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Multiparametric Linear and Quadratic Programming

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In this work we present an algorithm for the solution of multiparametric linear and quadratic programming problems. With linear constraints and linear or convex quadratic objective functions, the optimal solution of these optimization problems is given by a conditional piecewise linear function of the varying parameters. This function results from first-order estimations of the analytical nonlinear optimal function. The core idea of the algorithm is to approximate the analytical nonlinear function by affine functions, whose validity is confined to regions of feasibility and optimality. Therefore, the space of parameters is systematically characterized into different regions where the optimal solution is an affine function of the parameters. The solution obtained is convex and continuous. Examples are presented to illustrate the algorithm and to enhance its potential in real-life applications.

1.1
Introduction

Variability and uncertainty are widely recognized as crucial topics in the design and operation of processes and systems [34]. Fluctuations in resources, market requirements, prices, and during plant operation make imperative the study of possible consequences of uncertainty and variability in the feasibility and economics of a project. In the optimization models, variability and uncertainty correspond to the inclusion of varying parameters.

According to the parameters' description, different solving approaches have been proposed: (i) multiperiod optimization [11, 42, 46], (ii) stochastic programming [4, 5, 10, 13, 20, 26, 40], and (iii) parametric programming. In the multiperiod optimization approach, the time horizon is discretized into time periods, associated with forecasts of the parameters. For instance, if the forecast is a demand of a specific chemical product in the ensuing years, the objective is to find a planning strategy for producing these chemicals, which maximizes the net present value. If the probability distribution function of the parameters is known, the stochastic programming identifies the optimal solution which corresponds to the maximum

Multi-Parametric Programming. Edited by E. Pistikopoulos, M. Georgiadis, and V. Dua

ISBN: 978-3-527-31691-5

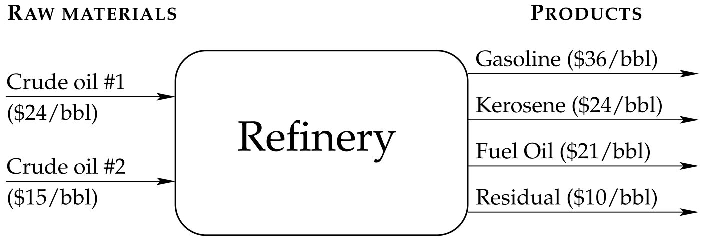

Fig. 1.1 Crude oil refinery.

expected profit. At last, the parametric programming approach aims to obtain the optimal solution as an explicit function of the parameters. In this chapter we will discuss techniques based upon the fundamentals of parametric programming.

Parametric programming is based on the sensitivity analysis theory, distinguishing from the latter in the targets. Sensitivity analysis provides solutions in the neighborhood of the nominal value of the varying parameters, whereas parametric programming provides a complete map of the optimal solution in the space of the varying parameters. Theory and algorithms for the solution of a wide range of parametric programming problems have been reported in the literature [1, 3, 15–18, 22, 25, 33, 45].

◇**Example 1** [19]

A refinery blending and production process is depicted in Fig. 1.1. The objective of the company is to maximize the profit by selecting the optimal combination of raw materials and products. Operating conditions are presented in Table 1.1, where θ_1 and θ_2 are parameters representing an additional maximum allowable production of gasoline and kerosene, respectively.

This problem formulates as a multiparametric linear programming problem (1.1), where x_1 and x_2 are the flow rates of the crude oils 1 and 2 in bbl/day, respectively, and the units of profit are \$/day.

$$\text{Profit} = \max_x 8.1x_1 + 10.8x_2, \tag{1.1a}$$

$$\text{s.t.}\quad 0.80x_1 + 0.44x_2 \le 24\,000 + \theta_1, \tag{1.1b}$$

$$0.05x_1 + 0.10x_2 \le 2000 + \theta_2, \tag{1.1c}$$

Table 1.1 Refinery data.

	Volume % yield		Maximum allowable
	Crude 1	Crude 2	production (bbl/day)
Gasoline	80	44	$24\,000 + \theta_1$
Kerosene	5	10	$2000 + \theta_2$
Fuel oil	10	36	6000
Residual	5	10	–
Processing cost (\$/bbl)	0.50	1.00	–

$$0.10x_1 + 0.36x_2 \le 6000, \tag{1.1d}$$

$$x_1 \ge 0,\ \ x_2 \ge 0, \tag{1.1e}$$

$$0 \le \theta_1 \le 6000, \tag{1.1f}$$

$$0 \le \theta_2 \le 500. \tag{1.1g}$$

The importance of solving this problem is as follows:

(i) the optimal policy for selecting the crude oil source is known as a function of θ_1 and θ_2;

(ii) substituting the value of θ_1 and θ_2 into the parametric profiles we know directly the optimal profit;

(iii) the sensitivity of the profit to the parameters is identified. The board of the company foresees more sensitive operating regions, making the management more efficient.

◇**Example 2** [12]

A Dutch agriculture cooperative society has to deal with the excess of milk produced. Since some high-valued products can be processed, this cooperative society has to set either the quantities, taking into account the demand (z), and prices (x) for each product. This specific cooperative society considers but four types of products: milk for direct consumption, butter, fat cheese, and low fat cheese (Fig. 1.2).

The capacity constraints are

$$0.026z_1 + 0.800z_2 + 0.306z_3 + 0.245z_4 \le 119, \tag{1.2a}$$

$$0.086z_1 + 0.020z_2 + 0.297z_3 + 0.371z_4 \le 251, \tag{1.2b}$$

$$z_1 \ge 0, \tag{1.2c}$$

$$z_2 \ge 0, \tag{1.2d}$$

$$z_3 \ge 0, \tag{1.2e}$$

$$z_4 \ge 0. \tag{1.2f}$$

Obviously, consumer demand depends critically on the price of the product, where a negative relation is expected:

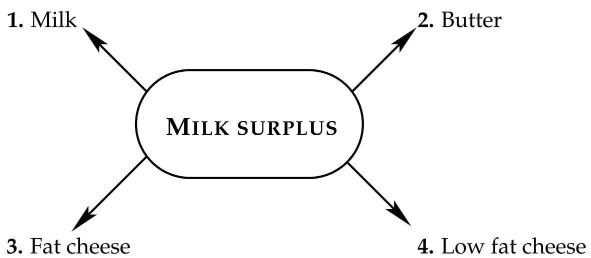

Fig. 1.2 Possible products from the milk surplus.

$$z_1 = -1.2338x_1 + 2139 + w_1, \tag{1.3a}$$

$$z_2 = -0.0203x_2 + 135 + w_2, \tag{1.3b}$$

$$z_3 = -0.0136x_3 + 0.0015x_4 + 103 + w_3, \tag{1.3c}$$

$$z_4 = +0.0016x_3 - 0.0027x_4 + 19 + w_4, \tag{1.3d}$$

where w_1, w_2, w_3, and w_4 are uncertainties associated with the consumer demand.

