

# Chaos

*A Mathematical Introduction*

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# 1

## MAKING PREDICTIONS

*When ancient societies wished to discover what the future held for them they consulted their soothsayers. The methods by which the soothsayers made their predictions now appear to us as quite strange: observing the entrails of animals, noting the position of the planets or viewing the gleam of sacred stones.*

*Nowadays we prefer predictions to be based on scientific theories. Most branches of science embody assumptions (or laws) which can be expressed as mathematical equations. Predictions are made by solving the mathematical equations and then interpreting their solutions in terms of the original scientific problem. In this chapter we illustrate these ideas by drawing on just one area of science: the theory of population growth. This area is sufficiently familiar that its basic assumptions can be understood easily. At the same time, the equations to which it leads can have solutions with extremely complicated behaviour patterns, leading to chaos.*

*The chaotic behaviour of the solutions has far reaching implications for the future of scientific endeavour: for many scientific experiments, accurate predictions of the long term outcomes may not be possible.*

## 1.1 MATHEMATICAL MODELS

In most scientific theories the assumptions (or laws), and the equations to which they lead, do not represent the original problem with complete accuracy. Hence the equations (and the assumptions) are only a *model* whose purpose is to capture the essential features of the original problem while ignoring incidental details.

Predictions about the outcome of the problem are then made by solving the equations. If the predictions of the theory do not agree with the observed outcomes to the desired accuracy, the assumptions of the model are modified to bring them more closely into line with reality, new predictions are made and the process repeated.

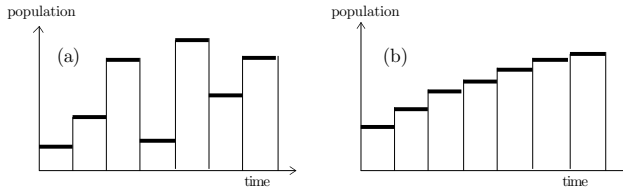
### Population growth

Many different mathematical models have been constructed for population growth and they will be our main concern in this chapter. Each of these models is an equation which expresses the rate at which the population is growing in terms of the size of the population.

From the mathematical model we hope to be able to determine the number  $N$  of individuals in the population at any later time  $t$ . Thus we are assuming that  $N$  is a function of  $t$  and we would like to be able to find this function as explicitly as possible. A simple formula expressing  $N$  in terms of  $t$  would be the ideal solution, but if this is not possible we would at least like to have a graph showing how  $N$  behaves as  $t$  increases.

Since a population grows by successive addition of individuals,  $N$  must be a discontinuous function of  $t$ . Between two consecutive times at which individuals are added, the population  $N$  will remain constant. Hence the graph of  $N$  against  $t$  must be that of a step function.

Figure 1.1.1 shows graphs of two step functions. The values of step function (b) have smaller jumps than those of step function (a). Although step functions are discontinuous, in practice they can approximate continuous functions very closely. It is not hard to imagine that if all of the jumps were small enough then the graph of the step function would be indistinguishable, for practical purposes, from that of a continuous function.

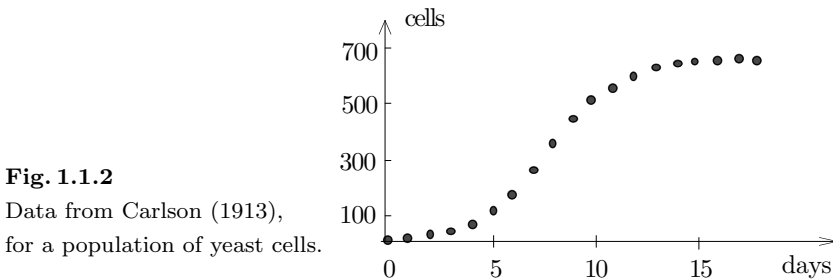


**Fig. 1.1.1** Two step functions (a) and (b). In each case, the population stays constant during a time-interval, and then jumps to a new constant value.

These considerations lead us to consider two main types of models for population growth: *continuous* and *discrete*. We now show how to recognize each type.

