CHAPTER 1

MATHEMATICS AND CULTURE

The idea that mathematical objects are in some sense eternal and independent of the flux of history and culture has its roots in the ancient worlds of Pythagoras and Plato. It has survived into modern times, and in one form or another has played a role in the thinking of most students of knowledge and science. The resistance to a sociology of mathematics has not, however, always rested on the naive notion of a "real" Platonic realm of Ideals. It has often stemmed from a fear of or resistance to the relativistic implications of any sociology of knowledge. But the conception of mathematics as a social fact does not entail relativism. Resolving the apparent contradiction between the fact of a recalcitrant reality and the idea that reality is socially constructed requires seeing such notions as mind, consciousness, knowledge, and nature in a new way. One of my objectives is to point in the direction of just such a new perspective. But I will be cautious about claiming that this still embryonic perspective is transparent.

Neither relativists nor realists will find support for their viewpoints in my explorations. The realists may be ready to claim me as a comrade when I announce that I am no enemy of the real world, truth, and objectivity. But the relativists are just as likely to embrace me when I argue that truth, and objective statements are shaped by human hands and brains in arenas of social production. I expect that in the course of reading this book readers will develop an appreciation for, if not a transparent understanding of, the idea that all of our thoughts and actions are social constructs. In any case, however, I ask readers to keep in mind that when I argue that mathematics is social through and through I do not mean that it is somehow "arbitrary" or "random".

A natural starting point for any sociology of mathematics is Oswald Spengler's thesis that each "Culture" has its own conception of number. This is the most dramatic expression of an idea adumbrated in Emile Durkheim's reflections on logical concepts as collective representations, and mirrored in various forms in the ruminations on the "anthropology" of mathematics by Ludwig Wittgenstein and others. It is not necessary to endorse Spengler's concept of the "soul" of a civilization and associated metaphysical postures (often misunderstood, in any case), nor his brand of nationalism (often incorrectly interpreted as "Hitlerian") in order to appreciate his uncompromising explanatory and materialist approach to mathematics.

Spengler's discussion of numbers occurs prominently in Chapter Two of the first volume of *The Decline of the West.* He chooses *number* "to exemplify the way in which a Soul seeks to actualize itself in the picture of its outer world – to show, that is, in how far culture in the 'become' state can express or portray an idea of human existence" (Spengler, 1926: 56). *Number*, "the primary element on which all mathematics rests", is *specifically* chosen "because mathematics, accessible in its full depth only to the very few, holds a quite peculiar position amongst the creations of the mind". Mathematics is "peculiar" because it is simultaneously a "science" ("fuller" and "more comprehensive" than logic), a "true art", and a "metaphysic". There is no a priori reason to agree with Spengler that mathematics is unique in this three-fold way; most if not all students of the sociology or natural history of mathematics assume mathematics is in some way unique among modes of knowing. What is significant is that Spengler's analysis is a formidable attack on the privileged status of mathematics as an intellectual or scholarly discipline. Before proceeding further, it will prove useful to briefly review some of the basic terms Spengler uses in his analysis of history.

Spengler makes several axiomatic distinctions: (1) "becoming" and "the become" (roughly, "process" and "result", or "experience as lived" and "experience as learned"); (2) "alien" (the outer world of sensation) and "proper" (the inner life of feeling); (3) "soul" (the possible, the future, "the still to be accomplished") and "world" (the actual, the past, "the accomplished"); and (4) "Nature" ("the numerable") and "History" ("everything unrelated to mathematics"). Spengler (1926: 59) writes:

> An actuality is Nature in so far as it assigns things-becoming their place as things-become, and History in so far as it orders things-become with reference to their becoming.

"Life" is "the form in which the actualizing of the possible is accomplished" (Spengler, 1926: 59). These ideas are introduced as part of "an immediate inward certainty", that is, basic or elemental facts of consciousness. Waking-consciousness is conceived "structurally" as a "tension of contraries" (Spengler, 1926: 54). These contraries share two important features: (1) they are each units or totalities (and together they form a totality), and (2) they are polarities which by virtue of being extremes establish that there is a potential for many types of "realities" (Spengler, 1926: 55):

> The possibilities that we have of possessing an "outer world" that reflects and attests our proper existence are infinitely numerous and exceedingly heterogeneous, and the purely organic and the purely mechanical worldview ... are only the extreme members of the series.

Finally, it is important to understand that Spengler uses Culture in a specific sense, a sense different from that associated with the anthropological concept of culture. When Spengler claims that "primitive man" has no Culture, he means that a "real knowledge of history and nature" is lacking. Only when the ensemble of self, history, and nature becomes separated for the waking-consciousness can we speak of Culture (Spengler, 1926: 55). In much the same way, Marx distinguished all human activity up to the threshold of communism as "prehistory", and communism as the beginning of "human history". This does not mean that there has not been any "history" in the conventional sense; nor does Spengler mean that there are human societies without "culture".

In order to appreciate Spengler's notion of number, it is important to understand that he conceives of a "fundamental connexion between *the become* (*the hard set*) and *Death*" (Spengler, 1926: 54). He then argues (Spengler, 1926: 56–57):

> The real secret of all things-become, which are *ipso facto* things extended (spatially and materially), is embodied in mathematical number as contrasted with chronological number. Mathematical number contains in its very essence the notion of a *mechanical demarcation*, number being in that respect akin to *word*, which, in the very fact of its comprising and denoting, fences off world-impressions.
> In number, then, as the *sign of completed demarcation*, lies the *essence* of everything actual, which is cognized, is delimited, and has become all at once – as Pythagoras and certain others have been able to see with complete inward certitude by a mighty and truly religious intuition.

The connection between mathematics and religion suggested in Spengler's reference to the religious intuition of Pythagoras is implied in the conception of mathematics as a world view.

Number, according to Spengler, is "the symbol of causal necessity". Number and the conception of God both contain "the ultimate meaning of the world-as-nature". The deep affinity between religion and mathematics is clearly evident in the Pythagoreans and the Platonists. But it is also present in Descartes, Pascal, and Leibniz (Spengler, 1926: 66). Religious intuition, Spengler argues, is behind the great mathematical discoveries of the greatest mathematical thinkers – "the creative artists of the realm of numbers" – in all Cultures. These people who experience the spirit of number living within themselves realize that they "know God"; Number is akin to God, and it is related to myth insofar as it originated in the "naming process" associated with the will to "power over the world" (Spengler, 1926: 56–57). There is a clear rationale for Spengler's conjecture on the relationship between mathematics and religion in the cases he cites as well as in such cases as the relationship between the medieval discourses on infinity motivated by theological questions and the development of the calculus, the differences between early British and modern algebra, and the relationship between theology and mathematics in the works of Boole, Cantor, and others. These cases will be discussed later in this book.

