FOREWORD

The introduction of differentials by Leibniz and Newton might be considered as the first appearance of infinitesimals in the mathematics of modern times, famous also because these objects were soon derided by Berkeley as ghosts of a highly unsure existence. Infinitesimals, i.e., nonvanishing positive 'numbers' smaller than any given fraction, had still been very useful for Euler and Cauchy; they became ruled out only later on, in the wake of the putatively rigorous tendencies which took over mathematical analysis during the 19th century. Since, the best known renewal of that venerable concept has been carried out in the preceding four decades by Abraham Robinson and others, who designed this nonstandard analysis to deal with infinitely small and infinitely large entities in a truly rigorous manner. Because of making a rather unrestricted use of classical logic and set theory and, in particular, of the axiom of choice, Robinson's theory in its full-fledged form has widely been suspected to be nonconstructive from the outset. In addition, the nonstandard idea of discretising the continuum seems to be even less compatible with the intuitionistic concept of a continuum in the true sense of the word, than with the classical atomistic notion.

The distance between constructive and nonstandard mathematics, however, is actually much smaller than it appears to be. Indications for this are that nonstandard practice often looks rather constructive, and that very small numbers unknown to vanish are indispensable to distinguish constructive mathematics from its traditional counterpart. At least from any naive point of view, it is therefore no wonder that constructive mathematics eventually proved its capability to tackle also relatively abstract objects such as infinitesimals. This progress cannot be thought of without the revival of constructive thinking since the 1960es, initiated by the work of Errett Bishop, Per Martin-Löf, and others as well as by the development of digital computers, which has eventually lead to today's pragmatic way in which constructive mathematics sees itself. Some far-reaching approaches to constructive nonstandard mathematics have indeed been undertaken quite recently, whence time was ripe for the first meeting dedicated simultaneously to constructive and nonstandard mathematics—and, of course, to the reunion of these seeming antipodes.

Consisting of peer-reviewed articles written on the occasion of such an event, this volume offers views of the continuum from various standpoints. Including historical and philosophical issues, the topics of the contributions range from the foundations, the practice, and the applications of constructive and nonstandard mathematics, to the interplay of these areas and the development of a unified theory.

Further Talks. The following talks were given at the conference but will be published elsewhere.

PETR ANDREYEV (Nizhnii Novgorod State University, Russia), *Definable standardness predicates in Internal Set Theory*

DOUGLAS BRIDGES (University of Canterbury, Christchurch, New Zealand), *Constructive investigations of functions of bounded variations*

NIGEL J. CUTLAND (University of Hull, England), *Constructive aspects of nonstandard methods in fluid mechanics*

DIRK VAN DALEN (Rijksuniversiteit Utrecht, The Netherlands), *Indecomposable subsets of the continuum*

JENS ERIK FENSTAD (University of Oslo, Norway), *Computability theory over the nonstandard reals*

JAMES HENLE (Smith College, Northhampton, Massachusetts), *Non-nonstandard analysis: category, measure, and integration*

CHRIS IMPENS (University of Gent, Belgium), *Some thoughts on nonstandard geometry*

HAJIME ISHIHARA (Japan Advanced Institute of Science and Technology), *A note on the Gödel-Gentzen translation*

H. JEROME KEISLER (University of Wisconsin, Madison), *Nonstandard methods in o-minimal structures*

M. ALI KHAN (John Hopkins University, Baltimore, Maryland), *Modelling 'neglibility' in mathematical economics: an application of Loeb spaces*

P. EKKEHARD KOPP (University of Hull, England), *Hyperfinite discretisations and convergence in option pricing models*

XIAOAI LIN (National University of Singapore), *On the almost independence of correspondences on Loeb space*

WILHELMUS A. J. LUXEMBURG (California Institute of Technology, Pasadena), *A Schauder type theorem for internal linear operators*

PER MARTIN-LÖF (University of Stockholm, Sweden), *Nonstandard type theory*

JOAN R. MOSCHOVAKIS (Occidental College, Los Angeles; University of Athens, Greece), *The intuitionistic continuum as an extension of the classical one*

JUHA OIKKONEN (University of Helsinki, Finland), *Some geometric ideas related to Brownian motion*

HERVÉ PERDRY (Université de Franche-Comte, Besançon, France), *Computing in the constructive henselisation of a valued field*

HANS PLOSS (Universität Wien, Austria), *On the rearrangement of series*

MICHAEL REEKEN (Universität Wuppertal, Germany), *Discretising the continuum*

HERMANN RENDER (Universität Duisburg, Germany), *Borel measure extensions defined on sub-σ-algebras*

GIOVANNI SAMBIN (Università di Padova, Italy), *Real numbers in formal topology*

PETER SCHUSTER (Universität München, Germany), *Elementary choiceless constructive analysis*

BAS SPITTERS (Katolieke Universiteit Nijmegen, The Netherlands), *A constructive converse of the mean value theorem*

YENENG SUN (National University of Singapore), *Asymptotic, hyperfinite, and continuum models*

WIM VELDMAN (Katolieke Universiteit Nijmegen, The Netherlands), *On some sets that are not positively Borel*

HANS VERNAEVE (University of Gent, Belgium), *Reducing distributions to hyperreal functions*

MANFRED WOLFF (Universität Tübingen, Germany), *On the approximation of operators and their spectra*

Further Participants

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NONSTANDARD CONSTRUCTION OF STABLE TYPE EUCLIDEAN RANDOM FIELD MEASURES

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Abstract: A nonstandard construction of stable type Euclidean random fields via hyperfinite flat integrals and stable white noise is given. Moreover, a brief account on an extension of Cutland's flat integral formula for (centered) Gaussian measures on the Hilbert space l_2 to the case of Banach spaces $l_p, 1 \leq p < \infty$, is presented.

Introduction

The aim of this paper is to derive a nonstandard flat integral representation for certain stable type Euclidean random field measures. In the case of Gaussian Euclidean random field measures, this was done in [3] where a flat integral formula for Nelson's free field measure has been given. In his seminal paper [6], Cutland studied the nonstandard flat integral representation of Wiener measure on the classical Wiener space $C_0[0,1]$, which gives a nonstandard justification of Donsker's (heuristic) "flat integral". He then used such representation to give a fairly simple

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and intuitive nonstandard proof of a Schilder's large deviation principle for the Wiener measure. Furthermore, in [7, 8, 9, 10], he extended such investigations to various (centered) Gaussian measures, which provides a shorter (and nonstandard) version of the large deviation principle discussed in general, for centered Gaussian measures on separable Banach spaces, in section III.3.4 of [11]. Let us also mention the interesting work [19] by Osswald, where he presents a further nonstandard construction of Brownian motion in abstract Wiener spaces based on [6]. In the last section of this paper, we will present shortly an extension of Cutland's work [9] on flat integral representation for measures on l_2 to the case of the Banach spaces $l_p, 1 \leq p < \infty$.

The study of large deviations for (non Gaussian) Euclidean random field measures seems delicate but possible. Also, one may expect to be able to discuss the scaling limits for such random field measures. To this purpose, a flat integral formula is apparently useful. Our strategy is to find an appropriate (hyperfinite) lattice setting and to construct certain lattice measures via inverse Fourier transform, then utilizing the fact that the stable Euclidean random field measures are induced by Euclidean random fields independent at each point to make a product measure of all such lattice measures, and finally to use Loeb measure structure to get the flat integral formula. This idea has been further utilized in [4] to investigate a functional integral realization for the class of Euclidean random field models for constructive quantum field theory developed in recent years in all space-time dimensions by Albeverio, Gottschalk and Wu.

In this paper, we take for granted the familiarity with the preliminaries on nonstandard analysis and the Loeb measure construction presented e.g. in [1], [5] and [18].

