ANNA SFARD

THERE IS MORE TO DISCOURSE THAN MEETS THE EARS: LOOKING AT THINKING AS COMMUNICATING TO LEARN MORE ABOUT MATHEMATICAL LEARNING

ABSTRACT. Traditional approaches to research into mathematical thinking, such as the study of misconceptions and tacit models, have brought significant insight into the teaching and learning of mathematics, but have also left many important problems unresolved. In this paper, after taking a close look at two episodes that give rise to a number of difficult questions, I propose to base research on a metaphor of *thinking-as-communicating*. This conceptualization entails viewing learning mathematics as an initiation to a certain well defined *discourse*. Mathematical discourse is made special by two main factors: first, by its exceptional reliance on symbolic artifacts as its *communication-mediating* tools, and second, by the particular *meta-rules* that regulate this type of communication. The meta-rules are the observer's construct and they usually remain tacit for the participants of the discourse. In this paper I argue that by eliciting these special elements of mathematical communication, one has a better chance of accounting for at least some of the still puzzling phenomena. To show how it works, I revisit the episodes presented at the beginning of the paper, reformulate the ensuing questions in the language of thinking-as-communication, and re-address the old quandaries with the help of special analytic tools that help in combining analysis of mathematical content of classroom interaction with attention to meta-level concerns of the participants.

In the domain of mathematics education, the term *discourse* seems these days to be on everyone's lips. It features prominently in research papers, it can be heard in teacher preparation courses, and it appears time and again in a variety of programmatic documents that purport to establish instructional policies (see e.g. NCTM, 2000). All this could be interpreted as showing merely that we became as aware as ever of the importance of mathematical conversation for the success of mathematical learning. In this paper, I will try to show that there is more to discourse than meets the ears, and that putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned. Above all, I will be arguing that communication should be viewed not as a mere aid to thinking, but as almost tantamount to the thinking itself. The *communicational approach to cognition*, which is under scrutiny in this paper, is built around this basic theoretical principle.

In what follows, I present the resulting vision of learning and explain why this conceptualization can be expected to make a significant con-

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tribution to both theory and practice of mathematics education. I begin with taking a close look at two episodes that give rise to a number of difficult questions. The intricacy of the problems serves as the immediate motivation for a critical look at traditional cognitive research, based on the metaphor of learning-as-acquisition, and for the introduction of an additional conceptual framework, grounded in the metaphor of learning-as-participation. In the last part of this article, in order to show how the proposed conceptualization works, I revisit the episodes presented at the beginning of the paper, reformulate the longstanding questions in the new language, and re-address the old quandaries with the help of specially designed analytic tools.

1. Questions We Have Always Been Asking About Mathematical Thinking and Are Still Wondering About

In spite of its being a relatively young discipline, the study of mathematical thinking has a rich and eventful history. Since its birth in the first half of the 20th century, it has been subject to quite a number of major shifts (Kilpatrick, 1992; Sfard, 1997). These days it may well be on its way toward yet another reincarnation. What is it that makes this new field of research so prone to change? Why is it that mathematics education researchers never seem truly satisfied with their own past achievements?

There is certainly more than one reason, and I shall deal with some of them later. For now, let me give a commonsensical answer, likely to be heard from anybody concerned with mathematics education – teachers, students, parents, mathematicians, and just ordinary citizens concerned about the well-being of their children and their society. The immediate suspect, it seems, is the visible gulf between research and practice, expressing itself in the lack of significant, lasting improvement in teaching and learning that the research is supposed to bring. It seems that there is little correlation between the intensity of research and research-based development in a given country and the average level of performance of mathematics students in this country (see e.g. Macnab, 2000; Schmidt et al., 1999; Stigler and Hiebert, 1999). This, in turn, means that as researchers we may have yet a long way to go before our solutions to the most basic problems asked by frustrated mathematics teachers and by desperate students become effective in the long run. The issues we are still puzzled about vary from most general questions regarding our basic assumptions about mathematical learning, to specific everyday queries occasioned by concrete classroom situations. Let me limit myself to just two brief examples of teachers' and researchers' dilemmas.

A function g(x) is partly represented by the table below. Answer the questions in the

x	$g(x)$
0	-5
1	0
2	5
3	10
4	15
5	20

(1) What is $g(6)$? ________
(2) What is $g(10)$? ________
(3) The students in grade 7 were asked to write an expression for the function g(x).
Evan wrote $g(x) = 5(x - 1)$
Amy wrote $g(x) = 3(x - 3) + 2(x - 2)$
Stuart wrote $g(x) = 5x - 5$
Who is right? Why?

Figure 1. *Slope* episode – The activity sheet.

