-module is isomorphic to one of the type $A \oplus C$ constructed in (a). □

Exercise 1D. Let ${}_SB_T$ be a bimodule, and form the ring $R = S^{\mathrm{op}} \otimes_{\mathbb{Z}} T$, where S^{op} denotes the *opposite ring* of S. (That is, S^{op} is the same abelian group as S, but with the opposite multiplication: The product of s_1 and s_2 in S^{op} is $s_2 s_1$.) Show that B can be made into a right R-module where $b(s \otimes t) = sbt$ for all $s \in S$, $t \in T$, $b \in B$, and that the right R-submodules of B are precisely its (S,T)-sub-bimodules. Conversely, show that every right R-module can be made into an (S,T)-bimodule. $\square$

• THE HILBERT BASIS THEOREM •

A large class of examples of noetherian rings (particularly, commutative ones) is revealed by this famous theorem. There are several different proofs available; we sketch one that we shall adapt later for skew polynomial rings.

Theorem 1.9. [Hilbert's Basis Theorem] *Let $S = R[x]$ be a polynomial ring in one indeterminate. If the coefficient ring R is right (left) noetherian, then so is S.*

Proof. The two cases are symmetric; let us assume that R is right noetherian and prove that any right ideal I of S is finitely generated. We need only consider the case when $I \neq 0$.

Step 1. Let J be the set of leading coefficients of elements of I, together with 0. More precisely,

$$J = \{r \in R \mid rx^d + r_{d-1}x^{d-1} + \cdots + r_0 \in I \text{ for some } r_{d-1}, \ldots, r_0 \in R\}.$$

Then check that J is a right ideal of R. (Note that if $r, r' \in J$ are leading coefficients of elements $s, s' \in I$ with degrees d, d', then, after replacing s and s' by $sx^{d'}$ and $s'x^d$, we may assume that s and s' have the same degree.)

Step 2. Since R is right noetherian, J is finitely generated. Let $r_1, \ldots, r_k$ be a finite list of generators for J; we may assume that they are all nonzero. Each r_i occurs as the leading coefficient of a polynomial $p_i \in I$ of some degree n_i. Set $n = \max\{n_1, \ldots, n_k\}$ and replace each p_i by $p_i x^{n-n_i}$. Thus, there is no loss of generality in assuming that all the p_i have the same degree n.

Step 3. Set $N = R + Rx + \cdots + Rx^{n-1} = R + xR + \cdots + x^{n-1}R$, the set of elements of S with degree less than n. This is not an ideal of S, but it is a left and right R-submodule. Viewed as a right R-module, N is finitely generated, and so it is noetherian by Corollary 1.4. Now $I \cap N$ is a right R-submodule of N, and consequently it must be finitely generated. Let $q_1, \ldots, q_t$ be a finite list of right R-module generators for $I \cap N$.

Step 4. We claim that $p_1, \ldots, p_k, q_1, \ldots, q_t$ generate I. Let I_0 denote the right ideal of S generated by these polynomials; then $I_0 \subseteq I$ and it remains to show that any polynomial $p \in I$ actually lies in I_0. This is easy if p has degree less than n, since in that case $p \in I \cap N$ and $p = q_1 a_1 + \cdots + q_t a_t$ for some $a_j \in R$.

Step 5. Suppose that $p \in I$ has degree $m \geq n$ and that I_0 contains all elements of I with degree less than m. Let r be the leading coefficient of p. Then $r \in J$, and so $r = r_1 a_1 + \cdots + r_k a_k$ for some $a_i \in R$. Set $q = (p_1 a_1 + \cdots + p_k a_k) x^{m-n}$, an element of I_0 with degree m and leading coefficient r. Now $p - q$ is an element of I with degree less than m. By the induction hypothesis, $p - q \in I_0$, and thus $p \in I_0$.