1. EUCLIDEAN RANDOM FIELD MEASURES

Let $\mathcal{D} := C_0^\infty(\mathbb{R}^d)$ be the vector space consisting of all C^∞-smooth functions on $\mathbb{R}^d$ $(d \in \mathbb{N})$ with compact support endowed with Schwartz topology and $\mathcal{D}'$ its (topological) dual space. Let $\mathcal{B}$ denote the Kolmogorov σ-algebra on $\mathcal{D}'$ generated by cylindrical sets of $\mathcal{D}'$ (which coincides with the topological σ-algebras generated by the strong or weak topologies of $\mathcal{D}'$). Let $p \in (0, 2]$ be arbitrarily fixed. From [13],

$$f \in \mathcal{D} \mapsto e^{-\int_{\mathbb{R}^d} |f(x)|^p dx} \in \mathbb{R}(\subset \mathbb{C})$$

is a characteristic functional on the nuclear space $\mathcal{D}$. By the well-known Bochner-Minlos' theorem (see e.g. [13]), there exists a unique probability measure μ on $(\mathcal{D}', \mathcal{B})$ such that

$$C(f) := \int_{\mathcal{D}'} e^{i<f,\omega>} d\mu(\omega) = e^{-\int_{\mathbb{R}^d} |f(x)|^p dx}, \quad f \in \mathcal{D}. \tag{1.1}$$

Moreover, there is a Euclidean random field[1] $F : \mathcal{D} \times (\mathcal{D}', \mathcal{B}, \mu) \to \mathbb{R}$ determined by $F(f, \omega) = < f, \omega >, f \in \mathcal{D}, \omega \in \mathcal{D}'$. We call F a stable random field and its probability law μ a stable random field measure.

Now let $m > 0$ if $d = 1, 2$ and $m \geq 0$ is $d \geq 3$ and let $\alpha \in (0, 1)$. Then the following stochastic pseudo-differential equation

$$(-\Delta + m^2)^\alpha X = F$$

induces a stable type Euclidean random field $X : \mathcal{D} \times (\mathcal{D}', \mathcal{B}, \mu_X) \to \mathbb{R}$ via $X(f, \omega) = < f, \omega >, f \in \mathcal{D}, \omega \in \mathcal{D}'$, where $\mu_X(D) := \mu((\Delta + m^2)^{-\alpha} D), D \in \mathcal{B}$, whose characteristic functional

$$C_X(f) := \int_{\mathcal{D}'} e^{i<f,\omega>} d\mu_X(\omega) = e^{-\int_{\mathbb{R}^d} |(\Delta+m^2)^{-\alpha} f(x)|^p dx}, \quad f \in \mathcal{D}. \tag{1.2}$$

Let us point out that if $p = 2$, μ and μ_X are Gaussian measures on $\mathcal{D}'$ supported by certain Sobolev spaces with negative indices (while μ_X is just Nelson's free field measure if $\alpha = \frac{1}{2}$, which was already studied using methods of nonstandard analysis in [1] and [3]). Also F introduced here is an interesting special case of infinitely divisible (Euclidean) random fields discussed e.g. in [2] (in the terminology of [13], an infinitely divisible random field is called "a generalized random process with independent value at every point"). Our main aim here is to give a representation formula for (the non Gaussian measures) μ and μ_X. Since there is no inverse Fourier transform for probability measures on ∞-dimensional spaces, we will realize our aim by using nonstandard analysis.

Similar methods will also be used to discuss Gaussian measures on l_p for $1 \leq p < \infty$, see Section 3.

2. NONSTANDARD CONSTRUCTION OF μ AND μ_X

Let us first give a hyperfinite representation of $\mathcal{D}'$ by following [14, 15]. Fix a polysaturated nonstandard model. Let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ be arbitrarily

[1] By "Euclidean", we mean that the probability law is invariant under the (proper) Euclidean transformation group.

fixed and $\delta := \frac{1}{N}$, an infinitesimal. We set

$$T := \{-N, -N + \frac{1}{N}, ..., -\frac{1}{N}, 0, \frac{1}{N}, ..., N - \frac{1}{N}, N\} \subset {}^*\mathbb{R}$$

and $\mathcal{L} := T^d \equiv \underbrace{T \times \cdots \times T}_{d\ times} \subset {}^*\mathbb{R}^d, d \in \mathbb{N}$. Let ${}^*\mathbb{R}^{\mathcal{L}}$ stands for the internal space of all internal functions from $\mathcal{L}$ into ${}^*\mathbb{R}$. We set

$$< f, g > := \sum_{t \in \mathcal{L}} \delta^d f(t) g(t), \quad f, g \in {}^*\mathbb{R}^{\mathcal{L}} .$$

Definition 2.1 *(Keßler [15]) $f \in {}^*\mathbb{R}^{\mathcal{L}}$ is called S-continuos whenever g is infinitesimal in ${}^*\mathcal{D}(K)$ for some compact set $K \subset \mathbb{R}^d$ implies that $< f, g >$ is infinitesimal in ${}^*\mathbb{R}$. Moreover, $f \in {}^*\mathbb{R}^{\mathcal{L}}$ is said to be $\mathcal{D}'$-nearstandard if $< f, \cdot > |_{{}^*\mathcal{D}(K)}$ is S-continuous for any compact set $K \subset \mathbb{R}^d$.*

$< f, \cdot >$ being linear on ${}^*\mathbb{R}^{\mathcal{L}}$ for $f \in {}^*\mathbb{R}^{\mathcal{L}}$, the necessary and sufficient condition for $< f, \cdot > |_{{}^*\mathcal{D}(K)}$ to be S-continuous for any compact set $K \subset \mathbb{R}^d$ is that $< f, g >$ is finite whenever g is finite in ${}^*\mathcal{D}(K)$. Thus, $f \in {}^*\mathbb{R}^{\mathcal{L}}$ is $\mathcal{D}'$-nearstandard if $< f, g >$ is finite for any compact set $K \subset \mathbb{R}^d$ and for any $\mathcal{D}(K)$-finite $g \in {}^*\mathcal{D}(K)$ (where $g \in {}^*\mathcal{D}(K)$ is said to be $\mathcal{D}(K)$-finite if the internal suprema $sup_{x \in {}^*K} |g^{(n)}(x)|, n \in \mathbb{N}$, are finite).

We denote by $Ns({}^*\mathbb{R}^{\mathcal{L}})$ the totality of $\mathcal{D}'$-nearstandard functions. We define the (weak) standard part mapping $st : Ns({}^*\mathbb{R}^{\mathcal{L}}) \to \mathcal{D}'$ via duality:

$$< st(f), g > = {}^\circ(< f, {}^*g >) , \quad \forall g \in \mathcal{D} .$$

$< st(f), \cdot >$ defines a distribution essentially because of the definition of the linear induction limit topology. The standard part mapping is continuous on each $\mathcal{D}(K)$ and hence on $\mathcal{D}$. On the other hand, from [14], every standard distribution $g \in \mathcal{D}'$ has a hyperfinite representation $f \in Ns({}^*\mathbb{R}^{\mathcal{L}}) : st(f) = g$. Therefore $st[Ns({}^*\mathbb{R}^{\mathcal{L}})] = \mathcal{D}'$.