Example 1: Why do children succeed or fail in mathematical tasks? What is the nature and the mechanism of the success and of the failure?

Or, better still, why does mathematics seem so very difficult to learn and why is this learning so prone to failure? This is probably the most obvious among the frequently asked questions, and it can be formulated at many different levels. The example that follows provides an opportunity to observe a 'failure in the making' – an unsuccessful attempt at learning that looks like a rather common everyday occurrence.

Figure 2 shows an excerpt from a conversation between two twelve year old boys, Ari and Gur, grappling together with one of a long series of problems supposed to usher them into algebraic thinking and to help them in learning the notion of function.[1] The boys are dealing with the first question on the worksheet presented in Figure 1. The question requires finding the value of the function $g(x)$, represented by a partial table, for the value of x that does not appear in the table ($g(6)$). Before proceeding, the reader is advised to take a good look at Ari and Gur's exchange and try to answer the most natural questions that come to mind in situation like this: What can be said about the boys' understanding from the way they go about the problem? Does the collaboration contribute in any visible way to their learning? If either of the students experiences difficulty, what is the nature of the problem? How could he be helped? What would be an effective way of overcoming – or preventing altogether – the difficulty he is facing?

While it is not too hard to answer some of these questions, some others seem surprisingly elusive. Indeed, a cursory glance at the transcript is enough to see that while Ari proceeds smoothly and effectively, Gur is unable to cope with the task. Moreover, in spite of Ari's apparently adequate algebraic skills, the conversation that accompanies the process of solving does not seem to help Gur. We can conclude by saying that while Ari's performance is fully satisfactory, Gur does not 'pass the test'.

BERT VAN OERS

EDUCATIONAL FORMS OF INITIATION IN MATHEMATICAL CULTURE

. "Seule l'histoire peut nous débarrasser de l'histoire"
Pierre Bourdieu (1982), **Leçon sur la leçon** (p.9)

ABSTRACT. A review of literature shows that during the history of mathematics education at school the answer of what counts as 'real mathematics' varies. An argument will be given here that defines as 'real mathematics' any activity of participating in a mathematical practice. The acknowledgement of the discursive nature of school practices requires an in-depth analysis of the notion of classroom discourse. For a further analysis of this problem Bakhtin's notion of speech genre is used. The genre particularly functions as a means for the interlocutors for evaluating utterances as a legitimate part of an ongoing mathematical discourse. The notion of speech genre brings a cultural historical dimension in the discourse that is supposed to be acted out by the teacher who demonstrates the tools, rules, and norms that are passed on by a mathematical community. This has several consequences for the role of the teacher. His or her mathematical attitude acts out tendencies emerging from the history of the mathematical community (like systemacy, non-contradiction etc.) that subsequently can be imitated and appropriated by pupils in a discourse. Mathematical attitude is the link between the cultural historical dimension of mathematical practices and individual *mathematical* thinking.

KEY WORDS: activity, tool, discourse, participation, genre, attitude

1. What is Really Mathematical?

'Math' is widely acknowledged as an undisputed part of the school curriculum. Over the past fifty years the classroom approach to mathematics has changed radically from a drill-and-practice affair to a more insight-based problem oriented approach. Every form of mathematics education makes assumptions about what the subject matter of mathematics really is, and – consequently – how the learning individual should relate to other members of the wider culture in order to appropriate this allegedly 'real mathematics', or to put it more directly, to appropriate what is taken to be mathematics in a given community. Part of a school's responsibility is to induct students into communities of knowledge and the teaching of mathematics can be seen as a process of initiating students in the culture of the mathematical community. In fact, students are from the be-

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ginning of their life a member of a community that extensively employs embodiments of mathematical knowledge. The school focuses attention on these embodiments and their underlying insights, and by so doing draws young children into a new world of understanding, with new conventions, rules and tools. So, basically, here is a process of reacculturation in which a student is assisted to switch membership from one culture to another. Buffee's (1993) insightful analysis of this process describes reacculturation as mostly a complex and usually even painful process: "Reacculturation involves giving up, modifying, or renegotiating the language, values, knowledge, mores and so on that are constructed, established, and maintained by the community one is coming from, and becoming fluent instead in the language and so on of another community" (Buffee, 1993, p. 225).