### Continuous models

The assumption underlying these models is that *the size of a population varies continuously with time*. As noted above, this assumption can never be strictly true. It is, nevertheless, a reasonable approximation for large populations with no preferred breeding season: for example, human populations or large populations of yeast cells. In such cases the addition of a few individuals will make so little difference to the overall population that an illusion of continuous variation will be produced.

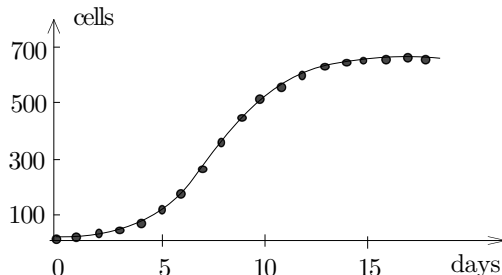


**Fig. 1.1.2**  
Data from Carlson (1913),  
for a population of yeast cells.

In Figure 1.1.2 the observed values for the number of yeast cells in a growing population are plotted. Since this is a population for which a continuous growth model is appropriate, we get the population–time graph by drawing a continuous curve through these points.

**Fig. 1.1.3**

Drawing a continuous curve through the points gives  $N$  as a continuous function of  $t$ .



An elongated-S (or *sigmoidal*) curve, as shown in Figure 1.1.3, is often associated with a continuously growing population in an environment with only limited food supply. A continuous mathematical model – one which predicts the sigmoidal shape of the curve – will be discussed in the next section.

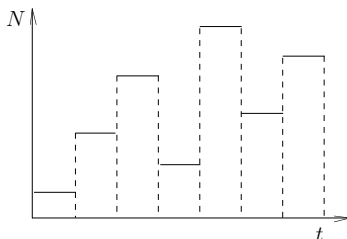
### Discrete models

These models are appropriate for populations which have specific breeding seasons. Because large numbers of the population all breed at the same time there is a perceptible jump in the size of the population at the end of each breeding season. Thus the assumption underlying these models is that *the size of a population changes abruptly at equally spaced times*. Many types of insects breed in this way, at equally spaced intervals of time.

*Graphs:* For discrete models there are various ways of representing population against time on a graph. One way is as a step function.

**Fig. 1.1.4**

The breeding seasons are equally spaced, hence the steps have equal width.



Note that there is no change in population between consecutive breeding seasons. It is thus appropriate to measure the size of the population at the end of each breeding season.

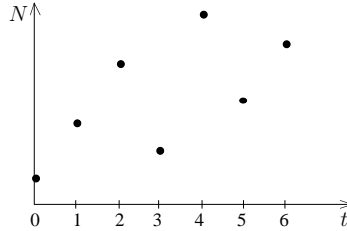
This allows us to regard the time  $t$  as being a non-negative integer  $0, 1, 2, \dots$ . Hence the graph of population against time consists of



discrete points corresponding to these measurements as illustrated in Figure 1.1.5.

**Fig. 1.1.5**

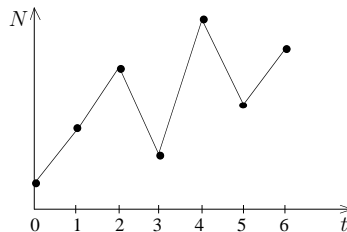
In a discrete model, we regard the time  $t$  as a non-negative integer. Hence the graph of the population  $N$  against the time  $t$  consists of a sequence of isolated points.



A graph which consists of discrete points, as in Figure 1.1.5, is a theoretically adequate way to represent the data for a discrete model. In practice, however, it is customary to join the dots by line segments to make the pattern of the dots more apparent. Joining the dots we obtain the graph in Figure 1.1.6.

**Fig. 1.1.6**

The line segments joining the dots are not part of the graph. They are there to help us see how  $N$  varies from one generation to the next.

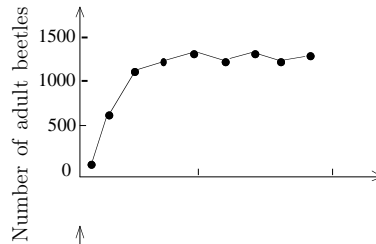


## Growth patterns

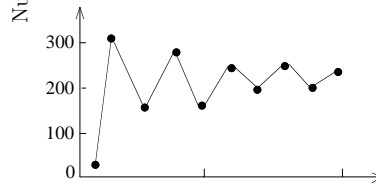
The growth curves for continuous models are typically predictable sigmoid curves whereas for discrete models there is a variety of possible growth patterns. They include not only monotonic behaviour but also oscillatory and chaotic behaviour.