The cooperative society wants to reward as much as possible their associates, and hence the objective is to maximize profit. Ignoring production costs, the objective function is written as

$$\text{Profit} = \max_{x} \sum_{i=1}^{4} x_i \cdot z_i, \tag{1.4}$$

which is a *quadratic function* of prices, x_i. The government avoids the escalation of the prices with an extra policy constraint:

$$0.0163x_1 + 0.0003x_2 + 0.0006x_3 + 0.0002x_4 \leq 10 + k, \tag{1.5}$$

where k refers to a possible price rise (e.g., $k = 0.1$ means a rise of 1% on the overall prices). This is regarded as a social constraint.

The optimization problem formulates as in (1.6).

$$\begin{aligned} \text{Profit} = \max_{x_1,x_2,x_3,x_4} & \sum_{i=1}^{4} x_i \cdot z_i, \\ \text{s.t.} \quad & 0.026z_1 + 0.800z_2 + 0.306z_3 + 0.245z_4 \leq 119, \\ & 0.086z_1 + 0.020z_2 + 0.297z_3 + 0.371z_4 \leq 251, \\ & 0.0163x_1 + 0.0003x_2 + 0.0006x_3 + 0.0002x_4 \leq 10 + k, \\ & z_1 = -1.2338x_1 + 2139 + w_1, \\ & z_2 = -0.0203x_2 + 135 + w_2, \\ & z_3 = -0.0136x_3 + 0.0015x_4 + 103 + w_3, \\ & z_4 = +0.0016x_3 - 0.0027x_4 + 19 + w_4, \\ & z_1 \geq 0, \\ & z_2 \geq 0, \\ & z_3 \geq 0, \\ & z_4 \geq 0, \\ & -150 \leq w_1 \leq 150, \\ & -5 \leq w_2 \leq 5, \\ & -6 \leq w_3 \leq 6, \\ & -2 \leq w_4 \leq 2, \\ & -1 \leq k \leq 1. \end{aligned} \tag{1.6}$$

The significance of such solution is as follows:

(i) the optimal price policy is known as a function of the uncertainty in the demand, w_i, and possible price rise, k;

(ii) sensitivity of the current best decision is known, and supports an efficient decision making.

As shown, this type of information is very useful for solving reactive or online optimization problems. Such problems usually require a repetitive solution of optimization problems; due to the varying conditions of most processes, the optimal decision/action changes with time. The key advantage of parametric programming is to obtain the optimal solution as a function of the varying parameters without exhaustively enumerating the entire parametric space.

A broad spectrum of process engineering applications has been identified: (i) hybrid parametric/stochastic programming [2, 27], (ii) process planning under uncertainty [35], (iii) scheduling under uncertainty [41], (iv) material design under uncertainty [14], (v) multiobjective optimization [31, 32, 39], (vi) flexibility analysis [6, 8], and (vii) computation of singular multivariate normal probabilities [7]. Although parametric programming has various applications, the online control problem [9, 37, 38, 44] is the most prolific application, where control variables are obtained as a function of the initial state of the system. This reduces the real-time optimal control problem to a simple function evaluation problem. Mathematically, such problems are formulated as multiparametric quadratic programs (mp-QP). Robust online control problems that can take into account uncertainty and disturbance can also be reformulated as mp-QPs to obtain the explicit robust control law [28, 29, 43].

The rest of the chapter organizes as follows. Section 1.2 describes the underlying mathematical background of the methodology, and finalizes with the algorithm; convexity/continuity properties of the solution are also proven. In section 1.3, some examples are solved in order to illustrate the procedure and to give an insight of the complexity involved.

1.2 Methodology

Consider the general parametric nonlinear programming problem:

$$
\begin{aligned}
& \min_x f(x,\theta), \\
\text{s.t.} \quad & g_i(x,\theta) \le 0, \quad \forall\ i = 1,\ldots,p, \\
& h_j(x,\theta) = 0, \quad \forall\ j = 1,\ldots,q, \\
& x \in X \subseteq \mathbb{R}^n, \\
& \theta \in \Theta \subseteq \mathbb{R}^m,
\end{aligned}
\tag{1.7}
$$

where f, g, and h are twice continuously differentiable in x and θ. The first-order Karush–Kuhn–Tucker (KKT) optimality conditions for (1.7) are given as follows:

$$\begin{aligned}
&\nabla \mathcal{L} = 0,\\
&\lambda_i g_i(x,\theta) = 0, \quad \lambda_i \geq 0, \quad \forall\, i = 1,\dots,p,\\
&h_j(x,\theta) = 0, \quad \forall\, j = 1,\dots,q,\\
&\mathcal{L} = f(x,\theta) + \sum_{i=1}^{p} \lambda_i g_i(x,\theta) + \sum_{j=1}^{q} \mu_j h_j(x,\theta).
\end{aligned} \tag{1.8}$$

The main sensitivity result for (1.7) derives directly from system (1.8), as shown in Theorem 1.

Theorem 1. *Basic sensitivity theorem [21]: Let x_0 be a vector of parameter values and (u_0, λ_0, μ_0) a KKT triple corresponding to (1.8), where λ_0 is nonnegative and u_0 is feasible in (1.7). Also assume that (i) strict complementary slackness (SCS) holds, (ii) the binding constraint gradients are linearly independent (LICQ: linear independence constraint qualification), and (iii) the second-order sufficiency conditions (SOSC) hold. Then, in the neighborhood of x_0, there exists a unique, once continuously differentiable function, $z(x) = [u(x), \lambda(x), \mu(x)]$, satisfying (1.8) with $z(x_0) = [u(x_0), \lambda(x_0), \mu(x_0)]$, where $u(x)$ is a unique isolated minimizer for (1.7), and*

$$\begin{pmatrix} \frac{du(x_0)}{dx} \\ \frac{d\lambda(x_0)}{dx} \\ \frac{d\mu(x_0)}{dx} \end{pmatrix} = -(M_0)^{-1} N_0, \tag{1.9}$$

where M_0 and N_0 are the Jacobian of system (1.8) with respect to z and x:

$$M_0 = \begin{pmatrix}
\nabla^2\mathcal{L} & \nabla g_1 & \cdots & \nabla g_p & \nabla h_1 & \cdots & \nabla h_q \\
-\lambda_1 \nabla^T g_1 & -g_1 & & & & & \\
\vdots & & \ddots & & & & \\
-\lambda_p \nabla^T g_p & & & -g_p & & & \\
\nabla^T h_1 & & & & & & \\
\vdots & & & & & & \\
\nabla^T h_q & & & & & &
\end{pmatrix},$$

$$N_0 = (\nabla^2_{xu}\mathcal{L}, -\lambda_1 \nabla^T_x g_1, \dots, -\lambda_p \nabla^T_x g_p, \nabla^T_x h_1, \dots, \nabla^T_x h_q)^T.$$

Note that the assumptions stated in the theorem above ensure the existence of the inverse of M_0 [30].

