Subjective Probability

The Real Thing

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2004

Printed in the United States of America

Typeface Computer Modern 10/12 pt. *System* LaTeX 2ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Jeffrey, Richard C.
Subjective probability : the real thing / Richard Jeffrey.
p. cm.
Includes bibliographical references and index.
ISBN 0-521-82971-2 – ISBN 0-521-53668-5 (pbk.)
1. Probability. 2. Judgment. I. Title.
BC141.J445 2004
121′.63–dc22 2003065689

ISBN 0 521 82971 2 hardback
ISBN 0 521 53668 5 paperback

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Probability Primer

Yes or no: Was there once life on Mars? We can't say. What about intelligent life? That seems most unlikely, but again, we can't really say. The simple yes-or-no framework has no place for shadings of doubt, no room to say that we see intelligent life on Mars as far less probable than life of a possibly very simple sort. Nor does it let us express exact probability judgments, if we have them. We can do better.

1.1 Bets and Probabilities

What if you were able to say exactly what odds you would give on there having been life, or intelligent life, on Mars? That would be a more nuanced form of judgment, and perhaps a more useful one. Suppose your odds were 1:9 for life, and 1:999 for intelligent life, corresponding to probabilities of 1/10 and 1/1000, respectively. (The colon is commonly used as a notation for "/", division, in giving odds—in which case it is read as "to".)

Odds m:n correspond to probability $\frac{m}{m+n}$

That means you would see no special advantage for either player in risking one dollar to gain nine in case there was once life on Mars; and it means you would see an advantage on one side or the other if those odds were shortened or lengthened. And similarly for intelligent life on Mars when the risk is a thousandth of the same ten dollars (1 cent) and the gain is 999 thousandths ($9.99).

Here is another way of saying the same thing: You would think a price of one dollar just right for a ticket worth ten if there was life on Mars and nothing if there was not, but you would think a price of only one cent right if there would have had to have been intelligent life on Mars for the ticket to be worth ten dollars. These are the two tickets:

Worth $10 if there was life on Mars.	Worth $10 if there was intelligent life on Mars.
Price: $1 Probability: 0.1	Price: 1 cent Probability: 0.001

So if you have an exact judgmental probability for truth of a hypothesis, it corresponds to your idea of the dollar value of a ticket that is worth 1 unit or nothing, depending on whether the hypothesis is true or false. (For the hypothesis of mere life on Mars the unit was $10; the price was a tenth of that.)

Of course you need not have an exact judgmental probability for life on Mars, or for intelligent life there. Still, we know that any probabilities anyone might think acceptable for those two hypotheses ought to satisfy certain rules, e.g., that the first cannot be less than the second. That is because the second hypothesis implies the first. (See the implication rule at the end of sec. **1.3** below.) In sec. **1.2** we turn to the question of what the laws of judgmental probability are, and why. Meanwhile, take some time with the following questions, as a way of getting in touch with some of your own ideas about probability. Afterward, read the discussion that follows.

Questions

1. A vigorously flipped thumbtack will land on the sidewalk. Is it reasonable for you to have a probability for the hypothesis that it will land point up?
2. An ordinary coin is to be tossed twice in the usual way. What is your probability for the head turning up both times?
   (a) 1/3, because 2 heads is one of three possibilities: 2, 1, 0 heads?
   (b) 1/4, because 2 heads is one of four possibilities: HH, HT, TH, TT?
3. There are three coins in a bag: ordinary, two-headed, and two-tailed. One is shaken out onto the table and lies head up. What should be your probability that it's the two-headed one?
   (a) 1/2, since it can only be two-headed or normal?

(b) 2/3, because the other side could be the tail of the normal coin, or either side of the two-headed one? (Suppose the sides have microscopic labels.)

4. *It's a goy!*[1]

   (a) As you know, about 49% of recorded human births have been girls. What is your judgmental probability that the first child born after time t (say, $t =$ the beginning of the 22nd century, GMT) will be a girl?

   (b) A *goy* is defined as a girl born before t or a boy born thereafter. As you know, about 49% of recorded human births have been goys. What is your judgmental probability that the first child born in the 22nd century will be a goy?