Some typical growth patterns for weevils – for which discrete models are appropriate – are shown in Figure 1.1.7.

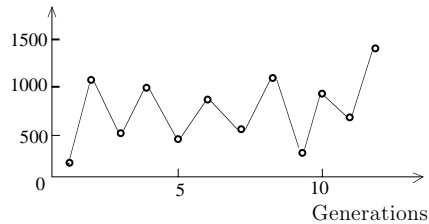
(a) *Callosobruchus chinensis*  
(Fujii, 1968);



(b) *Callosobruchus maculatus*  
(Utida, 1967);



(c) *Callosobruchus maculatus*  
(Fujii, 1968).



**Fig. 1.1.7** Population changes in laboratory cultures of three different strains of weevils. (Note that the axis which we normally label with the time  $t$  is here labelled *generations*. You can think of the first generation as occupying the time interval  $0 \leq t \leq 1$ . Then think of the population during generation 1 as the population when  $t = 0$ .)

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### Exercises 1.1

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1.1.1. Give examples of populations in each category:

- (a) a continuous model is appropriate,
- (b) a discrete model is appropriate.

1.1.2. For each of the graphs in Figure 1.1.7 :

- (a) for how many generations are the number of weevils plotted?
- (b) what is the number of weevils during the fifth generation?
- (c) what is the population when the time  $t = 3$  ?

## 1.2 CONTINUOUS GROWTH MODELS

This section contains two models for the continuous growth of a population. The first is a very simple model in which the effect of limited food and space on the growth are ignored. In the second model, these limitations are taken into account.

In each case we let  $N$  denote the number of individuals in the population at time  $t$ . As we are going to use a differential equation as the model, we ignore the fact that  $N$ , the number of individuals in the population, must be a whole number and we suppose instead that  $N$  is some differentiable function of the time  $t$ . In each model the derivative

$$\frac{dN}{dt}$$

is *the rate of increase of the population at time  $t$* . Hence the ratio

$$\frac{1}{N} \frac{dN}{dt}$$

where  $N > 0$  is *the rate of increase of the population per individual*. It is, on average, the number of offspring which each individual produces per unit time and is therefore called the *individual reproduction rate*. If deaths, as well as births, are to be considered then the above ratio is equal to  $\{\textit{birth rate}\} - \{\textit{death rate}\}$  per individual. It can then be negative, but not less than  $-1$ .

### Unlimited growth model

The simplest model of continuous growth is the assumption that, for a given type of population in a specified environment, *the individual reproduction rate remains constant*. This assumption can be written as a mathematical equation

$$\frac{1}{N} \frac{dN}{dt} = r,$$

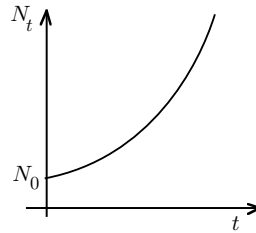
where  $r$  is a constant, and hence as

$$\frac{dN}{dt} = rN. \tag{1}$$

Thus we have derived a differential equation, which is the mathematical expression of our modelling assumption for population growth<sup>1</sup>. The differential equation has the solution

$$N_t = N_0 \exp(rt). \quad (2)$$

We assume enough data is available to determine the constant  $r$ . The solution then enables us to predict the size  $N_t$  of the population at any time  $t$  from its *initial value*  $N_0$ . For  $r > 0$  the population grows exponentially, as indicated in the graph of the solution in Figure 1.2.1.



**Fig. 1.2.1**

The unlimited growth which the differential equation (1) predicts.

Thomas Malthus (1766–1834) wrote extensively on the tendency of human populations to grow exponentially. For this reason, the above model is often referred to as the *Malthusian* model. ■

### Limited growth model

For exponential growth to continue indefinitely it is necessary to assume unlimited space and unlimited food supply. In real life, however, population growth is restricted by food, space and other necessities of living. The simplest and most familiar model taking this into account will now be described. It was first introduced in 1844 by Pierre-François Verhulst (1804–1849).