Corollary 1. *First-order estimation of $x(\theta)$, $\lambda(\theta)$, $\mu(\theta)$, near $\theta = \theta_0$ [22]: Under the assumptions of Theorem 1, a first-order approximation of $[x(\theta), \lambda(\theta), \mu(\theta)]$ in a neighborhood of θ_0 is*

$$\begin{bmatrix} x(\theta) \\ \lambda(\theta) \\ \mu(\theta) \end{bmatrix} = \begin{bmatrix} x_0 \\ \lambda_0 \\ \mu_0 \end{bmatrix} + (M_0)^{-1} \cdot N_0 \cdot \theta + o(\|\theta\|), \tag{1.10}$$

where $(x_0, \lambda_0, \mu_0) = [x(\theta_0), \lambda(\theta_0), \mu(\theta_0)]$, $M_0 = M(\theta_0)$, $N_0 = N(\theta_0)$, and $\phi(\theta) = o(\|\theta\|)$ means that $\phi(\theta)/\|\theta\| \to 0$ as $\theta \to \theta_0$.

Despite being a simple and linear expression, Eq. (1.10) may lead to complex computational problems, since in the general nonlinear case the Jacobians of system (1.8) are in most of the cases complex. Fortunately, it simplifies when (1.7) has a quadratic objective function, linear constraints, and the parameters appear on the right-hand side of the constraints:

$$\begin{aligned} & z(\theta) = \min_{x}\ c^T x + \tfrac{1}{2} x^T Q x, \\ \text{s.t.}\quad & Ax \le b + F\theta, \\ & x \in X \subseteq \mathbb{R}^n, \\ & \theta \in \Theta \subseteq \mathbb{R}^m, \end{aligned} \tag{1.11}$$

where c is a constant vector of dimension n, Q is an $(n \times n)$ symmetric positive definite constant matrix, A is a $(p \times n)$ constant matrix, F is a $(p \times m)$ constant matrix, b is a constant vector of dimension p, and X and Θ are compact polyhedral convex sets of dimensions n and m, respectively. Note that a term of the form $\theta^T Px$ in the objective function can also be addressed in the above formulation, as it can be transformed into the form given in (1.11) by substituting $x = s - Q^{-1} P^T \theta$, where s is a vector of arbitrary variables of dimension n and P is a constant matrix of dimension $(m \times n)$.

An application of Theorem 1 to (1.11) at $[x(\theta_Q), \theta_Q]$ gives the following result:

$$\begin{pmatrix} \frac{dx(\theta_Q)}{d\theta} \\ \frac{d\lambda(\theta_Q)}{d\theta} \end{pmatrix} = -(M_Q)^{-1} N_Q, \tag{1.12}$$

where

$$\begin{aligned} M_Q &= \begin{bmatrix} Q & A_1^T \cdots A_p^T \\ -\lambda_1 A_1 - V_1 & \\ \vdots & \ddots \\ -\lambda_p A_p & \quad -V_p \end{bmatrix}, \\ N_Q &= [Y, \lambda_1 F_1, \ldots, \lambda_p F_p]^T, \\ V_i &= A_i x(\theta_Q) - b_i - F_i \theta_Q, \end{aligned} \tag{1.13}$$

and Y is a null matrix of dimension $(n \times m)$. Thus, in the linear–quadratic optimization problem, the Jacobians reduce to a mere algebraic manipulation of the matrices declared in (1.11). In the neighborhood of the KKT point, $[x(\theta_Q), \theta_Q]$, Corollary 1 writes as follows:

$$\begin{bmatrix} x_Q(\theta) \\ \lambda_Q(\theta) \end{bmatrix} = -(M_Q)^{-1} N_Q (\theta - \theta_Q) + \begin{bmatrix} x(\theta_Q) \\ \lambda(\theta_Q) \end{bmatrix}. \tag{1.14}$$

Note that when assumptions in Theorem 1 are respected M_Q is always invertible.

This is where parametric programming detaches from the sensitivity analysis theory. Whilst sensitivity analysis stops here, where we know what happens if the process conditions deviate from the nominal values to some value in its neighborhood, parametric programming is concerned with the whole range of the parametric variability. The former associates with the uncertainty and the latter to the variability of the process.

The space of θ where this solution (1.14) remains optimal is defined as the critical region, CR^Q, and can be obtained by using feasibility and optimality conditions. Note that for convenience and simplicity in presentation, we use the notation CR to denote the set of points in the space of θ that lie in CR as well as to denote the set of inequalities which define CR. Feasibility is ensured by substituting $x_Q(\theta)$ into the inactive inequalities given in (1.11), whereas the optimality condition is given by $\tilde{\lambda}_Q(\theta) \geq 0$, where $\tilde{\lambda}_Q(\theta)$ corresponds to the vector of active inequalities, resulting in a set of parametric constraints. Let this set be represented by

$$\mathrm{CR}^R = \{\breve{A}x_Q(\theta) \leq \breve{b} + \breve{F}\theta, \tilde{\lambda}_Q(\theta) \geq 0, \mathrm{CR}^{\mathrm{IG}}\}, \tag{1.15}$$

where $\breve{A}, \breve{b}$, and $\breve{F}$ correspond to the inactive inequalities and $\mathrm{CR}^{\mathrm{IG}}$ represents a set of linear inequalities defining an initial given region. From the parametric inequalities thus obtained, the redundant inequalities are removed and a compact representation of CR^Q is obtained as follows:

$$\mathrm{CR}^Q = \Delta\{\mathrm{CR}^R\}, \tag{1.16}$$

where Δ is an operator which removes redundant constraints—for a procedure to identify redundant constraints see [25] (see Appendix A for a summary). Note that a CR^Q is a polyhedral region. Once CR^Q has been defined for a solution, $[x(\theta_Q), \theta_Q]$, the next step is to define the rest of the region, $\mathrm{CR}^{\mathrm{rest}}$, as proposed in [16] (see Appendix B for a summary):

$$\mathrm{CR}^{\mathrm{rest}} = \mathrm{CR}^{\mathrm{IG}} - \mathrm{CR}^Q. \tag{1.17}$$

Another set of parametric solutions in each of these regions is then obtained and corresponding CRs are obtained. The algorithm terminates when there are no more regions to be explored. In other words, the algorithm terminates when the solution of the differential equation (1.12) has been fully approximated by first-order expansions.

The main steps of the algorithm are outlined in Table 1.2. Note that while defining the rest of the regions, some of the regions are split and hence the same optimal

Table 1.2 mp-QP algorithm.