Discussion

1. Surely it is reasonable to suspect that the geometry of the tack gives one of the outcomes a better chance of happening than the other; but if you have no clue about which of the two has the better chance, it may well be reasonable for each to have judgmental probability 1/2. Evidence about the chances might be given by statistics on tosses of similar tacks, e.g., if you learned that in 20 tosses there were 6 up's you might take the chance of up to be in the neighborhood of 30%; and whether or not you do that, you might well adopt 30% as your judgmental probability for up on the next toss.

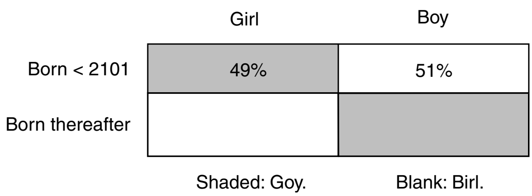
2,3. These questions are meant to undermine the impression that judgmental probabilities can be based on analysis into cases in a way that does not already involve probabilistic judgment (e.g., the judgment that the cases are equiprobable). In either problem you can arrive at a judgmental probability by trying the experiment (or a similar one) often enough, and seeing the statistics settle down close enough to 1/2 or to 1/3 to persuade you that more trials will not reverse the indications. In each of these problems it is the finer of the two suggested analyses that happens to make more sense; but any analysis can be refined in significantly different ways, and there is no point at which the process of refinement has to stop. (Head or tail can be refined to head–facing–north or head–not–facing–north or tail.) Indeed some of these analyses

[1] This is a fanciful adaptation of Nelson Goodman's (1983, 73–74) "grue" paradox.

seem more natural or relevant than others, but that reflects the probability judgments we bring with us to the analyses.

4. *Goys and birls.* This question is meant to undermine the impression that judgmental probabilities can be based on frequencies in a way that does not already involve judgmental probabilities. Since all girls born so far have been goys, the current statistics for girls apply to goys as well: these days, about 49% of human births are goys. Then if you read probabilities off statistics in a straightforward way your probability will be 49% for each of these hypotheses:

   (1) The first child born after t will be a girl.
   (2) The first child born after t will be a goy.



Thus $pr(1) + pr(2) = 98\%$. But it is clear that those probabilities should sum to 100%, since (2) is logically equivalent to

   (3) The first child born after t will be a boy,

and $pr(1) + pr(3) = 100\%$. To avoid this contradiction you must decide which statistics are relevant to $pr(1)$: the 49% of girls born before 2001, or the 51% of boys. And that is not a matter of statistics but of judgment—no less so because we would all make the same judgment: the 51% of boys.

1.2 Why Probabilities Are Additive

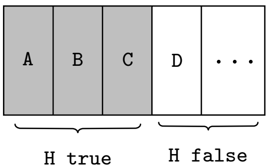
Authentic tickets of the Mars sort are hard to come by. Is the first of them really worth \$10 to me if there was life on Mars? Probably not. If the truth is not known in my lifetime, I cannot cash the ticket even if it is really a winner. But some probabilities are plausibly represented by prices, e.g., probabilities of the hypotheses about athletic contests and lotteries that people commonly bet on. And it is plausible to think that the general laws of probability ought to be the same for all hypotheses—about planets no less than about ball games. If that is so, we can justify laws of probability if we can prove all betting policies that violate them to be inconsistent. Such justifications are called "Dutch book arguments".[2] We shall give a Dutch book argument for the requirement that probabilities be additive in this sense:

> **Finite Additivity.** The probability of a hypothesis that can be true in a finite number of incompatible ways is the sum of the probabilities of those ways.

EXAMPLE 1, **Finite additivity.** The probability p of the hypothesis

(H) A sage will be elected

is $q + r + s$ if exactly three of the candidates are sages and their probabilities of winning are q, r, and s. In the following diagram, $A, B, C, D, \ldots$ are the hypotheses that the various different candidates win—the first three being the sages. The logical situation is diagrammed as follows, where the points in the big rectangle represent all the ways the election might come out, specified in minute detail, and the small rectangles represent the ways in which the winner might prove to be A, or B, or C, or D, etc.



Probabilities of cases A, B, C, D,...
are q, r, s, t,..., respectively.

1.2.1 Dutch Book Argument for Finite Additivity

For definiteness we suppose that the hypothesis in question is true in three cases, as in example 1; the argument differs inessentially for other examples, with other finite numbers of cases. Now consider the following four tickets.