Therefore $I = I_0$ and we are done. $\square$

It immediately follows that any polynomial ring $R[x_1, \ldots, x_n]$ in a finite number of indeterminates over a right (left) noetherian ring R is right (left) noetherian, since we may view $R[x_1, \ldots, x_n]$ as a polynomial ring in the single indeterminate x_n with coefficients from the ring $R[x_1, \ldots, x_{n-1}]$.

Corollary 1.10. *Let R be an algebra over a field k. If R is commutative and finitely generated as a k-algebra, then R is noetherian.*

Proof. Let $x_1, \ldots, x_n$ generate R as a k-algebra, and let $S = k[y_1, \ldots, y_n]$ be a polynomial ring over k in n independent indeterminates. Since R is commutative, there exists a k-algebra map $\phi : S \to R$ such that $\phi(y_i) = x_i$ for each i, and ϕ is surjective because the x_i generate R. Hence, $R \cong S/\ker(\phi)$. By the Hilbert Basis Theorem, S is a noetherian ring, and therefore R is noetherian. $\square$

Noncommutative finitely generated algebras need not be noetherian, as the following examples show.

Exercise 1E. Let k be a field.

(a) Let V be a countably infinite dimensional vector space over k with a basis $\{v_1, v_2, \ldots\}$. Define $s, t \in \operatorname{End}_k(V)$ so that $s(v_i) = v_{i+1}$ for all i while $t(v_i) = v_{i-1}$ for all $i > 1$ and $t(v_1) = 0$, and let R be the k-subalgebra of $\operatorname{End}_k(V)$ generated by s and t. Show that R is neither right nor left noetherian. [Hint: Define $e_1, e_2, \ldots$ in $\operatorname{End}_k(V)$ so that $e_i(v_i) = v_i$ for all i while $e_i(v_j) = 0$ for all $i \neq j$, and show that each $e_i \in R$. Then show that $\sum_i e_i R$ and $\sum_i R e_i$ are not finitely generated.]

(b) If F is the free k-algebra on letters X and Y, there is a unique k-algebra homomorphism $\phi : F \to R$ such that $\phi(X) = s$ and $\phi(Y) = t$. Since ϕ is surjective (by definition of R), we have $R \cong F/\ker(\phi)$, and so it is clear from part (a) that F cannot be right or left noetherian. Give a direct proof of this fact. [For instance, show that $\sum_i X^i Y F$ and $\sum_i F X Y^i$ are not finitely generated.] $\square$

Exercise 1F. Let R be an algebra over a field k, and suppose that R is generated by two elements x and y such that $xy = -yx$. Show that x^2 and y^2 are in the center of R, and that R is a finitely generated module over the subalgebra S generated by x^2 and y^2. [Hint: Use $1, x, y, xy$ to generate R.] Then apply Corollary 1.10 and Proposition 1.6 to conclude that R is noetherian.

Now suppose that, instead of $xy = -yx$, we have $xy = \xi yx$ for some scalar $\xi \in k^\times$ which is a root of unity, that is, $\xi^n = 1$ for some positive integer n. Modify the steps above to show that R is also noetherian in this case. $\square$

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• SKEW POLYNOMIAL RINGS TWISTED BY AUTOMORPHISMS •

In the Prologue we saw several examples of rings that look like polynomial rings in one indeterminate but in which the indeterminate does not commute with the coefficients – rather, multiplication by the indeterminate has been "skewed" or "twisted" by means of an automorphism of the coefficient ring, or a derivation, or a combination of such maps. To help the reader get used to constructing and working with such twisted polynomial rings, we begin here by concentrating on the case where the twisting is done by an automorphism. In Chapter 2, we move on to twists by derivations and then to general skew polynomial rings.