Cultures, according to Spengler, are incommensurable. Our present minds, he argues, are "differently constituted" than minds in earlier Cultures. Therefore, earlier mathematical events should not be viewed as stages in the development of "Mathematics" (Spengler, 1926: 57). The two major Cultures Spengler identifies are Classical and Western. He also identifies two minor Cultures: Babylonian-Egyptian and Arabian-Islamic (Indian and Chinese Cultures are also recognized in his schema). Each major Culture experiences the same birth-death sequence in its number-world: (1) *conception* of a new number form, (2) *zenith* of systematic development, and (3) *inward completion and conclusion* of a figure-world. In Classical Culture, the sequence is: (1) the Pythagorean conception of number as "magnitude"; (2) the achievement of the zenith between 450BCE and 350BCE in the works of Plato, Archytas, and Eudoxus; and (3) the inward completion in the works of Euclid, Apollonius, and Archimedes between 300BCE and 250BCE. In Western Culture, the sequence is: (1) the conception of number as "relation" (Descartes, Pascal, Fermat, Newton, Leibniz) in the seventeenth century; (2) the zenith achieved by Euler, Lagrange, and Laplace (1750–1800); and (3) the inward completion achieved from 1800 onwards by Gauss, Cauchy, and Riemann. Let us examine these differences in more detail.

Classical mathematics deals with number as magnitude, as the essence of what can be perceived through the senses, that is, viable, tangible units.

It is confined to facts in the present that are near, and small, with a focus on the properties of individual entities and their boundary surfaces (stereometry, or solid geometry). In general, it is confined to positive and whole numbers, and proportion as the nexus of magnitude. Western mathematics liberates geometry from the visual and algebra from magnitude. Numbers are images of "pure thought" (or "desensualized understanding"), and their abstract validity is self-contained. The focus is on whole classes of formal possibilities, groups of functions, and other *relations*; function is the nexus of relations. Whereas Classical mathematics affirms appearances, Western mathematics denies them; thus the opposition between fear of the irrational in Classical mathematics and the central role of the analysis of the infinite in Western mathematics. In Classical mathematics, the straight line is a measurable edge; in Western mathematics it is an infinite continuum of points – indeed, the core unit of Western mathematics is, Spengler argues, the "abstract space-element of the point", and the main theoretical objective is the interpretation of *space* (a "great and wholly religious symbol", in Spengler's view). Whereas enlargements and reductions of scale and the constancy of constituents are characteristic of Classical mathematics, Western mathematics is based on group transformations and the variability of constituents. In Classical mathematics, the equality sign in

$$3^2 + 4^2 = 5^2$$

establishes a rigid relationship between specific amplitudes, and signals that a problem is being worked out to a result. In Western mathematics

$$X^n + Y^n = Z^n$$

is Classical in appearance but is in reality a new kind of number. It is a picture and sign of a relation – the equality sign does not point to a result in the Classical sense (and because of this, Spengler argues, a new symbolism is needed in order to eliminate the vestigial and confusing parallels with Classical mathematics).

Spengler's characterization of Classical and Western polarizes differences and so underscores his view of the incommensurability between the Greek concept of number and the concept(s) of number fashionable among professional mathematicians from the late nineteenth century onward. But when Spengler contrasts Classical and Western, the latter label refers already to Descartes, Pascal, Fermat, and Leibniz. Earlier, the linkage between Classical and Western was much stronger.

CHAPTER 2

MATHEMATICS FROM THE GROUND UP

The social activities of everyday life gave rise to arithmetic and geometry, the two major forms of mathematical work, in all the ancient civilizations. The development of arithmetic was associated with problems in accounting, taxation, and trade, as well as with certain problems in religious, magical, artistic, and astronomical activities. Problems in land surveying, construction and engineering in general gave rise to geometry. Arithmetic and geometry appear in association with the rise of literacy. The emergence and development of these mathematical systems in particular civilizations is the product of both diffusion and independent invention.

It is now possible to *argue* that even fundamental geometrical shapes such as the circle and the square originated in social activities. According to A. Seidenberg's (1981) admittedly controversial conjecture, participants in early human societies identified with and imitated the motions of the stars. The ritual scene thus took on a circular shape; and this, Seidenberg argues, is the origin of the circle. The circle was eventually bisected and split into quadrants representing four basic organizational sectors within a society. Representatives of the four sectors "placed themselves about the center of the circle in positions corresponding to the positions of the four sections [of the circle], thereby giving rise to the square. The square was valued as a figure dual to the circle" (Seidenberg, 1981:324).

The *discipline* of mathematics emerged when and to the extent that sets of arithmetic and geometrical problems were assembled for purposes of codification and teaching, and to facilitate mathematical studies. Assembling problems was an important step toward unifying mathematics and stimulating abstraction. An even more important step was the effort to state general rules for solving all problems of a given type. A further step could be taken once problems were arranged so that they could be treated in more general and abstract terms. Problems that had arisen in practical settings could now be transformed into purely hypothetical puzzles, and problems could be invented without explicit reference to practical issues. The three famous puzzles proposed by Greek geometers of the 5th and 4th centuries BCE are among the earliest examples of such puzzles: to double the volume of a cube (duplication of the cube), to construct a square with the same area as a given circle (quadrature of the circle), and to divide a given angle into three equal parts (trisection of the angle). Such problems may have been related to the non-mathematical riddles religious oracles commonly posed for one another. One account of the origin of the problem of duplicating the cube, for example, is that the oracle at Delos, in reply to an appeal from the Athenians concerning the plague of 430 BCE, recommended doubling the size of the altar of Apollo. The altar was a cube. The early Hindu literature already refers to problems about the size and shape of altars, and these may have been transmitted to Greece by the Pythagoreans, a secret religious and political society. The problem is also a translation into spatial geometric algebra of the Babylonian cubic equation $x^3 = v$.