Step 1	In a given region solve (1.11) by treating θ as a free variable to obtain a feasible point $[\theta_Q]$
Step 2	Fix $\theta = \theta_Q$ and solve (1.11) to obtain $[x(\theta_Q), \lambda(\theta_Q)]$
Step 3	Compute $[-(M_Q)^{-1}N_Q]$ from (1.12)
Step 4	Obtain $[x_Q(\theta), \lambda_Q(\theta)]$ from (1.14)
Step 5	Form a set of inequalities, CR^R, as described in (1.15)
Step 6	Remove redundant inequalities from this set of inequalities and define the corresponding CR^Q as given in (1.16)
Step 7	Define the rest of the region, $\mathrm{CR}^{\mathrm{rest}}$ as given in (1.17)
Step 8	If no more regions to explore, go to next step, otherwise go to Step 1
Step 9	Collect all the solutions and unify the regions having the same solution to obtain a compact representation

solution may be obtained in more than one regions. Therefore, the regions with the same optimal solution are united and a compact representation of the final solution is obtained.

When θ is present on the right-hand side of the constraints, the solution space of (1.7) is convex and continuous [23]. Since (1.11) is a special case of (1.7), its solution has these properties as well. Due to its importance, we prove these properties specifically for (1.11) in the next theorem.

Theorem 2. *Consider the mp-QP (1.11) and let Q be positive definite, Θ convex. Then the set of feasible parameters $\Theta_f \subseteq \Theta$ is convex, the optimizer $x(\theta) : \Theta_f \mapsto \mathbb{R}^n$ is continuous and piecewise affine, and the optimal solution $z(\theta) : \Theta_f \mapsto \mathbb{R}$ is continuous, convex, and piecewise quadratic.*

Proof. We first prove convexity of Θ_f and $z(\theta)$. Take generic $\theta_1, \theta_2 \in \Theta_f$ and let $z(\theta_1), z(\theta_2)$ and x_1, x_2 be the corresponding optimal values and minimizers. Let $\alpha \in [0, 1]$ and define $x_\alpha \triangleq \alpha x_1 + (1-\alpha)x_2$, $\theta_\alpha \triangleq \alpha\theta_1 + (1-\alpha)\theta_2$. By feasibility, x_1, x_2 satisfy the constraints $Ax_1 \leq b + F\theta_1$, $Ax_2 \leq b + F\theta_2$. These inequalities can be linearly combined to obtain $Ax_\alpha \leq b + F\theta_\alpha$ and therefore x_α is feasible for the optimization problem (1.11). Since a feasible solution, $x(\theta_\alpha)$, exists at θ_α, an optimal solution exists at θ_α and hence Θ_f is convex.

The optimal solution at θ_α will be less than or equal to the feasible solution:

$$z(\theta_\alpha) \leq c^T x_\alpha + \frac{1}{2} x_\alpha^T Q x_\alpha$$

and hence,

$$z\,(\theta_\alpha) - [\alpha(c^T x_1 + \frac{1}{2}x_1^T Q x_1) + (1-\alpha)(c^T x_2 + \frac{1}{2}x_2^T Q x_2)] \tag{1.18a}$$

$$\leq c^T x_\alpha + \frac{1}{2}x_\alpha^T Q x_\alpha - [\alpha(c^T x_1 + \frac{1}{2}x_1^T Q x_1) + (1-\alpha)(c^T x_2 + \frac{1}{2}x_2^T Q x_2)] \tag{1.18b}$$

$$= \frac{1}{2}[\alpha^2 x_1^T Q x_1 + (1-\alpha)^2 x_2^T Q x_2 + 2\alpha(1-\alpha)x_2^T Q x_1 - \alpha x_1^T Q x_1 - (1-\alpha)x_2^T Q x_2] \tag{1.18c}$$

$$= -\frac{1}{2}\alpha(1-\alpha)(x_1 - x_2)^T Q(x_1 - x_2) \leq 0, \tag{1.18d}$$

which means that, (1.18e)

$$z(\alpha\theta_1 + (1-\alpha)\theta_2) \leq \alpha z(\theta_1) + (1-\alpha)z(\theta_2), \forall\theta_1, \theta_2 \in \Theta, \forall\alpha \in [0, 1], \tag{1.18f}$$

proving the convexity of $z(\theta)$ on Θ_f.

Within the closed polyhedral regions, CR^Q, in Θ_f the solution $x(\theta)$ is affine (Corollary 1). The boundary between two regions belongs to both closed regions. Because the optimum is unique the solution must be continuous across the boundary. The fact that $z(\theta)$ is continuous and piecewise quadratic follows trivially.

Remark 1. *Multiparametric linear program: Note that when Q is a null matrix, (1.11) reduces to a multiparametric linear program (mp-LP). This does not affect the solution* *procedure described above and the algorithm remains the same. This is because the results presented in the theorems are still valid as explained next. The results presented in Theorem 1 continue to hold true and SOSC is valid in spite of the fact that Q is a null matrix as discussed on page 71 in [22]. For mp-LPs x is an affine function of θ and λ remains constant in a CR as shown in Chapter 4 in [25] and therefore Corollary 1 can be used. Whilst the results of Theorem 2 regarding Θ_f and $x(\theta)$ are still valid, $z(\theta)$ simplifies to a continuous, convex, and piecewise linear function of θ as also shown in Chapter 4 in [25].*

Hence, at the end of the algorithm the solution obtained is a conditional piecewise function of the parameters and Theorem 2 implies that the optimal function computed, $z(\theta)$, is continuous and convex.

1.3 Numerical Examples

In this section, the solution steps are described in detail for the two illustrative examples presented before: the refinery problem and the surplus milk production. Additionally, we solve a mp-QP problem corresponding to a model-based predictive control problem [37].

1.3.1 Example 1: Crude Oil Refinery

Consider the mp-LP problem formulated for the crude oil refinery example:

$$\text{Profit} = \max_x 8.1x_1 + 10.8x_2, \tag{1.19a}$$

$$\text{s.t.}\quad 0.80x_1 + 0.44x_2 \le 24\,000 + \theta_1, \tag{1.19b}$$

$$0.05x_1 + 0.10x_2 \le 2000 + \theta_2, \tag{1.19c}$$

$$0.10x_1 + 0.36x_2 \le 6000, \tag{1.19d}$$

$$x_1 \ge 0, \tag{1.19e}$$

$$x_2 \ge 0, \tag{1.19f}$$

$$0 \le \theta_1 \le 6000, \tag{1.19g}$$

$$0 \le \theta_2 \le 500. \tag{1.19h}$$

The solutions steps are as follows.