Thus, let R be a ring, α an automorphism of R, and x an indeterminate. Let S be the set of all formal expressions $a_0 + a_1x + \cdots + a_nx^n$, where n is a nonnegative integer and the $a_i \in R$. It is often convenient to write such an expression as a sum $\sum_i a_ix^i$, leaving it understood that the summation runs over a finite sequence of nonnegative integers i, or by thinking of it as an infinite sum in which almost all of the coefficients a_i are zero. We define an addition operation in S in the usual way:

$$\Big(\sum_i a_ix^i\Big) + \Big(\sum_i b_ix^i\Big) = \sum_i (a_i + b_i)x^i.$$

As for multiplication, we would like the coefficients to multiply together as they do in R, and we would like the powers of x to multiply following the usual rules for exponents. We take the product of an element $a \in R$ with a power x^i (in that order) to be the single-term sum ax^i. It is in a product of the form x^ia that the twist enters. We define xa to be $\alpha(a)x$ and iterate that rule to obtain $x^ia = \alpha^i(a)x$. This leads us to define the following multiplication rule in S:

$$\Big(\sum_i a_ix^i\Big)\Big(\sum_j b_jx^j\Big) = \sum_{i,j} a_i\alpha^i(b_j)x^{i+j} = \sum_k \Big(\sum_{i+j=k} a_i\alpha^i(b_j)\Big)x^k.$$

Exercise 1G. Verify that the set S together with the operations defined above is a ring, and that when R is identified with the set of elements of S involving no positive powers of x, it becomes a subring of S. $\square$

Exercise 1H. Here is a more formal description of S, in which the symbol x does not make an a priori appearance.

Let $\overline{S}$ denote the set of those infinite sequences $a = (a_0, a_1, a_2, \dots)$ of elements of R in which $a_i = 0$ for all but finitely many indices i. For any $a, b \in \overline{S}$, define $a + b$ and ab to be the sequences in $\overline{S}$ with entries

$$(a+b)_i = a_i + b_i \qquad\qquad (ab)_k = \sum_{i+j=k} a_i \alpha^i(b_j)$$

for all i and k. Show that $\overline{S}$ with these operations is a ring, and that $\overline{S} \cong S$ via the rule $a \mapsto \sum_i a_i x^i$. This isomorphism makes it clear that x is just a name for a particular special element of S, corresponding to the sequence $(0, 1, 0, 0, 0, \dots)$ in $\overline{S}$. $\square$

We have glossed over an important point in our discussion of S – the question of when two formal expressions define the same element of S. Namely, we have taken it as understood that two elements of S are the same only if their coefficients are the same, that is, $\sum_i a_i x^i = \sum_i b_i x^i$ if and only if $a_i = b_i$ for all i. Missing coefficients are understood to be zero: In case the equation concerns finite sums and an index i occurs in the first sum but not in the second, equality of coefficients means that $a_i = 0$. Using the language of linear algebra, we can thus say that the elements $1, x, x^2, \dots$ in S are linearly independent over R. Since every element of S is a linear combination of these powers, S is thus a free left R-module with the powers of x forming a basis. This leads us to the following definition.

Definition. Let R be a ring and α an automorphism of R. We write

$$S = R[x; \alpha]$$

(where S and x may or may not already occur in the discussion) to mean that

(a) S is a ring, containing R as a subring;
(b) x is an element of S;
(c) S is a free left R-module with basis $\{1, x, x^2, \dots\}$;
(d) $xr = \alpha(r)x$ for all $r \in R$.

Thus, the expression $S = R[x; \alpha]$ can be used either to introduce a new ring S (constructed as above) or to say that a given ring S and element x satisfy conditions (a)–(d). Whenever $S = R[x; \alpha]$, we say that S is a *skew polynomial ring over* R.

In many algebra texts, rings of polynomials are introduced as specific rings resulting from special constructions. Note that the definition above is of a different type, since $R[x; \alpha]$ is defined to be any ring extension of R satisfying certain properties, rather than as any specific ring (although some construction is needed to guarantee that such skew polynomial rings exist). In particular, the element x in $R[x;\alpha]$ is just a ring element with certain special properties, not a mysterious "indeterminate."