The duplication, quadrature, and trisection problems were popular with the Sophists, who made a specialty out of debates of all kinds. A generation or two later, Plato introduced the constraint that the only valid solutions to these problems were those in which only an unmarked straightedge and a compass were used. This meant that special mechanical devices for drawing geometrical forms could not be used in mathematical competitions. The goal was apparently to control the competitive process and make it more rigorous by stressing intellectual means and "purely gentlemanly" norms. This development was related to social factors in the Platonist era. Plato's Academy was organized to help an elite group of intellectuals gain political power; and it represented the opposition of an aristocracy to democratization and commercialization. It is not surprising that this elite group of intellectuals developed an ideology of extreme intellectual purity, glorifying the extreme separation of hand and brain in the slave economy of classical Greece.

The three famous Greek puzzles and other problems became the basis of a mathematical game of challenge-and-response. Various forms of this game are important throughout most of the subsequent history of Western mathematics. Prior to the nineteenth and twentieth centuries, however, such competitions were often initiated, endorsed, or rewarded by patrons, scientific academies, and governments. Prizes were sometimes offered for solutions to practical problems. Economic concerns as well as governmental prestige were often mixed in with the struggles for intellectual preeminence.

At about the same time that they initiated mathematical contests, the Greek mathematicians took two further steps that led to new mathematical forms. They stipulated that a formal, logical mode of argument must be used in solving problems. This represented an extension of earlier methods of proof. And by extending the proof idea, the mathematicians created *systems* of interrelated proofs. This culminated in the *Elements* of Euclid shortly after 300 BCE. In addition to a collection of problems, Euclid presented an explicit body of abstractions in the form of definitions, postulates, and axioms. Euclid, like Aristotle, did not use the term "axiom" but something closer to "common notion". They both self-consciously worked at codifying past human experiences. The process of "systematization-and-abstraction" is one of the two major paths to new mathematical forms. The other major path is an "empirical" one.

The empirical path to new mathematical forms involves applying existing mathematical concepts and methods to new areas of experience. Most of the early Greek geometrical puzzles, for example, concerned flat figures. But the methods of plane geometry could be easily extended to solid geometry, and then to the properties of spheres or of conic sections; the work on conic sections eventually led to work on curves of various shapes. The intermittent periods of creativity in Alexandrian mathematics (especially from 300 to 200 BCE, and 150–200) were largely devoted to these extensions. No new level of abstraction (with the exception of trigonometry, considered below) was reached, but a number of new specialties appeared.

In arithmetical work the effort to find general rules for solving numerical problems led gradually to what we call algebra. Here again we see mathematicians developing the practice of posing problems primarily to challenge other mathematicians. For example, there is the famous problem, attributed to Archimedes (287 to 212 BCE): find the number of bulls and cows of various colors in a herd, if the number of white cows is one third plus one quarter of the total number of black cattle; the number of black bulls is one quarter plus one fifth the number of the spotted bulls in excess of the number of brown bulls, etc. Such problems, involving unknown quantities, led over a very long period to the introduction of various kinds of notations and symbolisms. These took quite different directions in ancient and medieval China and India, the Arab world, and later in medieval and Renaissance Europe. The creation of a highly abstract symbolism which could be mechanically manipulated to find solutions did not appear until the late 1500s and 1600s in Europe. To different degrees in different parts of the world, algebra developed through empirical extension. Problems were deliberately created to increase the number of unknowns, and to raise them to successively higher powers. Equations of the form $ax + b = c$ gave way to equations such as $ax^4 + by^3 + cz^2 = 9$. The complexity of these equations, of course, could be extended indefinitely (in the sixteenth century, for example, Vieta was challenged to solve an equation involving x^{45}) but the extensions also gave rise to efforts to find general rules for solving higher order equations. (That is, empirical extensions tended to promote abstract extensions). Of course, concrete problems in areas such as astronomy could lead to "complicated" equations. But the point here is that the social network itself could foster such developments more or less directly.

At the same time, arithmetic was developing in other directions. Elementary arithmetic (solving numerical problems in, for example, addition, subtraction, multiplication, and division) continued to stimulate efforts to find general rules for solving particular problems. There was tremendous variation from one system of numerical symbols and calculating rules to another in terms of the ease or difficulty with which they could be applied to solving practical problems. Most of the ancient forms of notation made working with large numbers, fractions, or complex operations like division or the extraction of roots difficult; the exposition of problems was usually rhetorical, that is, problems were expressed in words. A great deal of mathematical creativity went into the development of notational systems that could be readily manipulated. Among the most important of these innovations were the invention of decimal place notation and the zero sign in India; the standardization of positional methods for multiplication and division (in early seventeenth century Europe); and the invention of logarithms by Napier in 1614 for use in astronomy, navigation, and commerce.

A different development in arithmetic led to what we now call "number theory". This focused on the properties of numbers themselves. As early as Eratosthenes (ca. 230BCE), efforts were made to find prime numbers and to produce a general formula for doing so. There were also various propositions about how numbers are composed of other numbers. The work by the Pythagoreans on "triangular" and "square" numbers, for example, anticipated work that led to Fermat's famous theorem that every prime number of the form $4n + 1$ is a sum of two squares. Number theory was particularly popular in the Alexandrian period in an occultist, cabalistic form. In its more standard puzzle-solving form, it has remained popular among mathematicians from the Renaissance through the twentieth century.

One more branch of mathematics, based on a combination of arithmetic and geometry, developed in the Alexandrian period. Measuring angles and lines, and calculating their ratios, led to the creation of trigonometry (notably in the works of Hipparchus, ca. 140 BCE, and Menelaus ca. 100 BCE). Trigonometry spread to medieval India and the Arab world, and in Renaissance Europe provided the basis of Napier's development of logarithms.

The creation of new fields continued in modern Europe. They grew out of the processes of abstraction, the extension of results to new empirical areas, and the combination of existing mathematical fields into hybrid fields. The combination of algebra with a new coordinate representation in geometry by Descartes and Fermat produced analytic geometry. Consideration of the problems of motion and the study of curves gave rise to the calculus in the 1600s. Calculus was then applied to successively more complex functions (empirical extension); and eventually (in the 1800s) it was generalized into an abstract theory concerning such things as the rules for solving equations, and the general properties of all functions (abstract extension). The drive towards creating new fields by abstraction and extension seems to be characteristic of highly competitive periods.

Geometry itself experienced a rapid series of branchings around 1800 and thereafter, the best known being the non-Euclidean geometries. Other developments included descriptive geometry (Monge), projective geometry (Poncelet), higher analytical geometry (Plucker), modern synthetic geometry (Steiner and von Staudt), and topology (Mobius, Klein, and Poincaré). In the late nineteenth and early twentieth centuries, systems unifying these different geometries were developed by Klein, Hilbert, and E. Cartan.