Educational history teaches us that schools have tried to support this reacculturation process in a variety of ways. Underlying these approaches there are different assumptions concerning the nature of mathematics in the classroom, and concerning the way teachers should communicate with their pupils in the classroom. In this article I will try to apply Bakhtin's approach to the discourse in a mathematics classroom, especially focusing on the question of how the participants in this classroom are linked together and what common *background* is to be constructed in order to constitute a way of speaking and interacting that will be acknowledged as a *mathematical* discourse. The final aim is to find a way of describing some of the conditions that must be fulfilled in order to ascertain that the classroom's activity can really count as 'mathematical'. There is, however, no direct empirical way of achieving this just by observing a great number of existing classroom practices and describing the events in Bakhtinian terms. When we view the discipline of 'mathematics' as a "socially conventionalized discursive frame of understanding" (Steinbring, 1998, p. 364), we must also acknowledge – as Steinbring does – that not only factual technical mathematical operations are involved in mathematical activities in classrooms, but epistemological constraints and social conventions are also part of the process. The application of the Bakhtinian jargon requires that the hidden assumptions be brought into the open as they presumably co-determine the style and the course of the discursive process, and the authority and power relationships that are involved.

One of the values that are implicitly or explicitly applied in every mathematics classroom is an idea about what really counts as mathematical. On the basis of these notions mathematics education researchers, curriculum developers and teachers decide what is relevant or even compulsory for taking into account in the mathematics classes and courses. On the basis of their mathematical epistemology, teachers make observations of pupils' activities and select some actions as relevant or not, they value certain actions as 'good' or assess others as false or insignificant (van Oers, 2000b). Obviously, there is some normative idea at stake here about what mathematics really is, or – more modestly formulated – a norm that helps in deciding whether a particular action or utterance may count as 'mathematical' or not: one teacher focuses on number and numerals, another one on structures, while a third may stress the importance of problem solving. Introducing children in one way or another into the world of mathematics and its according speech genre probably implies teaching them the presumptions for identifying what is really mathematical and what isn't.

The idea of what mathematics really is, is of course not just an educational problem. Much of the engagement of the philosophy of mathematics is based on this very same query (see for example Rotman, 1988). Although there is probably often a relationship between the epistemological positions that can be taken with respect to mathematics as an intellectual discipline and one's view on mathematics education, I will directly focus here on the ideas about mathematics in education (school, curriculum).

As Bourdieu (1982) has already argued, education has a very important role to play in the institutionalization of a discipline through implicitly (hidden in the routines or habits of a particular community) or explicitly signaled values that create distinctions between people, and consequently mark some of them as (say) mathematicians or not, mathematically educated or not, etc. In a similar vein I shall argue here that the notion of what is mathematical and what not is developed in education, and the mastery of this value marks significantly those who will be acknowledged as mathematically educated (e.g. who may pass the exams) and who can't. Hence it is essential to find out what kind of conception of mathematics is used, and what the implications are for the relationship between teacher and pupils, as well as for the organization of the classroom discourse in mathematics. Presumably this notion of what is really mathematical in the classroom is one of the basic values that constitutes the speech genre of the mathematical classroom.

2. Views on Mathematics as Subject Matter in Schools

There exist a number of different conceptions about what the mathematical subject matter really is. The real mathematics manifests itself with different faces in the classroom, having different implications for the relationship between teacher (as a representative of culture) and pupils, and *a fortiori*, for the conception of communicating in the mathematics classroom.

STEPHEN LERMAN

CULTURAL, DISCURSIVE PSYCHOLOGY: A SOCIOCULTURAL APPROACH TO STUDYING THE TEACHING AND LEARNING OF MATHEMATICS

ABSTRACT. From a sociocultural perspective an object of research on mathematics teaching and learning can be seen as a particular moment in the zoom of a lens. Researchers focus on a specific part of a complex process whilst taking account of the other views that would be obtained by pulling back or zooming in. Researching teaching and learning mathematics must be seen in the same way. Thus in zooming out researchers address the practices and meanings within which students become school-mathematical actors, whilst zooming in enables a study of mediation and of individual trajectories within the classroom. In each choice of object of research the range of other settings have to be incorporated into the analysis. Such analyses aim to embrace the complexity of the teaching-learning process. This article will present a cultural, discursive psychology for mathematics education that takes language and discursive practices as central in that meanings precede us and we are constituted within language and the associated practices, in the multiple settings within which we grow up and participate.

KEY WORDS: cultural discursive psychology, learning theories, research in mathematics education, social practice, Vygotsky

INTRODUCTION

Researchers in mathematics teaching and learning draw on a range of intellectual resources for explanations, analyses and curriculum designs. The structures and meanings of mathematics (including historical and epistemological studies) and the methods and insights of psychology (especially constructivism) have provided rich theoretical fields for the mathematics education research community. They have not, however, enabled us to engage with schooling as reproduction, nor with culture or power, as they are manifest in the mathematics classroom. Sociology, anthropology and cultural studies provide intellectual resources to address these issues, and they have had their effect on psychology (e.g. Cole, 1996; Harré, 1995; Wells, 1999). In mathematics education, the last few years have seen a growing body of studies drawing on these resources (e.g. Dowling, 1998; Cooper and Dunne, 1999; de Abreu, 1998; Saxe, 1991; Nunes, Schliemann

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and Carraher, 1993; Lerman and Tsatsaroni, 1998; Evans, 2000; Adler, 2001; Lerman, 1998a) (for a more developed analysis see Lerman, 2000b).