In this model it is assumed that, due to the limitations of the environment, the population has a maximum sustainable size  $K$  and

*when the size of the population approaches  $K$ ,*  
*the individual reproduction rate approaches 0.*

It is also assumed that

*when the size of the population is close to 0,*  
*the individual reproduction rate is close to a number  $r > 0$ .*

<sup>1</sup>The differential equation (1), unlike the modelling assumption, still makes sense when  $N = 0$ . Putting  $N_t = 0$  for all  $t$  gives a *constant* solution of (2).

The simplest assumption for the individual reproduction rate consistent with these two assumptions is that it equals a linear function of  $N$  which drops from  $r$  to 0 as  $N$  increases from 0 to  $K$ . Hence

$$\frac{1}{N} \frac{dN}{dt} = r \left( 1 - \frac{N}{K} \right),$$

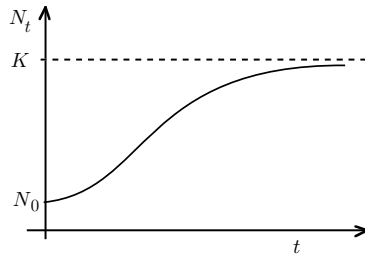
and so

$$\frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right). \quad (3)$$

Here  $K$  is called the *carrying capacity* of the environment and  $r$  is called the *intrinsic reproduction rate*. The differential equation (3) expresses Verhulst's model of population growth mathematically. It is called the *logistic equation*. Note that the differential equation has the constant solutions  $N_t = 0$  and  $N_t = K$ . This equation also has the solution

$$N_t = \frac{K}{1 + ((K/N_0) - 1)e^{-rt}}. \quad (4)$$

where  $N_0 > 0$  denotes the initial population, when  $t = 0$ . We assume enough data is available to determine the constants  $r$  and  $K$ . From the solution (4) we can then predict the size of the population at any time  $t$  provided that we are given its initial value  $N_0$ . Note that as the time  $t$  becomes indefinitely large, the population approaches the carrying capacity  $K$  and the growth rate approaches zero. The graph of a typical solution is given in Figure 1.2.2.



**Fig. 1.2.2**

The limited growth predicted by the differential equation (3).

How well does the logistic model predict the growth behaviour of actual populations? For example, how well does it predict the growth of a population of yeast cells given in Figure 1.1.2? According to [Em]<sup>2</sup> it is a reasonably good predictor for bacteria, yeast, and

<sup>2</sup>References are given at the end of each chapter.

protozoans. For laboratory populations of water fleas, *Daphnia*, fruit flies, *Drosophila* and sheep it gives a fair fit to data.

In both of our models, whether for limited or unlimited growth, the solutions of the differential equations are given by simple formulae which can be used to make the predictions. Because of this, we say that the solutions can be expressed in *closed form*.

---

**Exercises 1.2**

---

1.2.1. This exercise refers to the differential equation, with  $r$  constant,

$$\frac{dN}{dt} = rN.$$

(a) Show, by separating variables, that for  $N \neq 0$  the solution is

$$N_t = N_0 \exp(rt).$$

- (b) Show the general shape of the graph of a solution in the case  $r < 0$  and  $N_0 > 0$ .
- (c) What would you say is happening to the population being modelled by this differential equation when  $r < 0$ ?

1.2.2. This exercise refers to the following differential equation:

$$\frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right) \quad (r \text{ and } k \text{ constant}).$$

(a) Show, by separating variables, that for  $N \neq 0, K$  the solution is

$$N_t = \frac{K}{1 + ((K/N_0) - 1)e^{-rt}}.$$

(b) What does the logistic equation predict when the initial size of the population is greater than the carrying capacity?

1.2.3. For each of the differential equations in the above exercises, find all the constant solutions and discuss their biological interpretation.

### 1.3 DISCRETE GROWTH MODELS

This section is mainly about discrete versions of the two continuous models considered in the previous section.