Let us now turn to the construction of μ and μ_X. We begin to argue formally. In the hyperfinite lattice setting, we have $f = (f_t)_{t \in \mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}}$ and $g = (g_t)_{t \in \mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}}$ as hyperfinite sequences (or vectors). Since μ is the probability distribution of "a generalized random process with independent value at every point", we have $\mu = \prod_{t \in \mathcal{L}} \mu_t$, where $\mu_t := Proj_t \mu, t \in \mathcal{L}$, the marginal probability distribution of μ. Taking a hint from (1.1), we have for any $(f_t)_{t\in\mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}} \cap {}^*\mathcal{D}$ (i.e., the hyperfinite segment of ${}^*f$ for $f \in \mathcal{D}$),

$$\int_{{}^*\mathbb{R}^{\mathcal{L}}} e^{i\sum_{t\in\mathcal{L}}\delta^d f_t q_t} \prod_{t\in\mathcal{L}} d\mu_t(q_t) = e^{-\sum_{t\in\mathcal{L}}\delta^d |f_t|^p}$$

namely

$$\prod_{t\in\mathcal{L}} \int_{{}^*\mathbb{R}} e^{i\delta^d f_t q_t} d\mu_t(q_t) = \prod_{t\in\mathcal{L}} e^{-\delta^d |f_t|^p}$$

which further implies that

$$\int_{{}^*\mathbb{R}} e^{i\delta^d f_t q_t} d\mu_t(q_t) = e^{-\delta^d |f_t|^p} .$$

Setting $\mu_t^\delta(\cdot) := \mu_t(\delta^d \cdot)$, then

$$\int_{{}^*\mathbb{R}} e^{i f_t q_t} d\mu_t^\delta(q_t) = e^{-\delta^d |f_t|^p} .$$

Remarking that the above equality is a one-dimensional Fourier transform, one can take inverse Fourier transform to get the following expression for the density of μ_t (i.e. the Radon-Nikodym derivative with respect to one-dimensional Lebesgue measure)[2]

$$\frac{d\mu_t^\delta(q_t)}{dq_t} = \frac{1}{2\pi}\int_{{}^*\mathbb{R}} e^{-if_t q_t} e^{-\delta^d |f_t|^p} df_t\,, \quad (f_t)_{t\in\mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}} \cap {}^*\mathcal{D}$$

where dq_t and df_t stand for one-dimensional Lebesgue measure. Clearly this paves a way for us to construct μ.

Let $(\Omega, \mathcal{A}(\Omega), P)$ be a given internal probability space. The associated Loeb space is denoted by $(\Omega, \mathcal{A}_L(\Omega), P_L)$. Let $\{\eta_t(\omega) : \omega \in \Omega\}_{t\in\mathcal{L}}$ be an internal family of independent, identically distributed ${}^*\mathbb{R}$-valued random variables on $(\Omega, \mathcal{A}(\Omega), P)$, each $\eta_t : \Omega \to {}^*\mathbb{R}$ has (internal) density h given by

$$h(q_t) = \frac{1}{2\pi}\int_{{}^*\mathbb{R}} e^{-(if_t q_t + \delta^d |f_t|^p)} df_t, \ (f_t)_{t\in\mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}} \cap {}^*\mathcal{D}, (q_t)_{t\in\mathcal{L}} \in {}^*\mathbb{R}^{\mathcal{L}}$$

[2]Actually this inverse Fourier transform can be computed explicitly, for instance, by using the formulae of items 82 and 83 on page 25 of F. Oberhettinger: *Fourier Transforms of Distributions and Their Inverses. A Collection of Tables.* Academic Press, New York, London, 1973. But we do not need it here.

THE CONTINUUM IN SMOOTH INFINITESIMAL ANALYSIS

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Abstract: In this paper an investigation is made of the properties of the continuum in smooth infinitesimal analysis: it is shown that it differs in certain important respects from its counterpart in constructive analysis.

As presented in [1], *smooth infinitesimal analysis*, SIA, is a theory formulated within higher-order intuitionistic logic and based on the following axioms:

Axioms for the continuum, or smooth real line R. These are the usual axioms for a(n) (intuitionistic) field expressed in terms of two operations + and $\cdot$, and two distinguished elements $0, 1$.

Axioms for the strict order relation < on R. These are:

1 $a < b$ and $b < c$ implies $a < c$.

2 $\neg(a < a)$.

3 $a < b$ implies $a + c < b + c$ for any c.

4 $a < b$ and $0 < c$ implies $a \cdot c < b \cdot c$.

5 either $0 < a$ or $a < 1$.

6 $a \neq b$[1] implies $a < b$ or $b < a$.

[1]Here $a \neq b$ stands for $\neg a = b$. It should be pointed out that axiom 6 is omitted in some presentations of SIA, e.g. those in [3] and [4].

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The relation $\leq$ on $\mathbf{R}$ is defined by $a \leq b \Leftrightarrow \neg b < a$. The open interval (a, b) and closed interval $[a, b]$ are defined as usual, viz. $(a, b) = \{ x : a < x < b \}$ and $[a, b] = \{ x : a \leq x \leq b \}$; similarly for half-open, half-closed, and unbounded intervals.

Write Δ for the subset $\{ x : x^2 = 0 \}$ of $\mathbf{R}$; we use the letter ε as a variable ranging over Δ. Elements of Δ are called (nilsquare) *infinitesimals* or *microquantities.* Since, clearly, $0 \in \Delta$, Δ may be regarded as an *infinitesimal neighbourhood of* 0. Δ is subject to the

Microaffineness Principle. *For any map $g : \Delta \to \mathbf{R}$ there exist unique $a, b \in \mathbf{R}$ such that, for all ε, we have*

$$g(\varepsilon) = a + b \cdot \varepsilon.$$

Notice that then $a = g(0)$.

From these three axioms it follows that the continuum in SIA differs in certain key respects from its counterpart in *constructive analysis* CA, which is furnished with an elegant axiomatization in [2].

To begin with, the third basic property of the strict ordering relation $<$ in CA, given as axiom R2(3) on p. 102 of [2], and which may be written

$$(*) \qquad \neg(x < y \vee y < x) \to x = y$$

is incompatible with the axioms of SIA. For (*) implies

$$(**) \qquad \forall x \neg(x < 0 \vee 0 < x) \to x = 0.$$

But in SIA we have by Exercise 1.6 and Thm. 1.1 (i) of [1],

$$\forall x \in \Delta \neg(x < 0 \vee 0 < x) \wedge \Delta \neq \{0\},$$

which clearly contradicts (**).

Thus in CA the set Δ of infinitesimals would be degenerate (i.e., identical with $\{0\}$), while the nondegeneracy of Δ in SIA is one of its characteristic features.

Next, call a binary relation S on $\mathbf{R}$ *stable* if it satisfies

$$\forall x \forall y (\neg\neg x S y \to x S y).$$

In CA, the equality relation is stable, a fact which again follows from principle R2(3) referred to above. But in SIA it is not stable, for, as shown in Thm. 1.1 (ii) of [1], there we have $\forall x \in \Delta \neg\neg x = 0$. If $=$ were stable, it would follow that $\forall x \in \Delta\, x = 0$, in other words, that Δ is degenerate, which is not the case in SIA.

Axiom 6 of SIA, together with the transitivity and irreflexivity of $<$, implies that $<$ is stable. This may be seen as follows. Suppose $\neg\neg a < b$. Then certainly $a \neq b$, since $a = b \rightarrow \neg a < b$ by irreflexivity. Therefore $a < b$ or $b < a$. The second disjunct together with $\neg\neg a < b$ and transitivity gives $\neg\neg a < a$, which contradicts $\neg a < a$. Accordingly we are left with $a < b$. As can be deduced from assertion 8 on p. 103 of [2], the stability of $<$ implies *Markov's principle*, which is not affirmed in CA.[2]

A subset $A \subseteq \mathbf{R}$ is *indecomposable* if it admits only trivial partitionings, that is, if $A = U \cup V$ and $U \cap V = \emptyset$, then $U = \emptyset$ or $V = \emptyset$. Clearly A is indecomposable iff any map $f : A \rightarrow 2 = \{0, 1\}$ is constant.

In SIA one also assumes the

Constancy Principle. *If $A \subseteq \mathbf{R}$ is any closed interval on $\mathbf{R}$, or $\mathbf{R}$ itself, and $f : A \rightarrow \mathbf{R}$ satisfies $f(a + \varepsilon) = f(a)$ for all $a \in A$ and $\varepsilon \in \Delta$, then f is constant.*

As shown in Thm. 2.1 of [1], it follows in SIA from the Constancy Principle that $\mathbf{R}$ *itself and each of its closed intervals is indecomposable.* From this we can deduce that in SIA *all intervals in* $\mathbf{R}$ *are indecomposable.* To do this we employ the following

Lemma. Suppose that A is an inhabited subset of $\mathbf{R}$ satisfying

(*) for any $x, y \in A$ there is an indecomposable set B such that

$$\{x, y\} \subseteq B \subseteq A.$$

Then A is indecomposable.