In algebra, there was a parallel set of developments after 1800. The effort to find a general solution for the quintic and other higher order equations led to the creation of the theory of groups by Abel, Galois, Cauchy, and others. This theory focused on an abstract pattern among the coefficients of equations, and opened up a new area of inquiry in abstract mathematics.

Abstract algebras were created by Boole, Cayley, Sylvester, Hamilton, and Grassman. All of these new tools were applied to other branches of mathematics. Dedekind applied set theory to the calculus, Cantor applied it to the concept of infinity, and others applied it to topology, number theory, and geometry. These developments led to the creation of yet

CHAPTER 3

THE MATHEMATICS OF SURVIVAL IN CHINA

The earliest sign of mathematical activity in China is the legend of Yü the Great and the Lo River tortoise. According to this legend, Emperor Yü received a divine gift in the form of a magic square – the Lo Shu diagram – etched into the tortoise's shell. The earliest archeological indication of Chinese mathematical activity is the tally and code symbols carved on oracle bones in the Shang period, about thirty-four hundred years ago. By the time of Han (2nd century BCE to the 4th century), the Chinese had developed a codified notation and were computing with a counting board and rods. The ideographic script used by the Chinese may have encouraged their advances in notation and computing. Needham (1959: 13) conjectures (following Derek Price) that civilizations which had alphabets available for constructing numerals were prompted to break out of the "ten fingers" constraint and develop systems of more than 9 numerals. The Shang Chinese appear to be the first people able to express any number using no more than nine numerals.

The oldest extant Chinese manuscript which is of mathematical interest is the *Chou Pei Suan Ching* ("Arithmetic Classic of the Gnomon and the Circular Paths of Heaven"). It pre-dates the 3rd century BCE, and includes what we know as the "Pythagorean" theorem, empirical geometry, and basic operations with fractions using the concept of a common denominator. Numbers are expressed in words.

The practical significance of the "Pythagorean" theorem is explained in this manuscript by Shang Kao (Needham, 1959: 23). A plane triangle (right-angled) laid on the ground is used to construct things "straight and square" (using cords). Heights are observed using a recumbent triangle. A reversed triangle is used to measure depth. And a flat triangle serves to determine distance. A circle can be drawn by revolving a triangle; and squares and oblongs can be formed by uniting triangles; Shang Kao concludes:

He who understands the earth is a wise man, and he who understands the heavens is a sage. Knowledge is derived from the straight line (the shadow). The straight line is derived from the right angle. And the combination of the right angle with numbers is what guides and rules the ten thousand things.

The *Chou Pei* begins as a discussion involving the Duke of Chou, Chou Kung, and Shang Kao. Later, the dialogue is between Chhen Izu and Jung Fang. These speakers gradually fade, and are replaced by the phrases "According to the method" (Fa yeh), and "According to the art" (Shu yeh). These are, according to Needham (1959: 21), apparently intercalated texts. This sort of routinizing or objectifying process tends to occur whenever a certain degree of generational continuity occurs in a field. The substitution of general expressions for persons or attributed statements is a fundamental aspect of the abstraction process and the source of many of the difficulties we have understanding the grounds for pure or formal statements. I discuss this issue in later chapters.

One of the most celebrated texts in the Chinese mathematical literature is the *Chiu Chang Suan Shu*, variously referred to in English as "Arithmetic in Nine Sections", "Nine Chapters on Mathematical Art", and "Nine Chapters on Mathematical Techniques". It is also called the *Chiu-chang suan ching*, "Mathematical Manual in Nine Chapters". This text is said by some authorities to be a compilation produced in the third or second century BCE. Chang Ts'ang (fl. 165–152BCE) may have been the person who initially compiled this manual. But in fact the compilation may have been written as late as the period 50BCE to 100CE. Different sections, of course, may have been written at different times. The version we have now was probably extant by the first century CE at the latest. There were many commentaries on the manual, but the two that stand out are by Liu Hui (3rd century), and Li ch'un-feng (seventh century) (Needham, 1959: 24ff; Libbrecht, 1973: 267ff.; Yan and Du Shiran, 1987: 33ff.).

The *Chiu Chang Suan Shu* had the same sort of influence on Chinese mathematics that Euclid's *Elements* had on Greek mathematics, even to the extent of inhibiting mathematical developments. This was a result of the degree to which it systematized mathematics and became a "classic". It was studied by a wide range of scholars (Needham, 1959: 27). Liu Hui's commentary in particular influenced the course of Chinese mathematics for more than a millenium. He also wrote another important but much shorter work, the *Hai-tao suan-ching*, "Sea Island Mathematical Manual" in 263. Liu Hui's works were included in the *Suan-ching shih-shu*, "Ten mathematical Manuals", of 656. Li Shun-feng (602–670), the

T'ang mathematical worker and astronomer, annotated and added commentaries to these works, and they eventually became standard texts. According to official regulations, mathematics students had to spend three years studying Liu Hui's works. His works were also prescribed texts in the Japanese schools opened in 702. Very little is known about Liu Hui's life besides the facts that he lived in the kingdom of Wei toward the end of the Three Kingdoms period (221–265), and that he was well-known for his writings on mathematics.

The *Chiu Chang Suan Shu* was designed as a handbook for architects, engineers, officials, and merchants. Thus, many of the two-hundred and forty six problems in the nine chapters deal with building canals, dikes, and city walls, taxation, barter, public services, and other aspects of everyday life. The nine chapter headings are (1) Land Surveying (rules for finding areas: and for adding, subtracting, multiplying, and dividing fractions), (2) Millet and Rice (Percentages and proportions), (3) Distribution by Progression (distributing properties among partners according to given rates, taxing goods which vary in quality, and other problems involving arithmetical and geometrical progressions – all solved using proportions), (4) Diminishing Breadth (for example, finding the sides of a rectangle given the area and one side; finding the circumference of a circle given the area; extracting square and cube roots), (5) Consultations on Engineering Work (volumes of prisms, pyramids, wedges, cones, and other solids), (6) Impartial Taxation (deals with problems of pursuit and allegation such as how long it takes taxpayers to transport their grain contributions to the capitol; and ratio problems related to allocating tax burdens according to population size), (7) Excess and Deficiency (use of the rule of "false position" to solve problems of the type $ax + b = 0$), (8) Calculation by Tabulation (simultaneous linear equations, using positive and negative numbers), and (9) Right Angles (applications of the "Pythagorean" Theorem). There are several notable problems in this manuscript. Problem 18, Chapter 8, with its five unknowns and four equations, heralds the indeterminate equation. Problem 13, Chapter 9, is similar to a problem found in the writings of Brahmagupta (seventh century): "There is a bamboo 10 feet high. When bent, the upper end touches the ground 3 feet away from the stem. Find the height of the break". Problem 20, Chapter 9 seems to require solving the quadratic equation

$$x^2 + (20 + 14)x - 2(20)1775 = 0;$$

however, there is no method given for solving the problem.