The basic change is to consider the population only at a set of times which are equally spaced, say at unit time apart. Hence we regard  $N_t$ , the population at time  $t$ , as being defined only when  $t$  is restricted to the values  $0, 1, 2, \dots$ .

In place of the derivative, we now have the difference  $N_{t+1} - N_t$ . This represents *the rate of increase in population during the time interval from  $t$  to  $t + 1$* . Hence

$$\frac{1}{N_t} (N_{t+1} - N_t)$$

is the *the rate of increase of population per individual* during the given time interval.

#### Unlimited growth model

The simplest model of discrete growth assumes that *the individual reproduction rate is constant*; that is

$$\frac{1}{N_t} (N_{t+1} - N_t) = r$$

where  $r$  is constant, so  $N_{t+1} = N_t + rN_t$  and hence

$$N_{t+1} = N_t(1 + r). \tag{5}$$

This is an example of a *difference equation*. Since (5) is to hold for  $t = 0, 1, 2, 3, \dots$ , it is equivalent to the infinitely many equations

$$\begin{aligned} N_1 &= N_0(1 + r) \\ N_2 &= N_1(1 + r) \\ N_3 &= N_2(1 + r) \\ &\vdots \end{aligned} \tag{6}$$

A *solution* of the difference equation is an infinite sequence of numbers  $N_0, N_1, N_2, N_3, \dots$  which satisfies the equations (6).

A way to work out a solution is to pick a starting value  $N_0$  and then use the first of the equations (6) to calculate  $N_1$ . We repeat this process: use the second of the equations (6) to find  $N_2$ . In this way, calculate successively  $N_1, N_2, N_3, \dots$  in terms of  $N_0$ . This process of repeated substitution is called *iteration*.

**1.3.1 Example** *Let the sequence  $N_0, N_1, N_2, N_3, \dots$  be a solution of the difference equation (5). Use iteration to express  $N_1, N_2, N_3$  in terms of  $N_0$ .*

*Solution:* Substituting each of the equations in (6) into the equation which follows it gives in turn

$$\begin{aligned} N_1 &= N_0(1+r) \\ N_2 &= N_1(1+r) = N_0(1+r)(1+r) = N_0(1+r)^2 \\ N_3 &= N_2(1+r) = N_0(1+r)^2(1+r) = N_0(1+r)^3. \quad \blacksquare \end{aligned}$$

The above results suggest more generally that every element  $N_t$  of the solution can be obtained from the formula

$$N_t = N_0(1+r)^t. \quad (7)$$

The validity of this general formula can be proved by mathematical induction or, alternatively, by checking that

- (a) *it gives the correct initial value and*
- (b) *when substituted in the difference equation (5) it makes both sides equal.*

The following example illustrates this method.

**1.3.2 Example** *Show, by substitution, that the formula (7) gives the solution of the difference equation (5) with initial value  $N_0$ .*

*Solution:* Putting  $t = 0$  in (7) gives the initial value  $N_0(1+r)^0$ , which is correctly equal to  $N_0$ .



Substituting (7) into the difference equation (5) gives, moreover,

$$\begin{aligned}\text{LHS} &= N_{t+1} = N_0(1+r)^{t+1} \\ \text{RHS} &= N_t(1+r) = N_0(1+r)^t(1+r) = N_0(1+r)^{t+1}.\end{aligned}$$

Thus the two sides are equal. Hence (7) gives the solution of (5) satisfying the required initial condition. ■

We say that (5) has a *closed form* solution since we have been able to find a simple formula for the solution. It is only in exceptional cases, however, that we shall be able to do this.

### Limited growth model

The assumptions here are analogous to those in the limited growth model for continuous growth in the previous section. Hence

$$\frac{1}{N_t}(N_{t+1} - N_t) = r \left(1 - \frac{N_t}{K}\right)$$

and so

$$N_{t+1} = N_t + rN_t \left(1 - \frac{N_t}{K}\right). \quad (8)$$

This difference equation is our limited growth model for discrete growth.

### Other models

Other models are given by the difference equations

$$N_{t+1} = \lambda N_t(1 + aN_t)^{-\beta} \quad (9)$$

and

$$N_{t+1} = \lambda N_t \exp(-\alpha N_t). \quad (10)$$

In both (9) and (10)  $\lambda$  is the growth rate when the population is small and  $a$ ,  $\alpha$  and  $\beta$  are constants.