Proof. Suppose A satisfies (*) and $A = U \cup V$ with $U \cap V = \emptyset$. Since A is inhabited, we may choose $a \in A$. Then $a \in U$ or $a \in V$. Suppose $a \in U$; then if $y \in V$ there is an indecomposable B for which $\{a, y\} \subseteq B \subseteq A = U \cup V$. It follows that $B = (B \cap U) \cup (B \cap V)$, whence $B \cap U = \emptyset$ or $B \cap V = \emptyset$. The former possibility is ruled out by the fact that $a \in B \cap U$, so $B \cap V = \emptyset$, contradicting $y \in B \cap V$. Therefore $y \in V$ is impossible; since this is the case for arbitrary y, we conclude

[2]In versions of SIA that omit axiom 6 neither the stability of $<$, nor Markov's principle, can be derived.

that $V = \emptyset$. Similarly, if $a \in V$, then $U = \emptyset$, so that A is indecomposable as claimed.

We use this lemma to show that the open interval $(0,1)$ is indecomposable; similar arguments work for arbitrary intervals. In fact, if $\{x,y\} \subseteq (0,1)$, it is easy to verify that

$$\{x,y\} \subseteq [\tfrac{xy}{x+y}, \tfrac{1-xy}{2-x-y}] \subseteq (0,1).$$

Thus, in view of the indecomposability of closed intervals, $(0,1)$ satisfies condition (*) of the lemma, and so is indecomposable.

Aside from certain infinitesimal subsets to be discussed below, in SIA indecomposable subsets of $\mathbf{R}$ correspond to connected subsets of $\mathbf{R}$ in classical analysis, that is, to intervals. In particular, any puncturing of $\mathbf{R}$ is *decomposable*, for it follows immediately from Axiom 6 that

$$\mathbf{R} - \{a\} = \{\, x : x > a \,\} \cup \{\, x : x < a \,\}.$$

Similarly, the set $\mathbf{R} - \mathbf{Q}$ of irrational numbers is decomposable as

$$\mathbf{R} - \mathbf{Q} = [\{\, x : x > 0 \,\} - \mathbf{Q}] \cup [\{\, x : x < 0 \,\} - \mathbf{Q}\}.$$

This is in sharp contrast with the situation in *intuitionistic analysis* IA, that is, CA augmented by Kripke's scheme, the continuity principle, and bar induction. For it is shown in [5] that in IA not only is any puncturing of $\mathbf{R}$ indecomposable, but that this is even the case for the set of irrational numbers (further indecomposabilty results for IA may be found in [6].) This would seem to indicate that in some sense the continuum in SIA is considerably less "syrupy"[3] than its counterpart in IA.

It can be shown that the various "infinitesimal" subsets of $\mathbf{R}$ introduced in [1] are indecomposable. For example, the indecomposability of Δ can be established as follows. Suppose $f : \Delta \to \{0,1\}$. Then by Microaffineness there are unique $a, b \in \mathbf{R}$ such that $f(\varepsilon) = a + b \cdot \varepsilon$ for all ε. Now $a = f(0) = 0$ or 1; if $a = 0$, then $b \cdot \varepsilon = f(\varepsilon) = 0$ or 1, and clearly $b \cdot \varepsilon \neq 1$. So in this case $f(\varepsilon) = 0$ for all ε. If on the other hand $a = 1$, then $1 + b \cdot \varepsilon = f(\varepsilon) = 0$ or 1; but $1 + b \cdot \varepsilon = 0$ would imply $b \cdot \varepsilon = -1$ which is again impossible. So in this case $f(\varepsilon) = 1$ for all ε. Therefore f is constant and Δ indecomposable.

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[3] It should be emphasized that this phenomenon is a consequence of axiom 6: it cannot necessarily be affirmed in versions of SIA not including this axiom.

In SIA *nilpotent infinitesimals* are defined to be the members of the sets

$$\Delta_k = \{ x \in \mathbf{R} : x^{k+1} = 0 \},$$

for $k = 1, 2, \ldots$, each of which may be considered an infinitesimal neighbourhood of 0. These are subject to the

Micropolynomiality Principle. *For any $k \geq 1$ and any $g : \Delta_k \to \mathbf{R}$, there exist unique $a, b_1, \ldots, b_k \in \mathbf{R}$ such that for all $\delta \in \Delta_k$ we have*

$$g(\delta) = a + b_1\delta + b_2\delta^2 + \cdots + b_k\delta^k.$$

Micropolynomiality implies that no Δ_k coincides with $\{0\}$.

An argument similar to that establishing the indecomposability of Δ does the same for each Δ_k. Thus let $f : \Delta_k \to \{0, 1\}$; Micropolynomiality implies the existence of $a, b_1, \ldots, b_k \in \mathbf{R}$ such that $f(\delta) = a + \zeta(\delta)$, where $\zeta(\delta) = b_1\delta + b_2\delta^2 + \cdots + b_k\delta^k$. Notice that $\zeta(\delta) \in \Delta_k$, that is, $\zeta(\delta)$ is nilpotent. Now $a = f(0) = 0$ or 1; if $a = 0$ then $\zeta(\delta) = f(\delta) = 0$ or 1, but since $\zeta(\delta)$ is nilpotent it cannot =1. Accordingly in this case $f(\delta) = 0$ for all $\delta \in \Delta_k$. If on the other hand $a = 1$, then $1 + \zeta(\delta) = f(\delta) = 0$ or 1, but $1 + \zeta(\delta) = 0$ would imply $\zeta(\delta) = -1$ which is again impossible. Accordingly f is constant and Δ_k indecomposable.

The union $\mathbf{D}$ of all the Δ_k is the *set of nilpotent infinitesimals*, another infinitesimal neighbourhood of 0. The indecomposability of $\mathbf{D}$ follows immediately by applying the Lemma above.

The next infinitesimal neighbourhood of 0 is the closed interval $[0, 0]$, which, as a closed interval, is indecomposable. It is easily shown that $[0, 0]$ includes $\mathbf{D}$, so that it does not coincide with $\{0\}$.

It is also easily shown, using axioms 2 and 6, that $[0, 0]$ coincides with the set

$$\mathbf{I} = \{ x \in \mathbf{R} : \neg\neg x = 0 \}.$$

So $\mathbf{I}$ is indecomposable. (In fact the indecomposability of $\mathbf{I}$ can be proved independently of axioms 1–6 through the general observation that, if A is indecomposable, then so is the set $A^* = \{ x : \neg\neg x \in A \}$.)

Finally, we observe that the sequence of infinitesimal neighbourhoods of 0 generates a strictly ascending sequence of decomposable subsets containing $\mathbf{R} - \{0\}$, namely:

$$\mathbf{R} - \{0\} \subset (\mathbf{R} - \{0\}) \cup \{0\} \subset (\mathbf{R} - \{0\}) \cup \Delta_1 \subset (\mathbf{R} - \{0\}) \cup \Delta_2 \subset \ldots$$
$$(\mathbf{R} - \{0\}) \cup \mathbf{D} \subset (\mathbf{R} - \{0\}) \cup [0, 0].$$

CONSTRUCTIVE UNBOUNDED OPERATORS

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Abstract: The existence of points outside the domain of an unbounded linear mapping between normed spaces is investigated constructively.

Introduction

In this paper we begin a constructive investigation of unbounded operators by studying a number of questions about their domains. We intend this to be the start of a programme to investigate systematically the constructive theory of unbounded operators, and incidentally to counter, once and for all, some common misconceptions about the viability of such a theory (see [10] and [5]).

Since constructive analysis uses intuitionistic, rather than classical, logic, even at the base level of the theory of unbounded operators there are problems that are classically trivial but constructively significant. For example, the obvious contradiction argument, based on the closed graph theorem, merely shows that the domain of a closed unbounded operator is not complete; it does not enable us to construct a point of the initial Banach space at which the operator is not defined. In the general case we have only managed a partial, weak solution to this

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problem; but in the special case of a closed unbounded operator T on a Hilbert space, such that both T and T^* are densely defined, we can find $\xi \in H$ such that $T\xi$ is not defined.