The advantage of the type of definition just given is that, in many situations, we will be able to say that some ring *equals* a skew polynomial ring $R[x;\alpha]$, rather than having to say that it is *isomorphic to* $R[x;\alpha]$. This is useful even in the context of ordinary polynomial rings. For example, we can say that the subring of $\mathbb{R}$ generated by $\mathbb{Q}$ and π *is* a polynomial ring $\mathbb{Q}[\pi]$, instead of having to name an indeterminate x and a $\mathbb{Q}$-algebra isomorphism of $\mathbb{Q}[x]$ onto $\mathbb{Q}[\pi]$.

The discussion above shows that, given R and α, a skew polynomial ring $S = R[x;\alpha]$ does exist. As is the case for ordinary polynomial rings, we would like S to be unique, up to appropriate isomorphisms. We prove this with the help of the following *universal mapping property*, in which the map ψ may be thought of as an analog of an evaluation map on ordinary polynomials in the commutative theory. The main ingredients of the lemma may be displayed as in the following diagram.

$$\begin{tikzcd}
R[x;\alpha] \arrow[dr, dashed, "\exists!\ \psi"] & \\
R \arrow[u, "\subset"] \arrow[r, "\phi"] & T
\end{tikzcd}$$

Lemma 1.11. *Let R be a ring, α an automorphism of R, and $S = R[x;\alpha]$. Suppose that we have a ring T, a ring homomorphism $\phi : R \to T$, and an element $y \in T$ such that $y\phi(r) = \phi\alpha(r)y$ for all $r \in R$. Then there is a unique ring homomorphism $\psi : S \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$.*

Proof. Clearly any such map would have to be given by the rule

$$\psi\big(\sum_i a_i x^i\big) = \sum_i \phi(a_i) y^i,$$

and so there is at most one possibility for ψ. This rule does give a well-defined function $\psi : S \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$, and so we just need to show that ψ is a ring homomorphism. It is clear that ψ is additive and that $\psi(1) = 1$. The rule $y\phi(r) = \phi\alpha(r)y$ implies (by induction) that $y^i\phi(r) = \phi\alpha^i(r)y^i$ for all $i \in \mathbb{Z}^+$ and $r \in R$. Hence,

$$\begin{aligned}
\Big[\psi\big(\sum_i a_i x^i\big)\Big]\Big[\psi\big(\sum_j b_j x^j\big)\Big] &= \Big[\sum_i \phi(a_i) y^i\Big]\Big[\sum_j \phi(b_j) y^j\Big] \\
&= \sum_{i,j} \phi(a_i)\phi\alpha^i(b_j) y^{i+j} = \sum_k \Big(\sum_{i+j=k} \phi(a_i)\phi\alpha^i(b_j)\Big) y^k \\
&= \psi\Big[\sum_k \big(\sum_{i+j=k} a_i \alpha^i(b_j)\big) x^k\Big] = \psi\Big[\big(\sum_i a^i x^i\big)\big(\sum_j b_j x^j\big)\Big]
\end{aligned}$$

for all elements $\sum_i a^i x^i$ and $\sum_j b_j x^j$ in S. Therefore ψ is a ring homomorphism, as required. $\square$

Corollary 1.12. *Let R be a ring and α an automorphism of R. Suppose that $S = R[x;\alpha]$ and $S' = R[x';\alpha]$. Then there is a unique ring isomorphism $\psi : S \to S'$ such that $\psi(x) = x'$ and $\psi|_R$ is the identity on R.*

Proof. First, apply Lemma 1.11 with $\phi : R \to S'$ being the inclusion map; we obtain a unique ring homomorphism $\psi : S \to S'$ such that $\psi(x) = x'$ and $\psi|_R = \phi$. We may rephrase the last property by saying that $\psi|_R$ is the identity on R. By symmetry, Lemma 1.11 also provides a ring homomorphism $\psi' : S' \to S$ such that $\psi'(x') = x$ and $\psi'|_R$ is the identity on R.

Now $\psi'\psi : S \to S$ is a ring homomorphism such that $(\psi'\psi)(x) = x$ and $(\psi'\psi)|_R$ is the identity on R. The identity map on S enjoys the same properties. Hence, the uniqueness part of Lemma 1.11 (where now $T = S$ and $y = x$) implies that $\psi'\psi$ equals the identity map on S. Similarly, $\psi\psi'$ equals the identity map on S'.