It was easy to show that the difference equation (5) has a closed form solution. Deciding whether (8), (9) and (10) have closed form solutions is not so easy. In principle, however, we can solve them using iteration. Their solutions have a variety of types of behaviour: monotonic, cyclic, damped oscillatory and chaotic.

The growth model (8) is of limited interest to biologists. It is hard, in fact, to find population data which can be predicted by it. It has the unrealistic feature, moreover, that if  $N_t$  is large enough, then  $N_{t+1}$  will be negative. In the last two models, however, by choosing the parameters we can get solutions to fit any one of the three sets of observations given in Figure 1.1.7. Hence these models appear to be more realistic than (8). Each of the models in (8) and (9) may be derived from a particular set of biological assumptions.

### Scaling and parameters

The two constants  $r$  and  $K$  which appear in the difference equation (8) are called *parameters*. The idea behind this terminology is that (8) is essentially not just *one* difference equation but a *family* of such equations — one equation for each choice of  $r$  and  $K$ . This gives us flexibility in modelling, since we can choose these numbers in a way which best fits the data for a particular population in a given environment.

A standard technique in mathematical modelling is rescaling so as to lump together as many parameters as possible into a single parameter. We shall illustrate this by showing how the two parameters in (8) can be replaced by one parameter  $\mu$ .

To achieve this, the trick is to introduce a ‘scaled’ population  $x_t$  in place of  $N_t$  by putting

$$x_t = \frac{r}{(r+1)K} N_t \quad (t = 0, 1, 2, \dots).$$

Solving this equation for  $N_t$  and then substituting in (8) gives the difference equation

$$x_{t+1} = \mu x_t (1 - x_t) \quad (11)$$

where  $\mu = 1 + r$ . This gives the scaled population at the end of the  $(t+1)^{\text{th}}$  breeding season in terms of that at the end of the  $t^{\text{th}}$  breeding season. The difference equation (11) is called the *discrete logistic equation* (with parameter  $\mu$ .)

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**Exercises 1.3**

---

- 1.3.1. A solution  $x_0, x_1, x_2, x_3, \dots$  of a difference equation is said to be *constant* if all of the  $x$ 's are the same. Find all the constant solutions of the difference equation

$$x_{t+1} = x_t(1 - x_t) \quad (t = 0, 1, 2, 3, \dots).$$

- 1.3.2. Let  $\mu > 0$ . Repeat Exercise 1 for the difference equation

$$x_{t+1} = \mu x_t(1 - x_t).$$

- 1.3.3. For each of the difference equations given in the text, find all the constant solutions.

- 1.3.4. This exercise refers to the solution of the difference equation

$$x_{t+1} = (x_t)^2$$

which satisfies the initial condition  $x_0 = 2$ .

- Use iteration to find the next three elements  $x_1, x_2, x_3$  of the solution.
  - Guess a general formula for  $x_t$ . Check your answer by substituting it in the difference equation.
  - What happens to  $x_t$  as  $t$  approaches  $\infty$ ?
- 1.3.5. Repeat Exercise 4, but this time use the initial condition  $x_0 = 1$ . What do you notice about the solution?

- 1.3.6. This exercise refers to the solution of the difference equation

$$x_{t+1} = \sqrt{x_t}$$

which satisfies the initial condition  $x_0 = 64$ .

- Use iteration to find the next three elements  $x_1, x_2, x_3$  of the solution.
- Guess a general formula for  $x_t$ , valid for  $t = 0, 1, 2, 3, \dots$ . Check your answer by substituting it in the difference equation.
- What happens to  $x_t$  as  $t$  approaches  $\infty$ ?