[2] In British racing jargon a *book* is the set of bets a bookmaker has accepted, and a book *against* someone—a "Dutch" book—is one the bookmaker will suffer a net loss on no matter how the race turns out. I follow Brian Skyrms in seeing F. P. Ramsey as holding Dutch book arguments to demonstrate actual inconsistency. See Ramsey's "Truth and Probability" in his *Philosophical Papers,* D. H. Mellor (ed.), Cambridge University Press, 1990.

`Worth $1 if` H `is true.`	Price $\$p$
`Worth $1 if` A `is true.`	Price $\$q$
`Worth $1 if` B `is true.`	Price $\$r$
`Worth $1 if` C `is true.`	Price $\$s$

Suppose I am willing to buy or sell any or all of these tickets at the stated prices. Why should p be the sum $q + r + s$? Because no matter what it is worth—\$1 or \$0—the ticket on H is worth exactly as much as the tickets on A, B, C together. (If H loses it is because A, B, C all lose; if H wins it is because exactly one of A, B, C wins.) Then if the price of the H ticket is different from the sum of the prices of the other three, I am inconsistently placing different values on one and the same contract, depending on how it is presented.

If I am inconsistent in that way, I can be fleeced by anyone who will ask me to buy the H ticket and sell or buy the other three depending on whether p is more or less than $q + r + s$. Thus, no matter whether the equation $p = q + r + s$ fails because the left—hand side is more or less than the right, a book can be made against me. That is the Dutch book argument for additivity when the number of ultimate cases under consideration is finite. The talk about being fleeced is just a way of dramatizing the inconsistency of any policy in which the dollar value of the ticket on H is anything but the sum of the values of the other three tickets: To place a different value on the three tickets on A, B, C from the value you place on the H ticket is to place different values on the same commodity bundle under two demonstrably equivalent descriptions.

When the number of cases is infinite, a Dutch book argument for additivity can still be given—provided the infinite number is not too big! It turns out that not all infinite sets are the same size.

EXAMPLE 2, **Cantor's diagonal argument.** The positive integers (I^+'s) can be counted off, just by naming them successively: 1, 2, 3, ... On the other hand, the sets of positive integers (such as {1, 2, 3, 5, 7} or the set of all even numbers, or the set of multiples of 17, or {1, 11, 101, 1001, 10001} cannot be counted off as first, second, ..., with each such set appearing as n'th in the list for some finite positive integer n. This was proved by Georg Cantor (1895) as follows. Any set of I^+'s can be represented by an endless string of plusses and minuses ("signs"), e.g., the set of even I^+'s by the string $- + - + \ldots$ in which plusses appear at the even numbered positions and minuses at the odd, the set $\{2, 3, 5, 7, \ldots\}$ of prime numbers by an endless string that begins $- + + - + - +$, the set of all the I^+'s by an endless string of plusses, and the set of no I^+'s by an endless string of minuses. Cantor proved that no list of endless strings of signs can be complete. He used an amazingly simple method ("diagonalization") which, applied to any such list, yields an endless string $\bar{d}$ of signs which is not in that list. Here's how. For definiteness, suppose the first four strings in the list are the examples already given, so that the list has the general shape

$$s_1 : - + - + \ldots$$
$$s_2 : - + + - \ldots$$
$$s_3 : + + + + \ldots$$
$$s_4 : - - - - \ldots$$
$$\text{etc.}$$

Define the *diagonal* of that list as the string d consisting of the first sign in s_1, the second sign in s_2, and, in general, the n'th sign in s_n:

$$- + + - \ldots$$

And Define the *antidiagonal* $\bar{d}$ of that list as the result $\bar{d}$ of reversing all the signs in the diagonal,

$$\bar{d} : + - - + \ldots$$

In general, for any list $s_1, s_2, s_3, s_4, \ldots,$ $\bar{d}$ cannot be any member s_n of the list, for, by definition, the n'th sign in $\bar{d}$ is different from the n'th sign of s_n, whereas if $\bar{d}$ were some s_n, those two strings would have to agree, sign by sign. Then the set of I^+'s defined by the antidiagonal of a list cannot be in that list, and therefore no list of sets of I^+'s can be complete.

> **Countability.** A countable set is defined as one whose members (if any) can be arranged in a single list, in which each member appears as the n'th item for some finite n.

Of course any finite set is countable in this sense, and some infinite sets are countable. An obvious example of a countably infinite set is the set $I^+ = \{1, 2, 3, \ldots\}$ of all positive whole numbers. A less obvious example is the set I of all the whole numbers, positive, negative, or zero: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The members of this set can be rearranged in a single list of the sort required in the definition of countability:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

So the set of all the whole numbers is countable. Order does not matter, as long as every member of I shows up somewhere in the list.