Procedures for extracting square roots were known in the ancient Eastern and Western civilizations. The methods were derived in basically the same way, from the equation,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

The Greek method was geometrical; it was based entirely on the two-dimensional Euclidean figure. Apparently, no efforts were made to generalize this method by constructing three-dimensional figures and finding cube roots. Very little was known about extracting cube roots in the West until the sixteenth century. The Chinese method for extracting square roots was more advanced than the Greek method in some respects, and more abstract. It was thus possible for mathematical workers to eventually generalize the square root method – by introducing the coefficient of the highest power x^2 as the *hsia fa* (literally, the lowest divisor) – and applying it in the solution of any numerical equation. Liu Hui and Li Shun-feng, commentators on the *Chiu chang suan shu*, give some indication that cubical blocks may have been used to demonstrate the cube root method (Lam Lay Yong, 1977: 56).

The nine problems (chapters) of Liu Hui's *Hai-tao suan ching*, "The Sea Island Mathematical Manual", deal with everyday applications of a classical Chinese method for determining the distance from the sun to a flat earth using double triangulation. The manuscript was originally known as the *Ch'ung ch'a*, "Method of Double Difference" (referring to double or repeated application of proportions to sides of right triangles). It was appended to the *Chiu-chang suan-shu* as Chapter 10, then separated from it in the seventh century when the ten mathematical manuals were selected and renamed, probably, after the first problem which concerns an island. The problems involve finding (1) the elevation of an island and its distance from a pole; (2) the height of a tree on a hill; (3) the size of a distant walled city, (4) the depth of a valley, (5) the height of a tower on a plain as seen from a hill, (6) the width of a gulf seen from a distance on land, (7) the depth of water at the bottom of an abyss, (8) the width of a river seen from a hill, and (9) the size of a city seen from a mountain. All of these problems were worked out *rhetorically*; no algebraic generalizations were arrived at – none were thought necessary by the Chinese calculators.

Many Han mathematical texts are lost, including the *Li Li Suan Fa*, "Mathematical Methods Concerned With Pitchpipes and the Calendar" (author unknown); and the *Suan shu* books of Tu Chung and Hsu Shang (1st century BCE). (How much of this is due to the burning of the books in 213BCE is not clear; the Confucianist writings were the target, and books on practical subjects were not so much affected. One copy of each of the burned books was kept for the State Library, but that library was burnt down during the violence at the end of the Ch'in dynasty; Eberhard, 1971: 66). With the beginning of the Common Era, and the end of the Han dynasty, China entered a period during which very little progress was made in mathematics. There was very little significant mathematical activity between the third and sixth centuries. The mathematical work carried out during this period, including the writing of texts, followed (and repeated the mistakes in) the *Chiu chang suan shu* and did not further the achievements recorded in that manuscript. The *Sun Tsu Suan-ching* ("Arithmetical Classic of Sun Tsu") was written between 280 and 473.

Toward the end of the second century, the *Shu-shu chi-i*, "Memoir on Some Traditions of Mathematical Art", written by Hsu Yeh (or Hsu Yo), appeared. Because of references in this text to large numbers and their presentation in the form of arithmetical series, many scholars regard it as having been fabricated after the introduction of Indian ideas about *Kalpas*; the fabrication may even have been carried out by the commentator Chen Luan, who flourished between 560 and 580. Chen Luan was a Buddhist convert and there are references to Buddhism in his commentary. The text is also closer to Taoism and divination than earlier books. Needham dates this text around 190. He points out that the Indian system has very different progressions and nomenclatures. There is some evidence of Chinese opposition to associating mathematics and mysticism, and to Buddhist large numbers in particular. Needham (1950: 88) cites Shen Tso-Chi, a twelfth century scholar on this point:

> It is only things which are beyond shape and number (*hsiang shu*) which cannot be investigated. How can there be mathematics beyond the reach of shape and number?

The following passage from this text illustrates the influence of the counting board on how the Chinese thought and wrote about calculating:

> In making calculations we must first know the position (and structure) (*wei*) (of

CHAPTER 4

MATHEMATICS IN CONTEXT: THE ARABIC-ISLAMIC GOLDEN AGE

THE ARABIC WORLD BEFORE MOHAMMED

Arabia, on the eve of Mohammed's birth and the growth and spread of Islam, was a collection of regional cultures separated by language, religion, and politics. Neither Christianity nor Zoroastrianism nor the regional Arabian and African religions were capable of breaking down the regional cultural barriers and forging a civilizational area. The unifying mission of Mani in the third century had only limited success. Significantly, there were various political, economic, and intellectual resources spread throughout these regions. These resources were consolidated during the Mohammedan expansion and formed the basis of the cultural growth of the Arabic-Islamic world from the 700s on.

When Justinian closed the School of Athens in 529, the School's philosophers migrated to Baghdad. In Persia, Khosur the Holy, a contemporary of Justinian and a patron of science, invited Greek scholars to his court. Christian monasteries, centers of learning, were scattered throughout the Near East. As it spread, Islam linked Greek, Christian, Persian, and Babylonian traditions, as well as Indian and Chinese traditions.

The Arabs possessed no numerals before the time of Mohammed (sixth century); numbers were written in words. Before the end of the eighth century, they possessed a good numerical notation and Brahmagupta's work on arithmetic and algebra. Before the end of the ninth century, they were in possession of the major Greek mathematical works.

Agriculture and commerce arose in central Arabia at the beginning of the Christian era. Jews escaping Babylonian persecution and, later, the destruction of Jerusalem by Titus and Hadrian, entered Arabia as craftsworkers, goldsmiths, swordsmiths, and traders. Heretical Nestorians, Coptic Monophysites and other Christians also entered Arabia in the centuries before the Islamic expansion. Jews and Christians struggled to gain control of the Yemen spice trade. This struggle culminated in a raid on Mecca by Abyssinians around the time Mohammed was born.