- 1.3.7. Repeat Exercise 6, but this time use the initial value  $x_0 = 1$ . What do you notice about the solution?
- 1.3.8. We can define the sum  $s_n = 1 + 2 + 3 + \cdots + (n - 1) + n$  of the first  $n$  positive integers recursively by putting

$$\begin{aligned} s_1 &= 1 \\ s_{n+1} &= s_n + (n + 1) \quad (\text{for } n = 1, 2, 3, \dots) \end{aligned}$$

- (a) Check that the solution to this difference equation which satisfies the initial condition is obtained by putting

$$s_n = \frac{n(n + 1)}{2}$$

- (b) For the purpose of calculating, say  $s_{1000}$ , which do you think is the more convenient:
- iteration using the difference equation (1), or
  - using the quadratic formula for the solution from part (a) ?

- 1.3.9. Derive the scaled form (11) of the difference equation (8).

- (a) in the special case where  $r = 1$  and  $K = 2$ ,
- (b) in the general case where there is no restriction on these parameters.

[Hint. In each case first write the substitution so it expresses  $N_t$  in terms of  $x_t$ .]

- 1.3.10. By using a suitable substitution, show how to reduce the differential equation

$$\frac{dN}{dt} = rN \left( 1 - \frac{N}{K} \right)$$

to a differential equation involving just one parameter.

[Hint. This differential equation is the continuous analogue of the difference equation (8) in the text.]

## 1.4 NUMERICAL SOLUTIONS

In this section we look briefly at the behaviour of solutions of the discrete logistic equation

$$x_{t+1} = \mu x_t(1 - x_t) \quad (12)$$

As we mentioned in the previous section, iteration is used to work out solutions of discrete equations. It is known that (12) has a closed form solution when  $\mu = 2$  and  $\mu = 4$ . For any other values of  $\mu$ , we do not know whether (12) has a closed form solution. We use the next example and graphs of solutions to illustrate the behaviour of the solutions of (12) for various values of  $\mu$ .

**1.4.1 Example** *Let  $\mu = 1$ . Use iteration to find the first four elements of the solution  $(x_0, x_1, x_2, x_3, \dots)$  of (12) which satisfies the initial condition  $x_0 = 2$ . Guess what happens to  $x_t$  when  $t$  is large.*

*Solution:* Replacing  $\mu = 1$  in the formula and  $x_0 = 2$  we get

$$\begin{aligned}x_1 &= x_0(1 - x_0) = -2 \\x_2 &= x_1(1 - x_1) = -6 \\x_3 &= x_2(1 - x_2) = -42.\end{aligned}$$

We guess that as  $t$  increases  $x_t$  becomes increasingly large and negative, so we guess that  $x_t$  approaches  $-\infty$  as  $t$  approaches  $\infty$ . ■

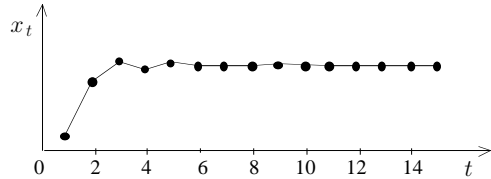
More generally, for initial values outside the interval  $[0, 1]$  the solutions very quickly approach minus infinity; hence we are normally interested only in solutions with initial values in  $[0, 1]$ .

### Graphs of solutions

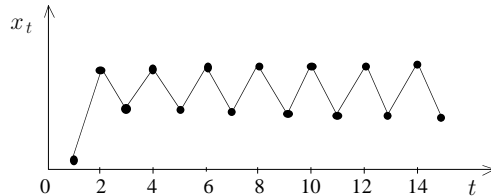
The graph of the solution of a difference equation is obtained by plotting *the value of  $x_t$  against  $t$* . Typical graphs of solutions of a difference equation are shown in Figure 1.4.1. If enough points are plotted on the graph, we can sometimes guess the behaviour of the solution from the pattern of the dots.

In Figure 1.4.1 we plot the solution of the discrete logistic equation (12) with initial value  $x_0 = .85$ , for different values of  $\mu$ . As  $\mu$  increases we observe different types of behaviour.

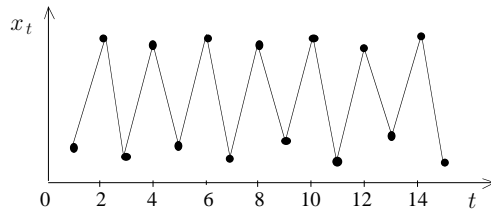
(a) For  $\mu = 2.5$  the solution tends toward a single value. (For  $1 \leq \mu \leq 3$ , the results are similar. The limiting value increases gradually as  $\mu$  increases.)