Step 1. Solve (1.19) by treating θ_1 and θ_2 as free variables. A feasible point obtained is $\theta_{Q\text{-}1} = [0, 0]^T$;

Step 2. Fix $\theta_{Q\text{-}1} = [0, 0]^T$ and solve (1.19). The solution is: $x_{Q\text{-}1} = [26\,207, 6896.6]^T$; $\lambda_{Q\text{-}1} = [4.655, 87.52, 0]$;

Step 3. Compute $[-M_{Q-1}^{-1} N_{Q-1}]$ from (1.13). The solution is given by

$$-M_{Q-1}^{-1} N_{Q-1} = \begin{bmatrix} 1.724 & -7.586 \\ -0.8621 & 13.79 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}.$$

Step 4. Compute $[x_{Q-1}(\theta), \lambda_{Q-1}(\theta)]$ from (1.14):

$$\begin{bmatrix} x_{Q-1}^1(\theta) \\ x_{Q-1}^2(\theta) \\ \lambda_{Q-1}^1(\theta) \\ \lambda_{Q-1}^2(\theta) \\ \lambda_{Q-1}^3(\theta) \end{bmatrix} = \begin{bmatrix} 1.724 & -7.586 \\ -0.8621 & 13.79 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix} \cdot (\theta - \theta_{Q_1}) + \begin{bmatrix} 26\,207 \\ 6896.6 \\ 4.552 \\ 87.52 \\ 0.0000 \end{bmatrix},$$

or,

$$\begin{cases} x_{Q-1}^1 = 1.724 \cdot \theta_1 - 7.586 \cdot \theta_2 + 26207, \\ x_{Q-1}^2 = -0.8621 \cdot \theta_1 + 13.79 \cdot \theta_2 + 6896.6, \\ \lambda_{Q-1}^1 = 4.555, \\ \lambda_{Q-1}^2 = 87.52, \\ \lambda_{Q-1}^3 = 0.0000. \end{cases}$$

Step 5. Form a set of inequalities corresponding to CR^R,

$$\mathrm{CR}^R = \begin{cases} \breve{A} x_{Q-1}(\theta) \le \breve{b} + \breve{F}\theta : -0.1380\theta_1 + 4.206\theta_2 \le 896.5, \\ \tilde{\lambda}_{Q-1}(\theta) \ge 0 : \begin{cases} 4.552 \ge 0, \\ 87.52 \ge 0, \\ 0.0000 \ge 0, \end{cases} \\ \mathrm{CR}^{\mathrm{IG}} : \begin{cases} 0 \le \theta_1 \le 6000, \\ 0 \le \theta_2 \le 500, \end{cases} \end{cases} \tag{1.20}$$

Step 6. Remove redundant constraints,

$$\mathrm{CR}^{\mathrm{rest}} = \begin{cases} -0.1380\theta_1 + 4.206\theta_2 \le 896.5, \\ 0 \le \theta_1 \le 6000, \\ 0 \le \theta_2. \end{cases} \tag{1.21}$$

Step 7. Define the rest of the region, $\mathrm{CR}^{\mathrm{rest}}$,

$$\mathrm{CR}^R = \begin{cases} -0.1380\theta_1 + 4.206\theta_2 \ge 896.5, \\ 0 \le \theta_1 \le 6000, \\ \theta_2 \le 500. \end{cases} \tag{1.22}$$

Table 1.3 Solution of the refinery example.

i	CR^i	Optimal solution
1	$-0.14\theta_1 + 4.21\theta_2 \leq 896.55$ $0 \leq \theta_1 \leq 6000$ $0 \leq \theta_2$	$\text{Profit}(\theta) = 4.66\theta_1 + 87.52\theta_2 + 286\,758.6$ $x_1 = 1.72\theta_1 - 7.59\theta_2 + 26\,206.90$ $x_2 = -0.86\theta_1 + 13.79\theta_2 + 6896.55$
2	$-0.14\theta_1 + 4.21\theta_2 \geq 896.55$ $0 \leq \theta_1 \leq 6000$ $\theta_2 \leq 500$	$\text{Profit}(\theta) = 7.53\theta_1 + 305\,409.84$ $x_1 = 1.48\theta_1 + 24\,590.16$ $x_2 = -0.41\theta_1 + 9836.07$

Step 8. There is a region to explore, region (1.22). Return to **Step 1** and include constraints (1.22) in the optimization problem (1.19). This problem terminates in the next iteration ending with two critical regions.

Step 9. Collect the two regions. Since they have different solutions, they are not merged.

The solution of this problem is given in Table 1.3 and Fig. 1.3. We can conclude the following:

(i) A complete map of all the optimal solutions, profit and crude oil flowrates as a function of θ_1 and θ_2, is available.

(ii) The space of θ_1 and θ_2 has been divided into two regions, CR^1 and CR^2, where the profiles of profit and flowrates of crude oils remain optimal and hence (a) one does not have to exhaustively enumerate the complete space of θ_1 and θ_2 and (b) the optimal solution can be obtained by simply substituting the value of θ_1 and θ_2 into the parametric profiles without any further optimization calculations.

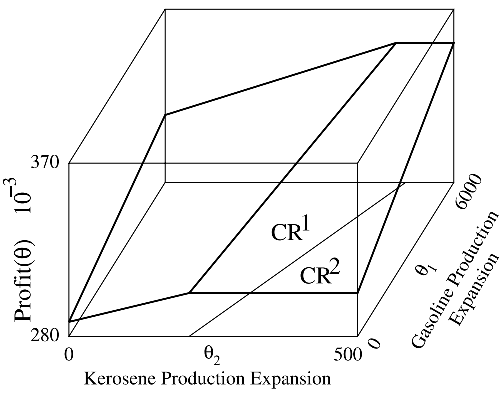

Fig. 1.3 Solution of refinery example.

(iii) The sensitivity of the profit to the parameters can be identified. In CR^1 the profit is more sensitive to θ_2, whereas in CR^2 it is not sensitive to θ_2 at all. Thus, for any value of θ that lies in CR^2, any expansion in kerosene production will not affect the profit.