Our setting is Bishop's constructive analysis (**BISH**—see [2] or [3]), which we can regard as mathematics carried out with intuitionistic logic ([14], [15]). At one stage we add Church's thesis, to enable us to prove a stronger result in recursive constructive analysis, which is simply one model of BISH. Further information about BISH and other varieties of constructive mathematics can be found in [8], [1], and [18].

Let T be a linear mapping between normed spaces X and Y. We say that T is

- **not bounded** if it is contradictory that it be bounded;
- **unbounded** if there exists a sequence (x_n) in X such that $\lim_{n\to\infty} x_n = 0$ and $\forall n\ (\|Tx_n\| > 1)$.

There is a constructive distinction between these two concepts: whereas the second clearly implies the first, the converse implication depends on **Markov's Principle**,

If (a_n) *is a binary sequence such that* $\neg\forall n\,(a_n = 0)$, *then* $\exists n\,(a_n = 1)$,

This principle, which represents an unbounded search, cannot be derived within, but is consistent with, Heyting arithmetic—that is, Peano arithmetic with intuitionistic logic; see [8], pages 137–138. For future reference, note that Markov's Principle is equivalent to the statement

$$\forall x \in \mathbf{R}\,(\neg\,(x = 0) \Rightarrow x \neq 0)\,,$$

where, in the context of a normed space X, $x \neq 0$ means $\|x\| > 0$. We shall return to the distinction between *bounded* and *not unbounded* at the end of the paper. We say that T is **strongly extensional** if

$$\forall x, x' \in X\ \left(Tx \neq Tx' \Rightarrow x \neq x'\right).$$

Markov's Principle implies that every linear mapping between normed spaces is strongly extensional. Without Markov's Principle the best we can prove constructively is that every linear mapping of a Banach space into a normed space is strongly extensional; see [6], Corollary 2.

1. CONSTRUCTING POINTS OUTSIDE DOMAINS

In this section we examine the problem of finding points outside the domain of an unbounded linear mapping between normed spaces. Our first result is connected with the **limited principle of omniscience (LPO)**,

> *If* (a_n) *is a binary sequence, then either* $a_n = 0$ *for all* n *or else there exists* n *such that* $a_n = 1$,

a weak form of the law of excluded middle that cannot be derived in intuitionistic logic (and that is false, even with classical logic, in the recursive model of BISH—see Chapter 3 of [8]).

Recall that every normed linear space X can be embedded as a dense subset of a Banach space $\widehat{X}$, the **completion** of X.

Proposition 1 *Let* T *be a strongly extensional unbounded linear mapping between normed spaces* X, Y. *Then for each binary sequence* (a_n) *there exists* $x \in \widehat{X}$ *such that if* $x \in X$, *then*

$$\forall n\ (a_n = 0) \vee \exists n\ (a_n = 1).$$

PROOF. We may assume that (a_n) has at most one term equal to 1. For each positive integer n choose a unit vector $x_n \in X$ such that $\|Tx_n\| > n^2$. Then $\sum_{n=1}^{\infty} a_n \|Tx_n\|^{-1} x_n$ converges to a sum x in the Banach space $\widehat{X}$, by comparison with $\sum_{n=1}^{\infty} n^{-2}$. Suppose that $x \in X$. Then either $Tx \neq 0$ or $\|Tx\| < 1$. In the first case, as T is strongly extensional, $x \neq 0$ and so there exists n such that $a_n = 1$. In the second, suppose that there exists N such that $a_N = 1$; then $Tx = \|Tx_N\|^{-1} Tx_N$ and so $\|Tx\| = 1$, a contradiction; whence $a_n = 0$ for all n. Q.E.D.

Ishihara [11] has shown that in recursive constructive mathematics—constructive mathematics plus Church's thesis—every linear mapping on a Banach space is sequentially continuous and therefore not unbounded; this result also holds in the intuitionistic model of constructive mathematics ([18], page 354, 2.8). Since BISH is consistent with classical (that is, traditional) mathematics, we cannot hope to prove that recursive/intuitionistic continuity result within BISH. What we can prove, however, is that the existence of unbounded linear mappings on a Banach space is essentially nonconstructive.

Corollary 2 *If there exists a strongly extensional unbounded linear mapping on a Banach space, then LPO holds.*

PROOF. This is an immediate consequence of Proposition 1. Q.E.D.

A linear mapping $T : X \to Y$ between normed spaces is said to be **closed** if its graph is a closed subset of $X \times Y$; that is, if

$$(x_n \to x \in X \text{ and } Tx_n \to y \in Y) \Rightarrow y = Tx.$$

The constructive **closed graph theorem** says that if X is complete and the graph of T is both closed and separable, then T is sequentially continuous ([12], Corollary 2). The separability hypothesis is not needed in classical analysis, where the conclusion is the stronger one that T is bounded.

Proposition 3 *If T is an unbounded closed linear mapping, with a separable graph, of a normed space X into a Banach space Y, then* $\neg\left(\widehat{X} = X\right)$.

PROOF. If $\widehat{X} = X$, then the closed graph theorem shows that T is sequentially continuous, a contradiction. Q.E.D.

Let H be a Hilbert space, and T a linear mapping of a dense subspace of H into H; then we say that T is a **densely defined operator on** H. The **adjoint** T^* of a densely defined operator T on H is defined as in classical mathematics. Thus the domain of T^* comprises those $x \in H$ for which there exists $y \in H$ (which is then uniquely defined by x) such that

$$\langle y, z\rangle = \langle x, Tz\rangle \quad (z \in H),$$

and for such an x we write $T^*x = y$. In contrast to the classical situation, T^* may not be defined even if T is a *bounded* operator defined on the whole space H; see [9].

When dealing with unbounded linear operators on linear subsets of a Hilbert space, we can obtain substantial improvements upon Corollary 2. For the first of these we need the constructive **uniform boundedness theorem** (a contrapositive of the usual classical version of that result):

> *If (A_n) is a sequence of bounded linear mappings from a Banach space X into a normed space Y, and (x_n) is a sequence of unit vectors in X such that $\|A_n x_n\| \to \infty$ as $n \to \infty$, then there exists $x \in X$ such that $\|A_n x\| \to \infty$ as $n \to \infty$* ([16], page 61).

Proposition 4 *If H is a Hilbert space, and T is an unbounded densely defined operator on H with an adjoint, then there exists an element ξ of H such that $T\xi$ is undefined.*

PROOF. Let (x_n) be a sequence converging to 0 in H such that $\|Tx_n\| \to \infty$. Applying the uniform boundedness theorem to the linear functionals $x \mapsto \langle x, Tx_n \rangle$, we obtain a unit vector $\xi \in H$ such that $|\langle \xi, Tx_n \rangle| \to \infty$ as $n \to \infty$. If $T\xi$ is defined, then $\langle \xi, Tx_n \rangle = \langle T^*\xi, x_n \rangle \to 0$, a contradiction. Hence $T\xi$ is not defined. Q.E.D.

If T is an unbounded densely defined operator on a Hilbert space H, what can we say about its adjoint? It is closed (see [13]), but need not be unbounded; in fact, it can be 0 even though T is densely defined (see [17]). Classically, T^* is unbounded: for if it were bounded, we could extend it by continuity to a bounded operator on H whose adjoint would be bounded; whence T would have a bounded extension. When H is separable, the same result holds constructively, but, of course, with a direct proof.

