Cambridge Tracts in Mathematics
and Mathematical Physics

No. 36

RANDOM VARIABLES AND PROBABILITY DISTRIBUTIONS

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BY

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CAMBRIDGE
AT THE UNIVERSITY PRESS
1970

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First printed 1937
Second edition 1962
Reprinted 1963
Third edition 1970
First paperback edition 2004

A catalogue record for this book is available from the British Library

Library of Congress Catalogue Card Number: 74-92246

ISBN 0 521 07685 4 hardback
ISBN 0 521 60486 9 paperback

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PREFACE

The Mathematical Theory of Probability has lately become of growing importance owing to the great variety of its applications, and also to its purely mathematical interest. The subject of this tract is the development of the purely mathematical side of the theory, without any reference to the applications. The axiomatic foundations of the theory have been chosen in agreement with the theory given by A. Kolmogoroff in his work *Grundbegriffe der Wahrscheinlichkeitsrechnung*, to which I am greatly indebted. In accordance with this theory, the subject has been treated as a branch of the theory of completely additive set functions. The method principally used has been that of *characteristic functions* (or *Fourier-Stieltjes transforms*).

The limitation of space has made it necessary to restrict the programme somewhat severely. Thus in the first place it has proved necessary to consider exclusively probability distributions in spaces of a finite number of dimensions. With respect to the advanced part of the theory, I have found it convenient to confine myself almost entirely to problems connected with the so-called *Central Limit Theorem* for sums of independent variables, and with some of its generalizations and modifications in various directions. This limitation permits a certain uniformity of method, but obviously a great number of important and interesting problems will remain unmentioned.

My most sincere thanks are due to my friends W. Feller, O. Lundberg and H. Wold for valuable help with the preparation of this work. In particular the constant assistance and criticism of Dr Feller has been very helpful to me.

H. C.

Department of Mathematical Statistics
University of Stockholm
December 1936

PREFACE TO THE SECOND EDITION

This Tract has now been out of print for a number of years. Since there still seems to be some demand for it, the Syndics of the Cambridge University Press have judged it desirable to publish a new edition.

However, owing to the vigorous development of Mathematical Probability Theory since 1937, any attempt to bring the book up to date would have meant rewriting it completely, a task that would have been utterly beyond my possibilities under present conditions. Thus I have had to restrict myself in the main to a number of minor corrections, otherwise leaving the work—including the Bibliography—where it was in 1937.

Besides the minor corrections, most of which are concerned with questions of terminology, there are, in fact, only two major alterations. In the first place, a serious error in the statement and proof of Theorem 11 has been put right. Further, the contents of Chapter IV, § 4, which are fundamental for the theory of asymptotic expansions, etc., developed in Chapter VII, have been revised and simplified. This permits a new formulation of the important Lemma 4, on which the proofs of Theorems 24–26 are based. Finally a brief list of recent works on the subject in the English language has been added.

H.C.

University Chancellor's Office
Stockholm
March 1960

PREFACE TO THE THIRD EDITION

When this Tract was first published in 1937, an important part of it was Chapter VII, containing Liapounoff's classical inequality for the remainder in the Central Limit Theorem, as well as the theory of the related asymptotic expansions. For the Third Edition, this chapter has been partly rewritten, and now brings a proof of the sharper inequality due to Berry and Esseen. Moreover, several minor changes have been made, and the terminology has been somewhat modernized.

H. C.

Djursholm
January 1969

ABBREVIATIONS AND NOTATIONS

Symbol	Signification	Explanation
d.f.	Distribution function	*page* 11
pr.f.	Probability function	11
s.d.	Standard deviation	21
$E(X)$	Mean value (or mathematical expectation) of X	20
$D(X)$	Standard deviation of X	21
c.f.	Characteristic function	24
$F(x) = F_1(x) * F_2(x)$	$F(x) = \int_{-\infty}^{\infty} F_1(x-t)\, dF_2(t)$	37
convergence i.pr.	Convergence in probability	39
$(F(x))^{n*}$	$F(x) * F(x) * \ldots$ (n times)	53

The *union* or *sum* of any finite or enumerable sequence of sets $S_1, S_2, \ldots$ is denoted by

$$S = S_1 + S_2 + \ldots.$$

The *intersection* or *product* of the sets $S_1, S_2, \ldots$ is denoted by

$$S = S_1 S_2 \ldots.$$

The inclusion sign $\subset$ is used in relations of the type $S_1 \subset S$ indicating that S_1 is a subset of S, and also in relations of the type $x \subset S$ to express the fact that x is an element of the set S.