Mohammed helped to stabilize Arabia. Blood-feuds were pacified. The power focused on petty strifes that divided the population of the peninsula became concentrated in "one hand" ready and eager to attack the powerful neighbors who had despised the weak and barbaric desert people for two millennia (Darlington, 1969: 339)

ISLAMIC EXPANSION

Trade revived, and the status and power of the merchants improved as the Islamic conquests restored, extended, and decentralized the trade network lost by the troubled Roman Empire. Old cities such as Damascus, Alexandria, and Antioch were revived and served as models for new cities, especially the new capitols of Cairo, Baghdad, and Cordoba. These events parallel the flowering of "mother-cities" such as Miletus and more or less autonomous satellite cities during the Ionian commercial revolution of the seventh century BCE.

The early merchants were mostly Christians, Jews, and Zoroastrians; later Moslems and Arabs became the dominant traders (they disdained agriculture but not trade). Port cities such as Baghdad, al-Barash, Siraf, Cairo, and Alexandria became active commercial centers (Hitti, 1956: 343–347).

The Caspian Sea, close to the Persian cultural centers and the wealthy cities of Samarkand and Bukhara, was the major scene of maritime commerce in Arabia. Mediterranean trade was never very prominent; neither was trade across the Black Sea (though there was some land trade in the tenth century with the northern Volga regions). Moslem traders, according to Arab tradition, reached China from al-Varash during the reign of al-Mansur. In any case, several embassies had been exchanged between China and the Arabs by the middle of the eight century.

In 751, the Arab governors of Samarkand captured some visiting Chinese paper experts. The Arabs learned about paper-making and eventually made a linen paper from Khorasan flax that was tougher than the paper the Chinese made from bamboo fibers. Linen paper replaced costly papyrus and parchment, and palm leaves. This greatly facilitated the transcription and reproduction of the Arabic translations of the Greek manuscripts. By the end of the eighth century, Baghdad had its first paper mill.

Darlington (1969: 351) argues that while war accounts for the expansion of Islam, it does not account for Islam's stimulation of cultural growth. He draws attention to the fact that stimulation only occurred where there was "something valuable before the coming of Islam". In each case, the "flowering of culture" lasts about six generations. This may reflect a "recombination effect"; that is, each conquest linked mutually stimulating cultural traditions, and an immediate "recombination effect gave rise to cultural innovation". However, neither Islam nor the original cultures were organized to sustain cultural growth:

So it was that successively in Damascus and Baghdad, in Cordoba and Marrakesh, in Isfahan and Delhi, we see the characteristic flaw of the new hybrid Islamic civilizations always based on a precarious balance between conversion and non-conversion, hybridization and non-hybridization, a balance which Muslim violence was not fitted to sustain. When the conquest ceased, with the expulsion of the Almoravids from Granada, with the retreat of the Turks from Vienna, with the collapse of the Moguls in Delhi, the intellectual and artistic as well as the political life of Islam came to a standstill.

THE CALIPHS

The Omayyads ruled Arabia from 661 to 750. An Arabic currency was introduced between 685 and 705. Between 705 and 715, the Omayyads reached the height of their power. In 750, the Abbasid family overthrew the Omayyads. The Omayyad Abd-al-Hathman escaped, and established the Omayyads in Spain, at Cordoba. The Abbasids ruled Arabia from 750 until the Mongol invasion of Persia in 1256 and the sacking of Baghdad in 1258.

Abu'l-Abbas was the first Abbasid Caliph. He died in 754. Al-Mansur (who reigned from 754–775) is considered the actual founder of the Abbasid dynasty. He established his capitol at Baghdad in 762. Al-Mansur intended to make Baghdad a military post for himself and his family, and for his Khorasanian bodyguard. The post was built under the auspicious horoscope of the court astrologer. It was under the rule of Harun al-Rashid (786–809) that Arabic-Islamic mathematics began to develop, under Indo-Persian, Syrian, and especially Hellenic influences.

The location of Baghdad made it a center for the transmission of Persian culture into the Arab world. The Caliphate was modelled after the Sasanid Chosroism. Slowly, Persian ideas and ways of life penetrated the

Caliphate until it became, in Hitti's (1956: 294) words, "more of a revival of Iranian despotism and less of an Arabian Sheikdom". Within three generations after al-Mansur set the foundation stone for his military post at Baghdad, the major works of Aristotle (many of them spurious), the leading neo-Platonist treatises, most of Galen's writings, and many Persian and Indian scientific works were available in Arabic translations from the Persian, Sanskrit, Syriac, and Greek. Merchants were central figures in the Baghdad community. So were scholars, if we can judge from a contemporary account of the daily routine of Ishaq ibn Hsuein, one of the great translators, by Ibn-Kallikan (Hitti, 1956: 306).

HELLENISM, THE GREAT TRANSLATORS, AND INTELLECTUAL LIFE

The diffusion of Hellenism into Arabic-Islamic culture occurred through Arabic-Islamic contacts with Christian Syrians at Edessa, heathen Syrians (self-proclaimed Sabians after the ninth century) in Harran, the cities of Antioch and Alexandria, and the numerous cloisters of Syria and Mesopotamia. Raids into "the land of Romans", especially during the reign of Harun al-Rashid, netted Greek manuscripts (primarily in Amorium and Ancyra) among the booty.

The search for Greek manuscripts led al-Ma'mun's emissaries to distant lands such as Constantinople and to the court of the Armenian emperor, Leo. The Byzantine emperor responded to al-Mansur's requests by sending manuscripts of Euclid as well as other works (Hitti, 1956: 309–315). Greek was unknown in Arabia. Translations were made by Jewish, heathen, and in particular Syrian Nestorian Christian subjects. The Syrians translated the Greek works into Syrian and then Arabic. The Greek influence reached its height under al-Ma'mun.

ARABIC-ISLAMIC MATHEMATICS I: INTELLECTUAL CONTEXT

The Abbasid Caliphs were patrons of medicine, astronomy, and mathematics. They brought or invited foreign physicians and other scholars to Baghdad. In 772, a Hindu astronomer named Kankah (or Mankah) arrived at the court of al-Rashid with a text generally thought to be the *Sindhind*, the Hindu revision of Brahmagupta's *Siddhantas*. (There is some controversy about just when the Hindu astronomer arrived and when the text was translated). The astronomer Ibrahim al-Fazari, a personal friend of al-Mansur, is said to have translated the *Sindhind*; sometimes the translation is credited to his son. There is some doubt about whether the *Sindhind* was translated this early (O'Leary, 1949: 152–153). The Arabs found it necessary to translate Euclid and Ptolemy before they could understand and use the *Sindhind*. The *Elements* and the *Almagest* were thus among the earliest, if not the earliest manuscripts translated.