Proposition 5 *If T is a densely defined operator on a separable Hilbert space H such that T^* is unbounded, then T is unbounded.*

PROOF. Since it is a dense subset of a separable space, the domain of T is separable; so there exists an increasing sequence (V_n) of finite–dimensional subspaces whose union is dense in that domain and therefore in H. Let P_n be the projection of H on V_n. Given $K > 0$, choose a unit vector $\xi \in H$ such that $\|T^*\xi\| > K + 1$. Then choose n such that

$$\|T^*\xi - P_nT^*\xi\| = \rho(T^*\xi, V_n) < 1$$

and therefore $\|P_nT^*\xi\| > K$. Setting

$$z = \|P_nT^*\xi\|^{-1} P_nT^*\xi,$$

we see that $P_nz = z$, $\|z\| = 1$, and

$$K < \|P_nT^*\xi\| = \langle P_nT^*\xi, z \rangle = \langle \xi, TP_nz \rangle = \langle \xi, Tz \rangle \leq \|Tz\|. \quad \text{Q.E.D.}$$

Corollary 6 *Let T be an unbounded densely defined operator on a Hilbert space, such that T^* is densely defined. Then there exists $\xi \in H$ such that $T\xi$ is undefined.*

PROOF. Use the preceding two results. Q.E.D.

Returning to the context of Banach spaces, we show how to improve Corollary 2 under Church's thesis (that is, in the recursive model of BISH).

THE POINTS OF (LOCALLY) COMPACT REGULAR FORMAL TOPOLOGIES

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Abstract: In a paper appeared in 1990, C.J. Mulvey established a constructive characterization of completely prime filters on a compact regular locale L; although proved by intuitionistic logic, the result relies on a notion of maximality which contains an impredicative second-order quantification. In this note we present an alternative concept of maximality, entirely phrased in first-order terms, and give a predicative characterization of the points of a compact regular formal topology (equivalently, we give a characterization of the points of a compact regular locale which can be entirely carried out within Intuitionistic Type Theory). This result is then generalized to locally compact regular formal topologies (resp. locally compact regular locale).

Introduction

Formal Topology[1] was conceived with the aim of developing point-free topology (Locale Theory) in a constructive and predicative foundational setting, such as Martin-Löf's Intuitionistic Type Theory. Quite recently, the topological notion of regularity has been predicatively formulated in

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[1]Formal Topology was introduced in [15]; a more recent presentation is contained in [16], [17].

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this framework, and the class of compact regular formal topologies has shown to have nice and promising properties, particularly from a constructive standpoint (cf. e.g. [5], [3] and [14]). In this note we establish a constructive characterization of the points of a compact regular formal topology, in which formal points are shown to coincide with particular subsets of basic neighbourhoods, the *maximal regular* ones. The specific feature of this characterization is that regular subsets will be considered to be maximal according to an entirely first-order criterion of maximality.

This result can then be seen to improve a previous characterization appeared in the context of Locale Theory: in [9] indeed Chris Mulvey introduces a particular formulation of the notion of maximality for regular filters which allows to prove intuitionistically that the completely prime filters on a compact regular locale coincide with the maximal regular filters (cf. [9]). In such a notion of maximality, however, an impredicative second-order quantification appears which makes the result incompatible with foundational settings for constructive mathematics such as Martin-Löf's Intuitionistic Type Theory and Aczel's Constructive Set Theory. A natural relation then exists between Formal Topology and Locale Theory (cf. [15]) that allows to give the following reading of our result: a characterization of completely prime filters on a compact regular locale by means of maximal regular filters can be obtained intuitionistically *and* predicatively, and such a characterization can be entirely carried out within Intuitionistic Type Theory.

Few modifications allow to generalize this result to *locally* compact regular formal topologies (and thus to locally compact regular locales). Then, in particular, examples of topologies for which these characterizations are valid are (that giving rise to) the Continuum, Cantor space and the spaces $\mathcal{L}(\mathcal{A})$ of linear functionals of norm ≤ 1 from a semi-normed space A to the reals[2].

1. PRELIMINARIES

We recall the basic definitions of Formal Topology ([15]). The reader is referred to [15], [16] and [17] for a detailed account (the presentation we are to adopt appears in [16]). We use a special notation for subsets,

[2]Endowed with the *weak** topology, cf. [4].

introduced and motivated in [18], which allows to work within Intuitionistic Type Theory (henceforth simply Type Theory, cf. [8], [13]) with essentially the usual mathematical formalism: for the present purpose it will suffice to know that a *subset* U of a set S is a unary predicate (dependent type) on S, $U(x)(x \in S)$, and that a *set-indexed family of subsets* is a binary predicate $U(x,i)(x \in S, i \in I)$ on the sets S and I, where for each $\bar{i}$, $U(x,\bar{i})(x \in S)$ is the subset of index $\bar{i}$ (for simplicity, we will also use the traditional notations $\{a \in S : U(a)\}$, to indicate the subset U, and $U_i(i \in I)$ for a family of subsets). Finally, we will write $a \epsilon U$ to mean $a \in S$ *and* $U(a)$ *true* (in the expression $a \epsilon U$ the symbol 'ϵ' is used, instead of '$\in$', to recall that we are considering a subset, i.e. a propositional function, and not a set; cf. [18]).

1.1 A (formal) topology is a triple $\mathcal{S} \equiv (S, \lhd, \mathsf{Pos})$ where S is a set, called the *base*, $\lhd$ is a relation between elements and subsets of S which satisfies the following conditions:

$$(\textit{reflexivity}) \qquad \frac{a \,\epsilon\, U}{a \lhd U}$$

$$(\textit{transitivity}) \qquad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

$$(\downarrow\textit{-right}) \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

where

$$\begin{aligned} U \lhd V &\equiv (\forall u \,\epsilon\, U) u \lhd V \\ U \downarrow V &\equiv \{d : S \mid (\exists u \,\epsilon\, U)\ (d \lhd \{u\})\ \&\ (\exists v \,\epsilon\, V)\ (d \lhd \{v\})\} \end{aligned}$$

and Pos is a subset of S which satisfies

$$(\textit{monotonicity}) \qquad \frac{\mathsf{Pos}(a) \quad a \lhd U}{(\exists b \,\epsilon\, U)\mathsf{Pos}(b)}$$

$$(\textit{positivity}) \qquad \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}.$$

We will write $\mathsf{Pos}(U)$ for $(\exists a \,\epsilon\, U)\mathsf{Pos}(a)$. The relation $\lhd$ is called *cover* and Pos *positivity predicate* (we pronounce $a \lhd U$ as 'U covers a', and

$\mathsf{Pos}(a)$ as 'a is positive'). For simplicity, when a basic neighbourhood a is covered by a singleton subset $\{b\}$ we will often write $a \lhd b$ instead of $a \lhd \{b\}$.

One can think intuitively of the elements of S as of indexes for the basic neighbourhoods of a topological space; the cover relation can then be seen as a formal description of the inclusion between basic neighborhoods and subsets of S, and the predicate Pos as a positive way to express that a certain neighbourhood is non-empty. Then, for instance, 'monotonicity' has the following intuitive reading: if a non-empty neighbourhood is covered by a family of neighbourhoods (indexed by U) then there must be at least one member of the family which is non-empty.

An equivalent formulation of positivity (cf. [15]) which we will use in the following is

$$\frac{a \lhd U}{a \lhd U^+},$$

where $U^+ \equiv \{b \,\epsilon\, U : \mathsf{Pos}(b)\}$ (that is, only non-empty neighbourhoods contribute to the cover).

Finally, we recall that given two subsets U, V of S, $U =_S V$ means exactly that $U \lhd V \;\&\; V \lhd U$, and that for $U \subseteq S$, the *(pseudo-)complement* U^* of U is given by $U^* \equiv \{b : \neg\mathsf{Pos}(b \downarrow U)\}$.

1.2 In a formal topology $\mathcal{S}$ a *formal point* is a subset $\alpha \subseteq S$ such that

i. $(\exists a)(a \,\epsilon\, \alpha)$ *ii.* $(a \,\epsilon\, \alpha \;\&\; b \,\epsilon\, \alpha) \rightarrow (\exists c)(c \,\epsilon\, a \downarrow b \;\&\; c \,\epsilon\, \alpha)$

iii. $\dfrac{a \,\epsilon\, \alpha \quad a \lhd U}{(\exists b)(b \epsilon U \;\&\; b \,\epsilon\, \alpha)}$ *iv.* $a \,\epsilon\, \alpha \rightarrow \mathsf{Pos}(a)$.

