Preface

Spectral theory is an important part of functional analysis. It has numerous applications in many parts of mathematics and physics including matrix theory, function theory, complex analysis, differential and integral equations, control theory and quantum physics.

In recent years, spectral theory has witnessed an explosive development. There are many types of spectra, both for one or several commuting operators, with important applications, for example the approximate point spectrum, Taylor spectrum, local spectrum, essential spectrum, etc.

The present monograph is an attempt to organize the available material most of which exists only in the form of research papers scattered throughout the literature. The aim is to present a survey of results concerning various types of spectra in a unified, axiomatic way.

The central unifying notion is that of a regularity, which in a Banach algebra is a subset of elements that are considered to be "nice". A regularity R in a Banach algebra $\mathcal{A}$ defines the corresponding spectrum $\sigma_R(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin R\}$ in the same way as the ordinary spectrum is defined by means of invertible elements, $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathrm{Inv}(\mathcal{A})\}$.

Axioms of a regularity are chosen in such a way that there are many natural interesting classes satisfying them. At the same time they are strong enough for non-trivial consequences, for example the spectral mapping theorem.

Spectra of n-tuples of commuting elements of a Banach algebra are described similarly by means of a notion of joint regularity. This notion is closely related to the axiomatic spectral theory of Żelazko and Słodkowski.

The book is organized in five chapters. The first chapter contains spectral theory in Banach algebras which form a natural frame for spectral theory of operators.

In the second chapter the spectral theory of Banach algebras is applied to operators. Of particular interest are regular functions – operator-valued functions whose ranges (kernels) behave continuously. Applied to the function $z \mapsto T - z$ where T is a fixed operator, this gives rise to the important class of Kato operators and the corresponding Kato spectrum (studied in the literature under many names, e.g., semi-regular operators, Apostol spectrum etc.).

The third chapter gives a survey of results concerning various types of essential spectra, Fredholm and Browder operators etc.

The next chapter concentrates on the Taylor spectrum, which is by many experts considered to be the proper generalization of the ordinary spectrum of single operators. The most important property of the Taylor spectrum is the existence of the functional calculus for functions analytic on a neighbourhood of the Taylor spectrum. We present the Taylor functional calculus in an elementary way, without the use of sheaf theory or cohomological methods.

Further we generalize the concept of regular functions. We introduce and study operator-valued functions that admit finite-dimensional discontinuities of the kernel and range. This is closely related with stability results for the index of complexes of Banach spaces.

The last chapter is concentrated on the study of orbits of operators. By an orbit of an operator T we mean a sequence $\{T^n x : n = 0, 1, \dots\}$ where x is a fixed vector. Similarly, a weak orbit is a sequence of the form $\{\langle T^n x, x^* \rangle : n = 0, 1, \dots\}$ where $x \in X$ and $x^* \in X^*$ are fixed, and a polynomial orbit is a set $\{p(T)x : p \text{ polynomial}\}$. These notions, which originated in the theory of dynamical systems, are closely related to the invariant subspace problem. We investigate these notions by means of the essential approximate point spectrum.

All results are presented in an elementary way. We assume only a basic knowledge of functional analysis, topology and complex analysis. Moreover, basic notions and results from the theory of Banach spaces, analytic and smooth vector-valued functions and semi-continuous set-valued functions are given in the Appendix.

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Prague V. M.
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Preface to the Second Edition

Since this book was written several years ago, further progress has been made in some parts of the theory. I use the opportunity to include some of the new results, improve the arguments in other places, and also to correct some unfortunate errors and misprints that appeared in the first edition.

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