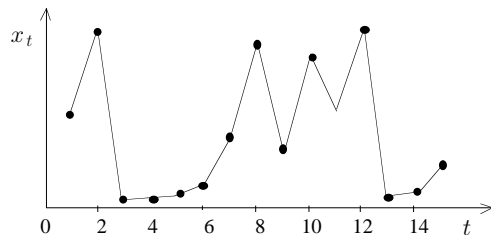
(b) For  $\mu = 3.1$  the solution eventually settles down to oscillate between two specific values.



(c) For  $\mu = 3.5$  the solution eventually settles down to oscillate between four specific values.



(d) For  $\mu = 4$  the solution appears to bounce around in a random fashion. This looks like chaos!



**Fig. 1.4.1** The above results suggest that the solutions of the difference equation become more complicated as  $\mu$  increases. In particular, the solution with initial condition  $x_0 = .85$  evolves from a constant (when  $\mu = 2.5$ ) to an oscillation between two specific values (when  $\mu = 3.1$ ) to an oscillation between four specific values (when  $\mu = 3.5$ ) and finally to chaos (when  $\mu = 4$ .)

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**Exercises 1.4**

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1.4.1.

- (a) Use iteration to find the solution of the difference equation

$$x_{t+1} = -x_t \quad (t = 0, 1, 2, 3, \dots)$$

with initial value  $x_0 = 1$ . Sketch the graph of the solution and describe its behaviour verbally.

- (b) Repeat part (a) when the initial condition is
- $x_0 = -1$
- .

1.4.2. Use iteration to find the solution of the difference equation

$$x_{t+1} = 4x_t(1 - x_t) \quad (\text{X})$$

which satisfies the initial condition  $x_0 = \frac{1}{2}$ . Sketch the graph of the solution and describe its behaviour verbally.

1.4.3. Consider the difference equation

$$y_{t+1} = 4y_t(1 - 2y_t). \quad (\text{Y})$$

- (a) Show that if you make the ‘change of variables’  $y_t = \frac{1}{2}x_t$  then you get the difference equation (X) of Exercise 2.
- (b) Which change of variables, when applied to the difference equation (X), gives the difference equation (Y) ?
- (c) Use the solution you found for the difference equation (X) and the change of variables from part (a) to deduce a solution of (Y).
- 1.4.4. Find  $c \in \mathbb{R}$  such that, under the change of variables  $x_t = y_t + c$ , the difference equation

$$x_{t+1} = 2x_t + 1$$

reduces to the difference equation

$$y_{t+1} = 2y_t.$$

## 1.5 DYNAMICAL SYSTEMS

In this chapter we have used models for population growth as an introduction to *dynamical systems*: the population changed with time and we wanted to make long-term predictions about its growth. We were interested in such questions as:

- (a) *Does the population become indefinitely large as time increases?*
- (b) *Does the population eventually die out ?*
- (c) *Does the population increase monotonically or does it oscillate about some value?*
- (d) *Are there values of the parameters for which the population grows in a chaotic, unpredictable way?*

To answer these questions, we set up mathematical models describing the rate of growth of the population and then solved, or attempted to solve, the resulting differential or difference equations.

Population growth, however, is only one of many areas in which the idea of a dynamical system arises. Devaney describes dynamical systems as ‘the branch of mathematics that attempts to describe processes in motion’. He goes on to say that

Such processes occur in all branches of science. For example, the motion of the stars and the galaxies in the heavens is a dynamical system, one that has been studied for centuries by thousands of scientists. The ups and downs of the stock market is another system that changes in time, as is the weather throughout the world. The changes chemicals undergo, the rise and fall of populations, and the motion of a simple pendulum are classical examples of dynamical systems in chemistry, biology and physics. Clearly dynamical systems abound [De2].

What is important about ‘dynamical systems’, of course, is that there is a law which describes their evolution over time. In the models of population growth studied in this chapter, the law was given by a differential equation or a difference equation.

To model real world situations, such as those described by Devaney, the first step often is to find the right differential or difference equation — one whose solutions model the observed evolution. As