Therefore ψ and ψ' are mutually inverse isomorphisms. $\square$

The proof of Corollary 1.12 illustrates a general principle, that objects with universal mapping properties are unique up to isomorphism. We shall see this principle in action a number of times later.

It is time to consider a specific example. Let $R = k[y]$ be an ordinary polynomial ring over a field k. Given a nonzero scalar $q \in k$, we can define a k-algebra automorphism α on R such that $\alpha(y) = qy$. (In function notation, $\alpha\big(p(y)\big) = p(qy)$ for $p(y) \in k[y]$.) Now let $S = R[x;\alpha]$. Then $xy = \alpha(y)x = qyx$, the basic "commutation rule" in S. Since the polynomials in R are just k-linear combinations of powers of y, elements of S can be written in the form $\sum_{i,j} \lambda_{ij} y^i x^j$ for scalars λ_{ij} (all but finitely many of which are zero), and multiplication in S follows the rule

$$\Big(\sum_{i,j} \lambda_{ij} y^i x^j\Big)\Big(\sum_{s,t} \mu_{st} y^s x^t\Big) = \sum_{i,j,s,t} \lambda_{ij}\mu_{st} q^{js} y^{i+s} x^{j+t}$$
$$= \sum_{l,m}\Big(\sum_{\substack{i+s=l\\ j+t=m}} \lambda_{ij}\mu_{st} q^{js}\Big) y^l x^m.$$

This example looks very much like one from the Prologue, which we now recall.

Definition. Let k be a field and $q \in k^\times$. The *quantized coordinate ring of* k^2 (corresponding to the choice of q) is a k-algebra, denoted $\mathcal{O}_q(k^2)$, presented by two generators x and y and the relation $xy = qyx$. In short,

$$\mathcal{O}_q(k^2) = k\langle x, y \mid xy = qyx \rangle.$$

In algebraic geometry, k^2 is the affine plane over k. Hence, $\mathcal{O}_q(k^2)$ is also known as a *coordinate ring of a quantum plane* (over k), or just as – the handiest abbreviation – a *quantum plane*.

In the example discussed just prior to the definition, S is a k-algebra, and it is generated by elements called x and y, which satisfy the relation $xy = qyx$. Does this mean that $\mathcal{O}_q(k^2)$ and S are the same? To answer this question, we must make clear exactly what is meant by the definition we have given for $\mathcal{O}_q(k^2)$. We do *not* mean "any algebra generated by two elements satisfying the given relation," since there are too many possibilities. For instance, the polynomial ring $k[x]$ is generated as a k-algebra by x and 0, and certainly $x0 = q0x$. Even more extreme, the base field k itself is generated as a k-algebra by 1 and 0, and $1 \cdot 0 = q0 \cdot 1$. These algebras have "unnecessary" relations – for instance, the second generator is zero in these algebras but not in S.

What is tacitly assumed in the definition of $\mathcal{O}_q(k^2)$ (and is encoded by using the term "presented") is that x and y satisfy no "extra" relations, i.e., no relations beyond those consequences of the given relation $xy = qyx$ forced by the axioms for a k-algebra (such as $xy^3 = q^3y^3x$). The way to make the idea of "no extra relations" precise is to start with a free algebra and factor out the minimum required to achieve the desired relations. Thus, if $k\langle X, Y\rangle$ is the free algebra on two letters X and Y (which satisfy no relations at all), and $\langle XY - qYX\rangle$ denotes the ideal of $k\langle X, Y\rangle$ generated by $XY - qYX$, we are declaring that

$$\mathcal{O}_q(k^2) = k\langle X, Y\rangle / \langle XY - qYX\rangle.$$

The elements x and y in the definition of $\mathcal{O}_q(k^2)$ are then the cosets of X and Y. It follows easily from this description that $\mathcal{O}_q(k^2)$ satisfies a universal mapping property and is therefore uniquely determined up to isomorphism of k-algebras, as follows.