1.3.2
Example 2: Milk Surplus

A reformulation of the milk surplus production problem is

$$
\begin{aligned}
&\text{Profit} = \max_x \\
&-1.2338x_1^2 - 0.0203x_2^2 - 0.0136x_3^2 - 0.0027x_4^2 + 0.0031x_3x_4 \\
&+2139x_1 + 135x_2 + 103x_3 + 19x_4 \\
&+x_1w_1 + x_2w_2 + x_3w_3 + x_4w_4, \\
\text{s.t.}\quad &-0.0321x_1 - 0.0162x_2 - 0.0038x_3 - 0.0002x_4 \le -80.5, \\
&-0.026w_1 - 0.800w_2 - 0.306w_3 - 0.245w_4, \\
&-0.1061x_1 - 0.0004x_2 - 0.0034x_3 - 0.0006x_4 \le 26.6, \\
&-0.086w_1 - 0.020w_2 - 0.297w_3 - 0.371w_4, \\
&1.2334x_1 \le 2139 + w_1, \\
&0.0203x_2 \le 135 + w_2, \\
&0.0136x_3 - 0.0015x_4 \le 103 + w_3, \\
&-0.0016x_3 + 0.0027x_4 \le 19 + w_4, \\
&0.0163x_1 + 0.0003x_2 + 0.0006x_3 + 0.0002x_4 \le 10 + k, \\
&-150 \le w_1 \le 150, \\
&-5 \le w_2 \le 5, \\
&-6 \le w_3 \le 6, \\
&-2 \le w_4 \le 2, \\
&-1 \le k \le 1.
\end{aligned}
\tag{1.23}
$$

Although formulation (1.23) has cross terms, x_iw_i, introducing an artificial variable s: $x = s - Q^{-1}P^T\theta$, the problem resumes to formulation (1.11). The solution for problem (1.23) is presented in Table 1.4.

Similar to the refinery company, the cooperative society has a complete map of the optimal solution, price of each product, as a function of the bounded parameters, demand and overall price rise. In this way, the cooperative society tackles the variability of the system in a more efficient way.

1.3.3
Example 3: Model-Based Predictive Control

This example is taken from [37] where MPC problems are reformulated as mp-QP problems. The vectors and matrices corresponding to (1.11) are as follows:

$$
c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \; Q = \begin{bmatrix} 0.0196 & 0.0063 \\ 0.0063 & 0.0199 \end{bmatrix};
$$

Table 1.4 Solution of mp-QP Example 2.

#CR	Optimal solution
1	$x_1 = +0.018\,222w_1 + 1.096\,03w_2 + 0.752\,233w_3 + 1.005\,84w_4 + 52.7399k + 418.119$ $x_2 = -1.662\,98w_1 - 50.015w_2 - 17.7007t3 - 15.2865w_4 - 149.711k + 3467.8$ $x_3 = +0.047\,9505w_1 - 6.899\,47w_2 - 11.389w_3 - 5.574\,06w_4 + 170.972k + 2736.15$ $x_4 = +0.865\,523w_1 + 6.3944w_2 - 0.588\,982w_3 - 42.3241w_4 + 413.347k + 2513.17$ Critical region $-150 \leq w_1 \leq 150$ $-5 \leq w_2 \leq 5$ $-6 \leq w_3 \leq 6$ $-2 \leq w_4 \leq 2$ $-1 \leq k \leq 1$ $0.008\,609w_1 + 0.2160w_2 + 0.088\,84w_3 + 0.078\,64w_4 + k \leq 3.089$
2	$x1 = +0.079\,3947w_1 + 0.088\,8125w_2 + 0.337\,98w_3 + 0.639\,184w_4 + 48.0772k + 432.52$ $x2 = +0.088\,8125w_1 + 0.099\,3474w_2 + 0.378\,071w_3 + 0.715\,004w_4 + 53.78k + 2839.29$ $x3 = +0.337\,98w_1 + 0.378\,071w_2 + 1.438\,76w_3 + 2.720\,98w_4 + 204.662k + 2632.1$ $x4 = +0.639\,184w_1 + 0.715\,004w_2 + 2.720\,98w_3 + 5.145\,89w_4 + 387.055k + 2594.38$ Critical region $w_1 \leq 150$ $w_2 \leq 5$ $w_3 \leq 6$ $-2 \leq w_4 \leq 2$ $k \leq 1$ $-0.008\,6087w_1 - 0.216\,013w_2 - 0.088\,843w_3 - 0.078\,635w_4 - k \leq -3.08864$

$$b = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix};\ A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix};\ F = \begin{bmatrix} 5.9302 & 6.8985 \\ -5.9302 & -6.8985 \\ 1.5347 & -6.8272 \\ -1.5347 & 6.8272 \end{bmatrix};$$

and $-1.5 \leq \theta_1 \leq 1.5, -1.5 \leq \theta_2 \leq 1.5$. The solution of this example is given in Table 1.5. This solution is transformed to obtain control variables as a function of state variables.

Concluding, the online model-based predictive control problem reduces to a function evaluation problem — see [37] for details.

1.4
Computational Complexity

Under the assumptions of Theorem 1, at the most n constraints can be active at a point in Θ. Thus, given a set of p constraints, all the possible combinations of active constraints are less than or equal to

$$\eta \triangleq \sum_{i=0}^{n} \binom{p}{I},$$

Table 1.5 Solution of mp-QP Example 2: $x(\theta)^i = W^i\theta + w^i$, $\mathrm{CR}^i : \Phi^i\theta \le \phi^i$.

$$W^1 = \begin{bmatrix} +0.000\,000 & +0.000\,000 \\ +0.000\,000 & +0.000\,000 \end{bmatrix} \qquad w^1 = \begin{bmatrix} +0.000\,000 \\ +0.000\,000 \end{bmatrix}$$

$$\Phi^1 = \begin{bmatrix} -1.0000 & -1.1633 \\ 1.0000 & 1.1633 \\ -1.0000 & 4.4486 \\ 1.0000 & -4.4486 \end{bmatrix} \qquad \phi^1 = \begin{bmatrix} 0.3373 \\ 0.3373 \\ 1.3032 \\ 1.3032 \end{bmatrix}$$

$$W^2 = \begin{bmatrix} 5.9302 & 6.8985 \\ 1.5347 & -6.8272 \end{bmatrix} \qquad w^2 = \begin{bmatrix} 2.0000 \\ -2.0000 \end{bmatrix}$$

$$\Phi^2 = \begin{bmatrix} -1.0000 & 0 \\ 0 & -1.0000 \\ 1.3655 & 1.0000 \\ -1.0000 & 1.3608 \end{bmatrix} \qquad \phi^2 = \begin{bmatrix} 1.5000 \\ 1.5000 \\ -0.2885 \\ -0.4006 \end{bmatrix}$$

$$W^3 = \begin{bmatrix} -0.4933 & 2.1945 \\ 1.5347 & -6.8272 \end{bmatrix} \qquad w^3 = \begin{bmatrix} 0.6429 \\ -2.0000 \end{bmatrix}$$

$$\Phi^3 = \begin{bmatrix} 0 & -1.0000 \\ -1.3655 & -1.0000 \\ 1.0000 & 0 \\ 1.3655 & 1.0000 \\ -1.0000 & 4.4486 \end{bmatrix} \qquad \phi^3 = \begin{bmatrix} 1.5000 \\ 0.2885 \\ 1.5000 \\ 0.5618 \\ -1.3032 \end{bmatrix}$$