We will denote by $\mathsf{Pt}(\mathcal{S})$ the collection of formal points. (Condition *iv.* is actually known to be derivable from *iii.* and positivity and could thus be skipped[3]).

1.3 The relation with Locale Theory can be sketched as follows (a detailed discussion of this subject is contained in [15], [17]): defining, for

[3]A proof is recalled in [12]. A generalized definition of Formal Topology can however be considered in which the positivity rule is not required, cf. [16].

$U \subseteq S$, $\mathcal{S}U$ to be the subset $\{a \in S : a \lhd U\}$, we say that U is *saturated* if $U = \mathcal{S}U$; denoting then by $Sat(\mathcal{S})$ the collection of saturated subset of $\mathcal{S}$, $Sat(\mathcal{S})$ endowed with the operations

$$\mathcal{S}U \wedge \mathcal{S}V \equiv \mathcal{S}U \cap \mathcal{S}V = \mathcal{S}(U \downarrow V)$$

and

$$\bigvee_{i\in I} \mathcal{S}U_i \equiv \mathcal{S}(\bigcup_{i\in I} U_i)$$

forms a *frame* (or locale, or complete Heyting algebra).

From a non-constructive point of view the converse is also valid (that is, any frame can be obtained as the frame of saturated subsets of a formal topology $\mathcal{S}$). Finally, the points of a formal topology $\mathcal{S}$ are easily shown to correspond to completely prime filters on $Sat(\mathcal{S})$.

1.4 A formal topology $\mathcal{S} \equiv (S, \lhd, \mathsf{Pos})$ is said to be *compact* if whenever $S \lhd U$ there exists a finite[4] subset $U_0 \subseteq U$ such that $S \lhd U_0$.

The notion of regularity have been recently introduced in Formal Topology as the predicative counterpart of that given in the context of locales (see for instance [6]): for a, b in S, we say that b is *well-covered* by a iff $S \lhd a \cup b^*$; defining $wc(a)$ to be the subset of neighbourhoods b which are well-covered by a, $wc(a) \equiv \{b : S \lhd a \cup b^*\}$, a formal topology $\mathcal{S}$ is then said to be *regular* if for all a in S, $a \lhd wc(a)$[5]. A topology $\mathcal{S}$ will be said to be *compact regular* if it is compact and regular.

The following lemmas will be used in the following, often without an explicit mention; they obtain in any formal topology $\mathcal{S}$.

Lemma 1.5. *Let V, W, Z be subsets of S, and $U_i (i \in I)$ be a family of subsets of S. We have*

i) $V \cup (W \downarrow Z) =_{\mathcal{S}} (V \cup W) \downarrow (V \cup Z)$,

ii) $(\bigcup_i U_i) \downarrow V = \bigcup_i (U_i \downarrow V)$.

[4]Note that here and in the following a set, or a subset, is considered to be 'finite' if its elements can be listed; cf. the notions of finite and sub-finite in [2].

[5]This definition appeared in [14]; in case of compactness, it is equivalent to the one introduced in [5], [3].

EMBEDDING A LINEAR SUBSET OF $\mathcal{B}(H)$ IN THE DUAL OF ITS PREDUAL

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Abstract: The embedding of a linear set of bounded operators on a separable Hilbert space as a dense subset of the dual of its predual is explored constructively.

In this paper we continue the study of spaces of operators on a Hilbert space within constructive mathematics, as part of a programme for the systematic constructive development of the theory of operator algebras [6, 7].

The constructive framework within which we operate was erected by the late Errett Bishop [2], under the requirement that "existence" must be strictly interpreted as "computability" relative to some notion of algorithm. By not specifying formally what he meant by an algorithm, other than insisting that it must be executable by a finite number of human beings or computers in a finite time, Bishop enabled his work to have a variety of interpretations; in particular, all theorems of Bishop's constructive mathematics hold within classical (that is, traditional) mathematics and Brouwer's intuitionistic mathematics, and under a recursive interpretation.

Note, incidentally, that Bishop's algorithmic interpretation of existence makes no demands about the complexity of the algorithms used; at present, constructive mathematics addresses questions of computability in principle, rather than computability in practice. (See, however, [13].)

The main practical difference between constructive and classical mathematics is one of logic. The algorithmic interpretation of existence forces us to re–examine the interpretation of each logical connective and quantifier, and leads, it seems inevitably, to the use of intuitionistic logic, as

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originally codified by Heyting [11]. We also have to modify the underlying non–logical principles—for example, the axioms for the real line **R** [5] or the axioms of Zermelo–Fraenkel set theory [14]—so as to ensure that one cannot derive from them classical principles, such as the law of excluded middle, that are independent of intuitionistic logic. Once these precautions have been taken, we are, to all (non–philosophical) intents and purposes, working with intuitionistic Zermelo–Fraenkel set theory using intuitionistic logic. (For additional background material on constructive mathematics see [1, 2, 3, 5, 9, 15].)

Let H be a Hilbert space (not necessarily separable), $\mathcal{B}(H)$ the space of bounded linear operators on H, and $\mathcal{B}_1(H)$ the unit ball of $\mathcal{B}(H)$. Recall that the **weak–operator** topology τ_w on $\mathcal{B}(H)$ is the weakest topology with respect to which the mapping $T \mapsto \langle Tx, y\rangle$ is continuous for all $x, y \in H$. This topology is determined by the seminorms $T \mapsto |\langle Tx, y\rangle|$, where x, y run through any dense subset of $\mathcal{B}_1(H)$. Classically, $\mathcal{B}_1(H)$ is τ_w–compact; but constructively the most we can say, in general, is that it is τ_w–totally bounded [6].

Let $\mathcal{R}$ be a linear subset of $\mathcal{B}(H)$, let $\mathcal{R}1 = \mathcal{R} \cap \mathcal{B}_1(H)$, and let $R_\sharp$ denote the linear space of all linear functionals on R that are **ultraweakly continuous**—that is, τ_w–uniformly continuous on $\mathcal{R}_1$. If $\mathcal{R}_1$ is τ_w–totally bounded, then

$$\|f\| = \sup\{|f(T)| : T \in \mathcal{R}_1\}$$

defines a norm on $\mathcal{R}_\sharp$; taken with this norm, $\mathcal{R}_\sharp$ is called the **predual** of $\mathcal{R}$.

For convenience, we denote by $f_{x,y}$ the ultraweakly continuous functional $T \mapsto \langle Tx, y\rangle$ on $\mathcal{B}(H)$. We also recall that the weak* topology on the dual X^* of a locally convex space is the topology defined by the seminorms $f \mapsto |f(x)|$ $(x \in X)$; see [9] and [10].

Theorem 1 *Let $\mathcal{R}$ be a linear subset of $\mathcal{B}(H)$ such that $\mathcal{R}_1$ is totally bounded in the weak–operator topology τ_w, and define a mapping ϕ of $\mathcal{R}$ into the dual space $\mathcal{R}_\sharp^*$ of $\mathcal{R}_\sharp$ by*

$$\phi(T)(f) = f(T) \quad (T \in \mathcal{R}).$$

Then ϕ is one–one and linear, and is uniformly continuous on $\mathcal{R}_1$. Moreover, $\phi(\mathcal{R}_1)$ is weak–dense in the unit ball of $\mathcal{R}_\sharp^*$, and the restriction of ϕ^{-1} to $\phi(\mathcal{R}_1)$ is uniformly continuous with respect to the weak*–topology on $\mathcal{R}_\sharp^*$ and the weak–operator topology on $\mathcal{R}_1$.*

PROOF. Since ϕ is clearly linear, to prove that it is one–one we need only show that its kernel is $\{0\}$. But if $\phi(T) = 0$, then we have $\langle Tx, y\rangle = \phi(T)\,(f_{x,y}) = 0$ for all $x, y \in H$; whence $T = 0$.