The Golden Age of Arabic-Islamic mathematics and science begins with the translations of the Greek works under al-Rashid and his son and successor, Caliph al-Ma'mun (809–833). Al-Hayyan (fl. 786–833) translated Euclid about 820; Apollonius was translated by al-Himshi and the Banu Musa (three brothers) around 875. Ishaq ibn Hunein and Tabit ibn Qurra translated Archimedes, Menelaus, Aristotle, and Ptolemy about 890. And Diophantes, Heron, Autoclycus, Theodosius, and Hypsicles were translated by Qusta ibn Luqa toward the end of the ninth century.

Ishaq ibn Hunein (809–873) was a Nestorian Christian, and is known as "the sheikh of the translators". Tabit ibn Qurra (836–901) led another group of translators from Harran, the center of the heathen Sabians. These Syrians had a long tradition of interest in astronomy and mathematics, rooted in star-worship. Under al-Mutawakkil, a school of philosophy and medicine, previously at Antioch and before that located in Alexandria, was established in Harran. Al-Hayyay is a forerunner of the Harran translators.

The Sabian Tabit was the first translator in a family line of translators. He was succeeded by his son (Sinan), two grandsons (Tabit, and Ibrahim), and one great grandson (al-Faraji). The most important Sabian intellectual after Tabit ibn Qurra was al-Battani, a convert to Islam.

The latter part of the tenth century saw the rise of Jacobite or Monophysite translators who concentrated on revising existing translations and preparing new translations of Aristotle. The introduction of neo-Platonic thought and mysticism into Arabia was due chiefly to these Monophysites.

Around 970, a school of popular philosophy influenced by Pythagoreanism flourished at al-Bashrah under "the brethren of sincerity" (Ikhwan al-Safa). There was also a branch in Baghdad. Like the original Pythagoreans, the brethren was a religious and political secret society. The group was opposed to the established government, and was linked to ultra-Shi'ites probably espousing Isma'ilite views. Al-Ghazzali was influenced by this school.

CHAPTER 5

INDIAN MATHEMATICS: A HISTORY OF EPISODES

A sociologically inspired history of Indian mathematics has yet to be written. And the conventional histories of Indian mathematics are often little more than lists of achievements, sometimes designed to illustrate and not infrequently to exaggerate India's contributions to "world mathematics". Nonetheless, a few sociologically important points can be made by reviewing the basic features of Indian mathematics.

In India as elsewhere, the connection between early mathematical activity and religion is evident. In ancient Vedic society, the male head of a household was required to maintain three fires, each one sheltered in a specially designed, precisely measured altar. He also had to transfer fires from one altar to another. Altars might be the same or different in shape, but their areas had to be related by a simple ratio. In ancient India as elsewhere in the ancient world, the construction of altars was an important source of mathematical problems. There are references to such problems in the *Rig Veda Samhita*, and specific details about "the science of the altar" in the *Taittireeya Samhita* and *Taittireeya Brahmana.*

The rules for constructing altars and transferring fires were codified in the *Sulvasutras*. These sutras are thought by some scholars to have been written as early as 800 to 500 BCE; others date them in the early centuries of the common era. In any case, they are an important document for historians of mathematics; they include such items as the "Pythagorean" theorem (for special cases), and the proposition that the diagonal of a rectangle divides it into equal parts.

"Sulva" seems to have originally meant "to measure", and later "rope" or "cord" (from "rope-measurers", probably more familiar in their Egyptian incarnation). For a long period, geometry was known as "sulva" or "rajjii" (rope). The *Sulvasutras* are not strictly speaking "mathematical" in the sense that the *Arya Bhateeya* (discussed below) is. But it is the earliest important codification of mathematical problems and findings in India. As in the case of many other achievements in Indian mathematics, there is little continuity between the results and findings recorded in these sutras and later developments in Indian mathematics. Other religious sutras such as the *Anuyoga Dwara Sutra* and the

Sthananga Sutra (300–500) also include important mathematical items. The former includes an enumeration of powers and roots, and rules for finding roots and powers. The latter lists the ten basic mathematical topics, including the four basic operations, geometry, applied arithmetic, and fractions.

Mahavira, the ninth century founder of Jainism, was a well-known mathematical worker in the court of Amoghavarsha Nripatunga, the Rashtrakoota King of what is today the state of Mysore. Ganitanuyoga, or "exposition of mathematical principles", and Samkbyana, "the science of arithmetic and astronomy", were important branches of Jainist literature. Mahavira's *Ganita Sara Sangraha* (written about 850) is the first "textbook" that presents arithmetic in its modern form. Mahavira's text includes problems such as the following (Srinivasiengar, 1967: 72):

> Three merchants found a purse in the way. One of them said "If I secure this purse, I shall become twice as rich as both of you with your moneys on hand". Then the second man said, "I shall become thrice as rich". The third man said, "I shall become 5 times as rich". What is the value of money in the purse, as also the money in hand with each of them?

Although he did not systematically study conics, Mahavira does refer briefly to the ellipse; he appears to be the only Indian mathematical worker to do so. His formula for the area of the ellipse is not correct, and was probably based on the formula for the area of the circle.

Indian mathematics, especially in the period before Greek astronomy was introduced (c.400), placed a relatively strong emphasis on large numbers. Geometry, arithmetic, number theory, and algebra were virtually ignored in favor of the use of numbers in social schemes. The *Upanishads* (c.700–500 BCE) contain numerous numerical descriptions: the 72,000 arteries, the 36,360 or 36,000 syllables, the 33,303 or 3306 gods, the 5, 6, 7, or 12 basic elements out of which the world is composed. The wisdom of the Buddha is illustrated by the gigantic numbers he can count out (on the order of eight times twenty-three series of 10^7), and his magnificence is illustrated by the huge number of boddhisatvas and other celestial beings who gather to set the scenes for his various sutras. The Hindu cosmology includes a cyclical view of time that enumerates great blocks of years called *yugas*. There are four *yugas* ranging from 432,000 to 1,728,000 years, all of which together make up one thousandth of a *kalpa* or 4,320,000,000 years.

