Matrix Calculus and Zero-One Matrices

This book presents the reader with mathematical tools taken from matrix calculus and zero-one matrices and demonstrates how these tools greatly facilitate the application of classical statistical procedures to econometric models. The matrix calculus results are derived from a few basic rules that are generalizations of the rules of ordinary calculus. These results are summarized in a useful table. Well-known zero-one matrices, together with some new ones, are defined, their mathematical roles explained, and their useful properties presented.

The basic building blocks of classical statistics, namely, the score vector, the information matrix, and the Cramer–Rao lower bound, are obtained for a sequence of linear econometric models of increasing statistical complexity. From these are obtained interactive interpretations of maximum likelihood estimators, linking them with efficient econometric estimators. Classical test statistics are also derived and compared for hypotheses of interest.

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Statistical and Econometric Applications

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521807883

First published 2002
This digitally printed first paperback version 2005

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Turkington, Darrell A.
Matrix calculus and zero-one matrices : statistical and econometric applications / Darrell A. Turkington.
p. cm.
Includes bibliographical references and index.
ISBN 0-521-80788-3
1. Matrices. 2. Mathematical statistics. I. Title.
QA188 .T865 2001
512.9′434 – dc21 2001025614

ISBN-13 978-0-521-80788-3 hardback
ISBN-10 0-521-80788-3 hardback

ISBN-13 978-0-521-02245-3 paperback
ISBN-10 0-521-02245-2 paperback

To Sonia

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Preface

This book concerns itself with the mathematics behind the application of classical statistical procedures to econometric models. I first tried to apply such procedures in 1983 when I wrote a book with Roger Bowden on instrumental variable estimation. I was impressed with the amount of differentiation involved and the difficultly I had in recognizing the end product of this process. I thought there must be an easier way of doing things. Of course at the time, like most econometricians, I was blissfully unaware of matrix calculus and the existence of zero-one matrices. Since then several books have been published in these areas showing us the power of these concepts. See, for example Graham (1981), Magnus (1988), Magnus and Neudecker (1999), and Lutkepohl (1996).

This present book arose when I set myself two tasks: first, to make myself a list of rules of matrix calculus that were most useful in applying classical statistical procedures to econometrics; second, to work out the basic building blocks of such procedures – the score vector, the information matrix, and the Cramer–Rao lower bound – for a sequence of econometric models of increasing statistical complexity. I found that the mathematics involved working with operators that were generalizations of the well-known vec operator, and that a very simple zero-one matrix kept cropping up. I called the matrix a shifting matrix for reasons that are obvious in the book. Its basic nature is illustrated by the fact that all Toeplitz circulant matrices can be written as linear combinations of shifting matrices.

The book falls naturally into two parts. The first part outlines the classical statistical procedures used throughout the work and aims at providing the reader with the mathematical tools needed to apply these procedures to econometric models. The statistical procedures are dealt with in Chap. 1. Chapter 2 deals with elements of matrix algebra. In this chapter, generalized vec and devec operators are defined and their basic properties investigated. Chapter 3 concerns itself with zero-one matrices. Well-known zero-one matrices such as commutation matrices, elimination matrices, and duplication matrices are defined and their properties listed. Several new zero-one matrices are introduced in this chapter. Explicit expressions are given for the generalized vec and devec of the commutation matrix, and the properties of these matrices are investigated in several theorems. Shifting matrices are defined and the connection among these matrices and Toeplitz and circulant matrices is explained. Moreover, the essential role they play in time-series processes is demonstrated. Chapter 4 is devoted to matrix calculus. The approach taken in this chapter is to derive the matrix calculus results from a few basic rules that are generalizations of the chain rule and product rule of ordinary calculus. Some of these results are new, involving as they do generalized vecs of commutation matrices. A list of useful rules is given at the end of the chapter.

The second part of the book is designed to illustrate how the mathematical tools discussed in the preceding chapters greatly facilitate the application of classical statistical procedures to econometric models in that they speed up the difficult differentiation involved and help in the required asymptotic work.

In all, nine linear statistical models are considered. The first three models (Chap. 5) are based on the linear-regression model: the basic model, the linear-regression model with autoregressive disturbances, and the linear-regression model with moving-average disturbances. The next three models (Chap. 6) are based on the seemingly unrelated regression equations (SURE) model: the basic model, the SURE model with vector autoregressive disturbances, and the SURE model with vector moving-average disturbances. The final three models (Chap. 7) are based on the linear simultaneous equations (LSE) model. We consider the basic LSE model and the two variations that come about when we assume vector autoregressive or vector moving-average disturbances.

For each model considered, the basic building blocks of classical statistics are obtained: the score vector, the information matrix, and the Cramer–Rao lower bound. Statistical analysis is then conducted with these concepts. Where possible, econometric estimators of the parameters of primary interest that achieve the Cramer–Rao lower bound are discussed. Iterative interpretations of the maximum-likelihood estimators that link them with the econometric estimators are presented. Classical test statistics for hypotheses of interest are obtained.

The models were chosen in such a way as to form a sequence of models of increasing statistical complexity. The reader can then see, for example, how the added complication changes the information matrix or the Cramer–Rao lower bound. There are, in fact, two such sequences in operation. We have in Chap. 5, for example, the basic linear-regression model followed by versions of this model with more complicated disturbance structures. Second, between chapters, we have sequences of models with the same characteristics assigned to the disturbances: for example, the linear-regression model with autoregressive disturbances followed by the SURE model and the LSE model with vector autoregressive disturbances.

It is assumed that the reader has a good working knowledge of matrix algebra, basic statistics, and classical econometrics and is familiar with standard asymptotic theory. As such, the book should be useful for graduate students in econometrics and for practicing econometricians. Statisticians interested in how their procedures apply to other fields may also be attracted to this work.

Several institutions should be mentioned in this preface: first, my home university, the University of Western Australia, for allowing me time off from teaching to concentrate on the manuscript; second, the University of Warwick and the University of British Columbia for providing me with stimulating environments at which to spend my sabbaticals. At Warwick I first became interested in matrix calculus; at British Columbia I put the finishing touches to the manuscript.

Several individuals must also be thanked: my teacher Tom Rothenberg, to whom I owe an enormous debt; Adrian Pagan, for his sound advice; Jan Magnus, for introducing me to the intricacies of zero-one matrices; my colleagues Les Jennings, Michael McAleer, Shiqing Ling, and Jakob Madsen for their helpful suggestions and encouragement; Helen Reidy for her great patience and skill in typing the many drafts of this work; finally, my family, Sonia, Joshua, and Nikola, for being there for me.