Exercise 1I. Let k be a field, $q \in k^\times$, and T a k-algebra. Suppose there are elements $u, v \in T$ satisfying the equation $uv = qvu$. Show that there is a unique k-algebra homomorphism $\phi : \mathcal{O}_q(k^2) \to T$ such that $\phi(x) = u$ and $\phi(y) = v$.

Conclude that if $\mathcal{O}_q(k^2)'$ is a k-algebra presented by two generators x' and y' and one relation $x'y' = qy'x'$, then $\mathcal{O}_q(k^2)' \cong \mathcal{O}_q(k^2)$. $\square$

To continue our discussion above, let us keep the symbols x and y as in the definition of $\mathcal{O}_q(k^2)$ but use new symbols $\hat{x}$ and $\hat{y}$ to rename the indeterminates in the skew polynomial ring S. By Exercise 1I, there is a unique k-algebra homomorphism $\phi : \mathcal{O}_q(k^2) \to S$ such that $\phi(x) = \hat{x}$ and $\phi(y) = \hat{y}$. Observe that ϕ is at least surjective, since $\hat{x}$ and $\hat{y}$ generate S. There are several ways to see that ϕ is actually an isomorphism; here are two.

Exercise 1J. (a) Use the relation $xy = qyx$ to show that every element of $\mathcal{O}_q(k^2)$ is a k-linear combination of the monomials y^ix^j. Then show that the monomials $\hat{y}^i\hat{x}^j$ in S are linearly independent over k. Since $\phi(y^ix^j) = \hat{y}^i\hat{x}^j$ for all i, j, conclude that the monomials y^ix^j are linearly independent and thus that ϕ is an isomorphism.

(b) Since $R = k[\hat{y}]$ is a polynomial ring over k, there is a unique k-algebra homomorphism $\eta : R \to \mathcal{O}_q(k^2)$ such that $\eta(\hat{y}) = y$. Show that $x\eta(r) = \eta\alpha(r)x$ for all $r \in R$, and conclude from Lemma 1.11 that η extends uniquely to a ring homomorphism $\psi : S \to \mathcal{O}_q(k^2)$ such that $\psi(\hat{x}) = x$. Finally, show that ϕ and ψ are inverses of each other. $\square$

Now that we have $\mathcal{O}_q(k^2) \cong S$, we can say that $\mathcal{O}_q(k^2)$ is a skew polynomial ring. Let us record this information in the following form.

Proposition 1.13. *Let k be a field and $q \in k^\times$. Then $\mathcal{O}_q(k^2) = k[y][x; \alpha]$, where $k[y]$ is a polynomial ring and α is the k-algebra automorphism of $k[y]$ such that $\alpha(y) = qy$.* $\square$

Of course, all this can be done with the variables in the reverse order. Thus,

$$\mathcal{O}_q(k^2) = k[x][y; \beta],$$

where β is the k-algebra automorphism of the polynomial ring $k[x]$ such that $\beta(x) = q^{-1}x$. We can also adapt the above discussion to any number of variables, as follows.

Definition. Let k be a field. A *multiplicatively antisymmetric matrix* over k is an $n \times n$ matrix $\mathbf{q} = (q_{ij})$ with entries $q_{ij} \in k^\times$ such that $q_{ii} = 1$ for all i and $q_{ji} = q_{ij}^{-1}$ for all i, j. Given such a matrix, the corresponding *multiparameter quantized coordinate ring of affine n-space*, or just *multiparameter quantum n-space*, is the k-algebra $\mathcal{O}_{\mathbf{q}}(k^n)$ presented by generators $x_1, \dots, x_n$ and relations $x_ix_j = q_{ij}x_jx_i$ for all i, j. For short, we write

$$\mathcal{O}_{\mathbf{q}}(k^n) = k\langle x_1, \dots, x_n \mid x_ix_j = q_{ij}x_jx_i \text{ for } 1 \le i, j \le n\rangle.$$

(The assumptions on $\mathbf{q}$ mean that the relation $x_ix_i = q_{ii}x_ix_i$ is trivial and that the relation $x_jx_i = q_{ji}x_ix_j$ duplicates the relation $x_ix_j = q_{ij}x_jx_i$. This prevents undesired relations, such as $x_i^2 = 0$, from occurring.)