This emphasis upon immense, cosmological numbers gives a distinctively Hindu view of the near-infinite stretches of being that surround the empirical world. It seems almost inevitable that the Hindus should have invented the "zero" as we know it and use it in modern mathematics, or *sunya* (emptiness) in Sanskrit. The concept *sunya*, developed about 100, was the central concept in Madhyamika Buddhist mysticism, and preceded the introduction of the mathematical zero about 600. Classical Indian world views are permeated with mathematics but of a special kind. It is a mathematics for transcending experience, but not in the direction of rationalistic abstraction. Instead, numbers are used for purposes of *mystification* or to convey the notion that some thing or being is *impressive*; they are symbols in a mathematical rhetoric designed to awe the listener into a religious posture. In general, numbers were used for *numerological* rather than mathematical purposes. The social roots of this distinctive mathematical system lie in the particularly exalted status of Indian religious specialists. The concrete as opposed to abstract nature of Hindu large numbers may also have been suggested by a social reality: the great variety of ethnic groups making up Indian society, institutionalized in the ramifications of the caste system (see Restivo, 1983: 60–63, for a discussion of the sociology of large number systems in India).

The connection between early number work and religion is universal. The reason for this is that religion is a somewhat loosely applied term for the everyday and ritual activities in which people create and recreate feelings of social solidarity and sustain "moral codes". In a context where the differentiation of social activities into role, organizational, or institutional sectors was minimal, number work inevitably was associated with both mundane counting and ritualistic ordering. The connection between religion, political power, and the calendar was another factor that linked number work; especially in astronomical matters, and religion.

A "GOLDEN AGE"

A "golden age" in Indian mathematics begins with the rise to power of King Gupta in 290, and ends in the midst of the political turmoil of the twelfth century. While the publication of Arya Bhata's *Arya Bhateeya* in 499 is the first major indicator of this golden age in mathematics, the "age of the Siddhantas" (Boyer, 1968: 231) is an important prelude. These works seem to have been a part of the renaissance in Sanskrit culture inaugurated under the Gupta dynasty. The *Siddhantas* all seem to have been works in which rules related to astronomy were listed in cryptic Sanskrit verse. There is little or no explanation of these rules, and no proofs are offered. While there is general agreement that the *Siddhantas* belong to the late fourth or early fifth century, there is some controversy about the origin of the facts they list. Some authorities claim that they reflect the influences of Paul of Alexandria and Ptolemy.

The *Surya Siddhantas*, the only one of the manuscripts that seems to be complete, has been dated ca. 400. This "system of the Sun" is written in epic stanzas, and identified as the work of Surya the Sun God. It includes a decimal system, positional notation, and ciphered forms for ten numerals. Not one of these is due to the Indians; but they are probably the first people to combine these three mathematical inventions. As in other manuscripts, references to predecessors are infrequent.

The *Arya Bhateeya* of 499 is the first treatise on mathematics as a more or less distinct activity published in India. It was written by Arya Bhata of Kusuma Pura (near modern Patina). There is some evidence that there was a "school" of mathematical workers at Kusuma Pura.

The *Arya Bhateeya* is a relatively brief text that reports what was known at the time about such topics as progressions, determining square and cube roots, and solving quadratic equations. Arya Bhata uses the sort of alphabet numerals used by the Greeks which are well-suited for the poetic style used in ancient writings on mathematics and other topics. He is the earliest mathematical worker-astronomer mentioned in the extant historical documents. Among his contributions was the proposal that the diurnal motion of the heavens is due to the rotation of the earth on its axis.

The most important mathematical workers of this "golden age" are, like Arya Bhata, astronomers. Varaha-Mihara is the author of an astronomical treatise, *Pancha Siddhantika* (505), and several works on astrology. And Brahmagupta (born in 598 and a member of the Ujjain School) wrote the *Brahma Sphuta Siddhanta*, the major source of Arabic-Islamic learning about Indian astronomy. A number of chapters in this manuscript are devoted to topics in arithmetic (Ganita) and algebra (Kuttaka, literally "pulverizer"). (The term Bija Ganita eventually came to mean "algebra": it appears for the first time in a mid-ninth century work by Prithudaka Swarmi, a commentator on Brahmagupta).

Brahmagupta is known for his solution of the indeterminate equation $Nx^2 + 1 + y^2$. He deals with interest problems, provides geometrical solutions for constructing right triangles that anticipate Fibonacci (1202) and Vieta (1580), and gives other results that anticipate the achievements of seventeenth century European mathematical workers.

At the age of 67, Brahmagupta wrote the *Khanda Khadyaka* (665), an expository astronomical treatise. Here he gives a rule ("Brahmagupta's rule") equivalent to what is generally known as the Newton-Sterling formula. He also has a claim to being the "inventor" of interpolation theory; but neither he nor his successors gave much attention to this topic.

The most celebrated mathematical worker in ancient India is Bhaskara (b.1104) "the gem of the circle of mathematicians". Bhaskara was a Brahmin, and (like Mahavira) from the area now known as Mysore. Bhaskara was a "pure mathematician" in the sense that he focused on simplifying, improving, and adapting the contributions of his predecessors. Unlike most earlier Indian mathematical workers, Bhaskara is aware of and refers to predecessors, including Brahmagupta and Mahavira, as well as Sridhara and Padmanabha (whose works are lost), Sripathi, Prithedaka Swarmi, and Varaha-Mihara.

Bhaskara's *Siddhanta Siromani* (1150) is a textbook that deals with arithmetic, algebra, the celestial globe, and the planets. The core text is written in a poetic style, with accompanying prose commentaries. It includes sections on the mechanical application of methods ("leelavati") and the theory underlying those methods ("bijaganita"). The text begins with a prayer to Lord Ganesa, and includes tables, the eight operations ($+$, $-$, $\times$, $\div$, x^2, x^3, $\sqrt[2]{}$, $\sqrt[3]{}$), fractions, zero, the rule of 3, interest, mensuration, and permutations. The Leelavati chapters are especially noted for their interesting "recreational" problems. A typical problem is formulated as follows (Srinivasiengar, 1967: 85):

> O Girl! out of a group of swans 7/2 times the square root of the number are playing on the shore of a tank. The two remaining ones are playing with amorous fight in the water. What is the total number of swans?

Comparable "interesting" problems are, of course, not uncommon in other mathematical traditions.

Bhaskara is the first writer to give $a/0 = \infty$. He did not use infinitesimals, but did work out some differential calculus results. Using summation methods used by mathematical workers from Archimedes to Kepler, Bhaskara determined the area and volume of the sphere.

Bhaskara is the last representative of the "golden age" of Indian