$$W^4 = \begin{bmatrix} 5.9302 & 6.8985 \\ -1.8774 & -2.1839 \end{bmatrix} \qquad w^4 = \begin{bmatrix} 2.0000 \\ -0.6332 \end{bmatrix}$$

$$\Phi^4 = \begin{bmatrix} -1.0000 & 0 \\ -1.0000 & 1.3608 \\ 1.0000 & -1.3608 \\ 1.0000 & 1.1633 \end{bmatrix} \qquad \phi^4 = \begin{bmatrix} 1.5000 \\ 0.7717 \\ 0.4006 \\ -0.3373 \end{bmatrix}$$

$$W^5 = \begin{bmatrix} 5.9302 & 6.8985 \\ 1.5347 & -6.8272 \end{bmatrix} \qquad w^5 = \begin{bmatrix} 2.0000 \\ 2.0000 \end{bmatrix}$$

$$\Phi^5 = \begin{bmatrix} -1.0000 & 0 \\ 1.0000 & -1.3608 \\ 1.3655 & 1.0000 \end{bmatrix} \qquad \phi^5 = \begin{bmatrix} 1.5000 \\ -0.7717 \\ -0.5618 \end{bmatrix}$$

$$W^6 = \begin{bmatrix} 5.9302 & 6.8985 \\ 1.5347 & -6.8272 \end{bmatrix} \qquad w^6 = \begin{bmatrix} -2.0000 \\ 2.0000 \end{bmatrix}$$

$$\Phi^6 = \begin{bmatrix} 1.0000 & 0 \\ 0 & 1.0000 \\ -1.3655 & -1.0000 \\ 1.0000 & -1.3608 \end{bmatrix} \qquad \phi^6 = \begin{bmatrix} 1.5000 \\ 1.5000 \\ -0.2885 \\ -0.4006 \end{bmatrix}$$

$$W^7 = \begin{bmatrix} -0.4933 & 2.1945 \\ 1.5347 & -6.8272 \end{bmatrix} \qquad w^7 = \begin{bmatrix} -0.6429 \\ 2.0000 \end{bmatrix}$$

$$\Phi^7 = \begin{bmatrix} 0 & 1.0000 \\ 1.3655 & 1.0000 \\ -1.0000 & 0 \\ -1.3655 & -1.0000 \\ 1.0000 & -4.4486 \end{bmatrix} \qquad \phi^7 = \begin{bmatrix} 1.5000 \\ 0.2885 \\ 1.5000 \\ 0.5618 \\ -1.3032 \end{bmatrix}$$

$$W^8 = \begin{bmatrix} 5.9302 & 6.8985 \\ -1.8774 & -2.1839 \end{bmatrix} \qquad w^8 = \begin{bmatrix} -2.0000 \\ 0.6332 \end{bmatrix}$$

$$\Phi^8 = \begin{bmatrix} 1.0000 & 0 \\ -1.0000 & 1.3608 \\ 1.0000 & -1.3608 \\ -1.0000 & -1.1633 \end{bmatrix} \qquad \phi^8 = \begin{bmatrix} 1.5000 \\ 0.4006 \\ 0.7717 \\ -0.3373 \end{bmatrix}$$

$$W^9 = \begin{bmatrix} 5.9302 & 6.8985 \\ 1.5347 & -6.8272 \end{bmatrix} \qquad w^9 = \begin{bmatrix} -2.0000 \\ -2.0000 \end{bmatrix}$$

$$\Phi^9 = \begin{bmatrix} 1.0000 & 0 \\ -1.0000 & 1.3608 \\ -1.3655 & -1.0000 \end{bmatrix} \qquad \phi^9 = \begin{bmatrix} 1.5000 \\ -0.7717 \\ -0.5618 \end{bmatrix}$$

where

$$\binom{p}{i} = \frac{p!}{(p-i)!i!}.$$

In the worst case, an estimate of η_r, the number of regions, CR, generated can be obtained as follows. The following analysis does not take into account (i) the reduction of redundant constraints, and (ii) possible empty sets are not further partitioned. The first critical region, CR^Q is defined by the constraints given in (1.15). For simplicity assume that CR^{IG} is unbounded. Thus, first CR^Q is defined by p constraints. From Appendix B, CR^{rest} consists of p convex polyhedra CR_l defined by at most p inequalities. For each CR_l, a new CR is determined which consists of $2p$ inequalities (the additional p inequalities come from the condition $CR \subseteq CR_l$), and therefore the corresponding CR^{rest} partition includes $2p$ sets defined by $2p$ inequalities. This way of generating regions can be associated with a search tree. By induction, it is easy to prove that at the tree level $k+1$ there are $k!p^k$ regions defined by $(k+1)p$ constraints. As observed earlier, each CR is the largest set corresponding to a certain combination of active constraints. Therefore, the search tree has a maximum depth of η, as at each level there is one admissible combination less. In conclusion, the number of regions is $\eta_r \leq \sum_{k=0}^{\eta-1} k!p^k$, each one defined by at most ηp linear inequalities.

The algorithm has been fully automated [36] and tested on a number of problems. The computational experience with test problems on a Pentium II-300 MHz computer is given in Tables 1.6 and 1.7.

Table 1.6 Computation time (seconds).

p	n/m	2	3	4	5
4	2	3.02	4.12	5.05	5.33
6	3	10.44	26.75	31.7	70.19
8	4	25.27	60.20	53.93	58.61

Table 1.7 Number of regions.

p	n/m	2	3	4	5
4	2	7	7	7	7
6	3	17	47	29	43
8	4	29	99	121	127

1.5 Concluding Remarks

A sensitivity analysis based algorithm has been presented for the solution of multiparametric linear and quadratic problems. These optimization problems have linear or convex quadratic objective function and linear constraints; the varying parameters are assumed to be additive linear terms on the constraints' right-hand side. Through a systematic partition of the parametric space, the algorithm provides a complete map of the optimal solution as a conditional piecewise linear function of the parameters. Each piecewise function derives from first-order estimation of the analytical nonlinear optimal function. Therefore, the piecewise linear functions are valid inside characteristic regions, defined using the optimality and feasibility conditions. Hence, the core idea of the algorithm is to approximate the analytical nonlinear function by affine functions, whose validity is optimally confined to critical regions. The solution obtained is convex and continuous.

In the context of online optimization, online model-based control and optimization problems involving parametric uncertainty can be reformulated as multiparametric optimization programs. Optimal control actions are computed off-line as functions of the state variables, and the space of state variables is subdivided into characteristic regions. Online optimization is then carried out by taking measurements from the plant, identifying the characteristic region corresponding to these measurements, and then calculating the control actions by simply substituting the values of the measurements into the expression for the control profile corresponding to the identified characteristic region. The online optimization problem thus reduces to a simple map-reading and function evaluation problem. The corresponding computational effort required by this kind of implementation is very small, as no optimization is done online. Benchmark examples have been presented to show the applicability and to describe the proposed procedure.