As a special case, fix $q \in k^\times$ and let $\mathbf{q}$ be the unique multiplicatively antisymmetric $n \times n$ matrix with $q_{ij} = q$ for all $i < j$. In this case, we use the subscript q in place of $\mathbf{q}$. Thus, $\mathcal{O}_q(k^n)$ is the k-algebra with generators $x_1, \dots, x_n$ and relations $x_ix_j = qx_jx_i$ for all $i < j$. It is called a *single parameter quantum n-space*.

Exercise 1K. Show that any quantum n-space can be expressed as an *iterated skew polynomial ring*, that is,

$$\mathcal{O}_{\mathbf{q}}(k^n) = k[x_1][x_2; \alpha_2][x_3; \alpha_3] \cdots [x_n; \alpha_n],$$

where $k[x_1]$ is an ordinary polynomial ring and α_i (for $i = 2, \dots, n$) is a k-algebra automorphism of $k[x_1][x_2; \alpha_2] \cdots [x_{i-1}; \alpha_{i-1}]$. $\square$

The simplest example of an enveloping algebra discussed in the Prologue arose from a 2-dimensional Lie algebra L with a basis $\{x, y\}$ such that $[yx] = x$. These elements generate the enveloping algebra $U(L)$, where the basic relation becomes $yx - xy = x$. This enveloping algebra can be exhibited as a skew polynomial ring in the following way.

Exercise 1L. Let A be the algebra over a field k presented by two elements x and y and the relation $yx-xy = x$. Show that $A = k[y][x; \alpha]$, where α is the k-algebra automorphism of the polynomial ring $k[y]$ such that $\alpha(y) = y-1$. $\square$

• SKEW-LAURENT RINGS •

The discusion of group algebras in the Prologue led us to the idea of a twisted version of a Laurent polynomial ring. Such a ring would look very much like the skew polynomial rings we have just developed, except that the indeterminate would now be invertible, i.e., negative as well as positive powers would occur. Making the obvious modifications to our definition of skew polynomial rings, we now define skew-Laurent (polynomial) rings.

Definition. Let R be a ring and α an automorphism of R. We write

$$T = R[x^{\pm 1}; \alpha]$$

to mean that

(a) T is a ring, containing R as a subring;
(b) x is an invertible element of T;
(c) T is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \dots\}$;
(d) $xr = \alpha(r)x$ for all $r \in R$.

When $T = R[x^{\pm 1}; \alpha]$, we say that S is a *skew-Laurent ring over* R, or a *skew-Laurent extension of* R.

Exercise 1M. Let α be an automorphism of a ring R.

(a) Show that a skew-Laurent ring $R[x^{\pm 1}; \alpha]$ exists.

(b) If $T = R[x^{\pm 1}; \alpha]$ and $S = \sum_{i=0}^{\infty} Rx^i \subseteq T$, show that S is a subring of T and that $S = R[x; \alpha]$. $\square$

Skew-Laurent rings satisfy a universal mapping property and are consequently unique up to isomorphism, as follows.

Exercise 1N. Let α be an automorphism of a ring R and $T = R[x^{\pm 1}; \alpha]$.

(a) Suppose that we have a ring U, a ring homomorphism $\phi : R \to U$, and a unit $y \in U$ such that $y\phi(r) = \phi\alpha(r)y$ for all $r \in R$. Show that there is a unique ring homomorphism $\psi : T \to U$ such that $\psi|_R = \phi$ and $\psi(x) = y$.

(b) If $U = R[y^{\pm 1}; \alpha]$, show that there is a unique ring isomorphism $\psi : T \to U$ such that $\psi(x) = y$ and $\psi|_R$ is the identity map on R.