For each $f \in \mathcal{R}_\sharp$ the mapping $T \mapsto \phi(T)(f)$ equals f and so is τ_w–uniformly continuous on $\mathcal{R}_1$. It follows immediately that ϕ is uniformly continuous as a mapping of $(\mathcal{R}_1, \tau_w)$ into $\mathcal{R}_\sharp^*$ (with the weak*–topology). Hence $K = \phi(\mathcal{R}_1)$ is weak*–totally bounded, and therefore located, in $\mathcal{R}_\sharp^*$ (see [7, 10]). Let u be an element of $\mathcal{R}_\sharp^*$, let $\{f_1, \ldots, f_N\}$ be a finitely enumerable subset of $\mathcal{R}_\sharp$, and let ε be a positive number. To prove that $\phi(\mathcal{R}_1)$ is weak*–dense in the unit ball $(\mathcal{R}_\sharp^*)_1$ of $\mathcal{R}_\sharp^*$, it is enough to show that K intersects the neighbourhood

$$V = \left\{v \in \left(\mathcal{R}_\sharp^*\right)_1 : |(u - v)\,(f_k)| < 3\varepsilon \;\; (1 \le k \le N)\right\}$$

of u in $(\mathcal{R}_\sharp^*)_1$. To this end, choose a finite–dimensional subspace $\mathcal{G}$ of $\mathcal{R}_\sharp$ such that

$$\inf\{\|f_k - g\| : g \in \mathcal{G}\} < \varepsilon \quad (1 \le k \le N)$$

([3], page 308, Lemma (2.5)); for each k $(1 \le k \le N)$, then choose $g_k \in \mathcal{G}$ such that $\|f_k - g_k\| < \varepsilon$. The dual space $\mathcal{G}^*$ of $\mathcal{G}$ is a finite–dimensional Banach space with respect to the usual norm defined by

$$\|v\|' = \sup\{|v(g)| : g \in \mathcal{G},\ \|g\| \le 1\}.$$

Since $K \subset (\mathcal{R}_\sharp^*)_1$, and $(\mathcal{R}_\sharp^*)_1$ is a subset of the unit ball of $(\mathcal{G}^*)_1$, we can regard u and, for each $T \in \mathcal{R}_1$, the functional $\phi(T)$ as elements of $(\mathcal{G}^*)_1$. Now suppose that

$$\inf\left\{\|u - \phi(T)\|' : T \in \mathcal{R}_1\right\} \neq 0. \qquad (5.1)$$

Note that K, being weak*–totally bounded, is located in $\mathcal{G}_1$. By Corollary (4.4) on page 341 of [3], there exists a linear functional ψ, with norm 1, on $\mathcal{G}^*$ such that

$$|\psi(u)| > \sup\{|\psi(v)| : v \in K\}.$$

Since $\mathcal{G}$ is finite–dimensional, ψ is weak*–uniformly continuous on $(\mathcal{G}^*)_1$; so, by Corollary (6.9) on page 357 of [3], there exists $g \in \mathcal{G}$ such that $\psi(v) = v(g)$ for all $v \in \mathcal{G}^*$. In particular,

$$\begin{aligned} |u(g)| &> \sup\{|v(g)| : v \in K\} \\ &= \sup\{|\phi(T)(g)| : T \in \mathcal{R}_1\} \\ &= \sup\{g(T) : T \in \mathcal{R}_1\} \\ &= \|g\|, \end{aligned}$$

which is absurd since $u \in (\mathcal{R}_\sharp^*)_1$. We conclude that (1) is false, and hence that

$$\|u - \phi(T_0)\|' < \frac{\varepsilon}{M+1}$$

for some $T_0 \in \mathcal{R}_1$, where

$$M = \max_{1 \le k \le N} \|g_k\| .$$

For each k $(1 \le k \le N)$ we now have

$$\begin{aligned} |(u - \phi(T_0))\,(f_k)| &\le |(u - \phi(T_0))\,(f_k - g_k)| + |(u - \phi(T_0))\,(g_k)| \\ &\le 2\,\|f_k - g_k\| + \|u - \phi(T_0)\|'\,\|g_k\| \\ &< 2\varepsilon + \frac{\varepsilon}{M+1} M \\ &< 3\varepsilon; \end{aligned}$$

in other words, $\phi(T_0) \in V$. Since $\phi(T_0) \in K$, this completes the proof that $\phi(\mathcal{R}_1)$ is dense in $(\mathcal{R}_\sharp^*)_1$.

Finally, the uniform continuity of the inverse mapping on K follows from the identity

$$|\langle Tx, y\rangle| = |\phi(T)\,(f_{x,y}))| \quad (x, y \in H;\ T \in \mathcal{R}_1)\,,$$

with reference to the definitions of the weak*– and weak–operator topologies, and to Proposition 1.2.8 on page 19 of [12]. Q.E.D.

We proved in [7] (see also [8]) that, under the hypotheses of Theorem 1, the ultraweakly continuous linear functionals on $\mathcal{R}$ extend to ultraweakly continuous linear functionals on $\mathcal{B}(H)$ and are precisely those functionals f_A mapping T to Trace(TA), with A a trace–class operator on H. The norm of f_A on $\mathcal{R}$ is

$$\|f_A\|_{\mathcal{R}} = \sup\{|\mathrm{Trace}(TA)| : T \in \mathcal{R}_1\}\,,$$

which in the case $\mathcal{R} = \mathcal{B}(H)$ equals the trace–class norm

$$\|A\|_2 = \mathrm{Trace}\,(A)$$

of A (see [4]).

Taken with Theorem 1, these observations lead to

Corollary 2 *Let $\mathcal{R}$ be a linear subset of $\mathcal{B}(H)$ such that $\mathcal{R}_1$ is totally bounded in the weak–operator topology τ_w. Let $\mathcal{T}(H)$ denote the set of trace–class operators on H, taken with the norm*

$$\|A\|_{\mathcal{R}} = \sup\{|\mathrm{Trace}(TA)| : T \in \mathcal{R}_1\},$$

Then

$$\Phi(T)(A) = \mathrm{Trace}(TA) \quad (T \in \mathcal{R},\ A \in \mathcal{T}(H))$$

defines a one–one linear mapping Φ of $\mathcal{R}$ into the dual space $\mathcal{T}(H)^*$ with the following properties.

(i) *$\Phi(\mathcal{R}_1)$ is dense in the unit ball $\mathcal{T}(H)^*_1$ of $\mathcal{T}(H)^*$.*

(ii) *Φ is uniformly continuous on $\mathcal{R}_1$.*

(iii) *the restriction of Φ^{-1} to $\Phi(\mathcal{R}_1)$ is uniformly continuous relative to the weak*–topology on $\Phi(\mathcal{R}_1)$ and the weak–operator topology on $\mathcal{R}_1$.*

Corollary 3 *Under the hypotheses of* Theorem 1 *and* Corollary 2, *the following conditions are equivalent.*

(i) *$\mathcal{R}_1$ is weak–operator complete.*

(ii) *ϕ maps $\mathcal{R}_1$ onto the unit ball of $\mathcal{R}_\sharp^*$.*

(iii) *Φ maps $\mathcal{R}_1$ onto the unit ball of $\mathcal{T}(H)^*$ relative to the norm $\|\cdot\|_{\mathcal{R}}$.*

PROOF. This is a special case of the following general lemma about metric spaces, whose straightforward proof we omit. Q.E.D.

Lemma 4 *Let X be a metric space, Y a complete metric space, and ϕ a one–one uniformly continuous mapping of X onto a dense subset of Y such that ϕ^{-1} is uniformly continuous on $\phi(X)$. Then X is complete if and only if $\phi(X) = Y$.*

Classically, any von Neumann algebra—that is, weak–operator closed *–subalgebra of $\mathcal{B}(H)$—can be identified, via the mapping ϕ, with the dual of its predual $\mathcal{R}_\sharp$ ([12], Vol. 2, page 482). If this were provable constructively, then we could use Theorem 1 to prove that $\mathcal{B}_1(H)$ is τ_w–complete, which, as mentioned above, we cannot do within constructive mathematics. Thus Theorem 1 appears to be the best general constructive result of its type.