In this article I will first describe some of the theories underpinning the move in psychology over the last decade or so to one which is fully cultural and focused on the way in which consciousness is constituted through discourse. I will argue that social practices are discursively constituted, and that people become part of practices as practices become part of them (Lerman, 2000b). Although I will return to this several times in the article, I want to emphasise here that "discourse" is to be taken to include all forms of language, including gesture, signs, artefacts, mimicking, and so on. If one focuses on learning in social practices and the manner in which the physical and cultural tools mediate learning, through all these forms of language, we can speak of 'discursive practices'.

In the second part I will focus on learning. Rather than seeing social factors as *causative* of learning, they can be seen as *constitutive* (Smith, 1993). Learning is about becoming, it is about participation in practices (Wenger, 1998). But people react differently in those practices, and perform their own trajectories through them. In arguing that people are discursively constituted the individual does not disappear; instead, the notion of individuality requires a reinterpretation. In this sense, I want to make it clear that there are a number of approaches to psychology as it relates to education. I find the perspective outlined in this article and well supported in the literature as the most persuasive and powerful, as well as fruitful for research, but other perspectives are also clearly well supported in the literature. The contrasts between sociocultural theories and individualistic ones have been well debated (e.g. Lerman, 1996; Steffe and Thompson, 2000; Lerman, 2000a) and have highlighted the contribution of each. Whilst a complementarity between some of these perspectives is sought by some (e.g. Sfard, 1998), I will take the view that many of these theories present their own world-view in terms of their understanding of human activity and consciousness and therefore notions that are familiar in one setting may need to be redefined in another. As I have argued elsewhere (Lerman, 1996, 2000a) incompatibilities lurk in incautious complementarities. I will, therefore, be advocating a particular view, that of a cultural, discursive psychology, towards which I have been working over a number of years (Lerman, 1998a, b), and not attempting to reach a complementarity with other theoretical frameworks, in particular individualistic psychologies, but I recognise that this is just one possible perspective.

Cobb and colleagues (e.g. Cobb, 2000) have developed what they see as an alternative approach, one that incorporates both psychological and sociocultural theories in a reflexive relationship. "... Each perspective constitutes the background against which mathematical activity is interpreted from the other perspective" (p. 64). The distinction is described as being about 'grain size', which has some similarities to the zoom metaphor that is employed in this article. The danger of their perspective, from my point of view, is that the social context, in the way they see it, cannot account for the forms of behaviour and activity of the individual, except in the important but superficial layer of classroom social norms (and socio-mathematical norms). 'Superficial' here is to be taken to mean the upper surface or layer of positioning in the classroom. Class, gender, ethnicity, race and other dimensions of identity seem to disappear with an appropriate social environment in the classroom. In this article I am arguing that we need an integrated account, one that brings the macro and micro together, one that enables us to examine how social forces such as a liberal-progressive position, affect the development of particular forms of mathematical thinking. I suggest that neither complementary nor emergent views can achieve this integration. In section 1.4 below I discuss a unit of analysis, from a largely Vygotskian position, that attempts to integrate the macro and the micro, and in section 2 I discuss the work of Basil Bernstein who offers an integrated sociological analysis.

In the mathematics classroom, interactions should not be seen as windows on the mind but as discursive contributions that may pull others forward into their increasing participation in mathematical speaking/thinking, in their zones of proximal development. Vygotsky's zone of proximal development is both a framework for the analysis of learning and a metaphor for the learning interaction. Elsewhere (Meira and Lerman, 2001) we have called it a symbolic space. I will outline a set of theoretical tools for the analysis of classroom interactions, drawing on this section. Readers can find initial attempts at such analyses in Lerman (2000c, 2001)

1. Discursive, Cultural Psychology

In the nineteenth century Durkheim and Marx challenged the image of the individual as the source of sense making and as the autonomous builder of her or his own subjectivity. Consciousness was to be seen as the result of social relations; in particular, relations to the means of production.

> It is not the consciousness of men that determines their being but, on the contrary, their social being that determines their consciousness. (Marx, 1859, p. 328/9)

Vygotsky's psychology was an application of Marx's theories to learning, providing a framework whereby the sociocultural roots of thought become internalised in the individual.