Acknowledgments

Financial support from EPSRC (GR/T02560/01) and Marie Curie European Project PRISM (MRTN-CT-2004-512233) is gratefully acknowledged.

Appendix A. Redundancy Check for a Set of Linear Constraints

Consider a system of linear constraints:

$$\sum_{j=1}^{N} g_{i,j}\theta_j \leq b_i, \quad i = 1, \ldots, k, \ldots, m. \tag{1.24}$$

Constraint k is redundant if there is a solution for the following problem:

$$\min_{\theta,\epsilon} \epsilon_k, \tag{1.25a}$$

$$\text{s.t.} \quad \sum_{j=1}^{N} g_{i,j}\theta_j + \epsilon_i = b_i, \quad i = 1, \ldots, m, \tag{1.25b}$$

$$\epsilon_i \in \mathbb{R}, \tag{1.25c}$$

such that $\epsilon_k \geq 0$. If $\{\min \epsilon_k\} > 0$, the constraint is said to be strongly redundant; if $\{\min \epsilon_k\} = 0$, simultaneously with another ϵ_i, one of them is said to be weakly redundant.

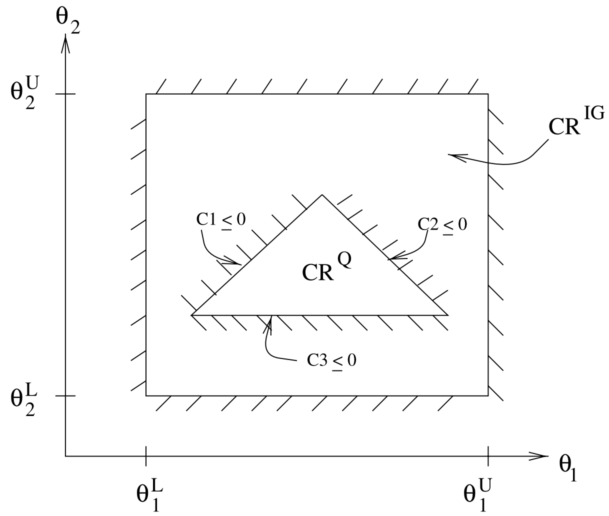

Fig. 1.4 Critical regions, CR^{IG} and CR^{Q}.

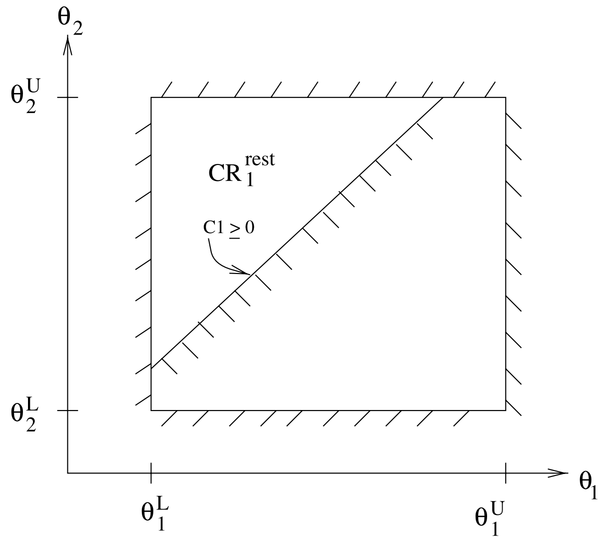

Fig. 1.5 Division of critical regions: Step 1.

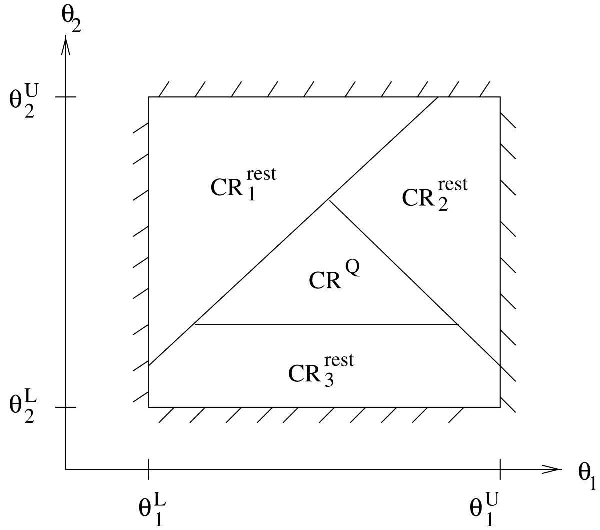

Fig. 1.6 Division of critical regions: rest of the regions.

Appendix B. Definition of Rest of the Region

Given an initial region, CR^{IG} and a region of optimality, CR^{Q} such that $CR^{Q} \subseteq CR^{IG}$, a procedure is described in this section to define the rest of the region, $CR^{rest} = CR^{IG} - CR^{Q}$. For the sake of simplifying the explanation of the procedure, consider the case when only two parameters, θ_1 and θ_2, are present (see Fig. 1.4), where CR^{IG} is defined by the inequalities: $\{\theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2 \leq \theta_2^U\}$ and CR^{Q} is defined by the inequalities: $\{C1 \leq 0, C2 \leq 0, C3 \leq 0\}$ where $C1, C2$, and $C3$ are linear in θ. The procedure consists of considering one by one the inequalities which define CR^{Q}. Considering, for example, the inequality $C1 \leq 0$, the rest of the region is given by, $CR_1^{rest} : \{C1 \geq 0, \theta_1^L \leq \theta_1, \theta_2 \leq \theta_2^U\}$, which is obtained by reversing the sign of inequality $C1 \leq 0$ and removing redundant constraints in CR^{IG} (see Fig. 1.5). Thus, by considering the rest of the inequalities, the complete rest of the region is given by: $CR^{rest} = \{CR_1^{rest} \cup CR_2^{rest} \cup CR_3^{rest}\}$, where CR_1^{rest}, CR_2^{rest} and CR_3^{rest} are given in Table 1.8 and are graphically depicted in Fig. 1.6. Note that for the

Table 1.8 Definition of rest of the regions.

Region	Inequalities
CR_1^{rest}	$C1 \geq 0, \theta_1^L \leq \theta_1, \theta_2 \leq \theta_2^U$
CR_2^{rest}	$C1 \leq 0, C2 \geq 0, \theta_1 \leq \theta_1^U, \theta_2 \leq \theta_2^U$
CR_3^{rest}	$C1 \leq 0, C2 \leq 0, C3 \geq 0, \theta_1^L \leq \theta_1 \leq \theta_1^U, \theta_2^L \leq \theta_2$

case when CR^{IG} is unbounded, simply suppress the inequalities involving CR^{IG} in Table 1.8.

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