EXAMPLE 3, **Countable additivity.** In example 1, suppose there were an endless list of candidates, including no end of sages. If H says that a sage wins, and $A_1, A_2, \ldots$ identify the winner as the first, second, ... sage, then an extension of the law of finite additivity to countably infinite sets would be this:

> **Countable Additivity.** The probability of a hypothesis H that can be true in a countable number of incompatible ways $A_1, A_2, \ldots$ is the sum $pr(H) = pr(A_1) + pr(A_2) + \ldots$ of the probabilities of those ways.

This equation would be satisfied if the probability of one or another sage's winning were $pr(H) = 1/2$, and the probabilities of the first, second, third, etc. sage's winning were $1/4, 1/8, 1/16$, etc., decreasing by half each time.

1.2.2 Dutch Book Argument for Countable Additivity

Consider the following infinite array of tickets, where the mutually incompatible A's collectively exhaust the ways in which H can be true (as in example 3).[3]

[3] No matter that there is not enough paper in the universe for an infinity of tickets. One small ticket can save the rain forest by doing the work of all the A tickets together. This eco-ticket will say: 'For each positive whole number n, pay the bearer \$1 if A_n is true.'

`Pay the bearer $1 if H is true.`	Price $*pr(H)*
`Pay the bearer $1 if A`$_1$` is true.`	Price $*pr(A*$_1$*)*
`Pay the bearer $1 if A`$_2$` is true.`	Price $*pr(A*$_2$*)*
...	...

Why should my price for the first ticket be the sum of my prices for the others? Because no matter what it is worth—$1 or $0—the first ticket is worth exactly as much as all the others together. (If H loses it is because the others all lose; if H wins it is because exactly one of the others wins.) Then if the first price is different from the sum of the others, I am inconsistently placing different values on one and the same contract, depending on how it is presented.

Failure of additivity in these cases implies inconsistency of valuations: a judgment that certain transactions are at once (1) reasonable and (2) sure to result in an overall loss. Consistency requires additivity to hold for countable sets of alternatives, finite or infinite.

1.3 Probability Logic

The simplest laws of probability are the consequences of finite additivity under this additional assumption:

> Probabilities are real numbers in the range from 0 to 1, with the endpoints reserved for certainty of falsehood and of truth, respectively.

This makes it possible to read probability laws off diagrams, much as we read ordinary logical laws off them. Let's see how that works for the ordinary ones, beginning with two surprising examples (where "iff" means *if and only if*):

De Morgan's Laws

(1) $\neg(G \wedge H) = \neg G \vee \neg H$ ("Not both true iff at least one false")

(2) $\neg(G \vee H) = \neg G \wedge \neg H$ ("Not even one true iff both false")

Here the bar, the wedge and juxtaposition stand for *not, or*, and *and.* Thus, if G and H are two hypotheses,

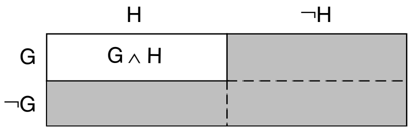
$G \wedge H$ (or GH) says that they are both true: G AND H

$G \vee H$ says that at least one is true: G OR H

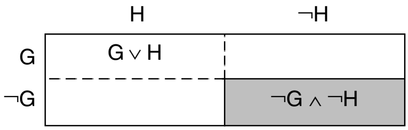
$\neg G$ (or $-G$ or $\bar{G}$) says that G is false: NOT G

In the following diagrams for De Morgan's laws the upper and lower rows represent G and $\neg G$ and the left- and right-hand columns represent H and $\neg H$. Now if R and S are any regions, $R \wedge S$ (or "RS"), is their intersection, $R \vee S$ is their union, and $\neg R$ is the whole big rectangle except for R.

Diagrams for De Morgan's laws (1) and (2):



(1) Shaded: $\neg(G \wedge H) = \neg G \vee \neg H$



(2) Shaded: $\neg(G \vee H) = \neg G \wedge \neg H$

Adapting such geometrical representations to probabilistic reasoning is just a matter of thinking of the probability of a hypothesis as its region's area, assuming that the whole rectangle, $H \vee \neg H$ $(= G \vee \neg G)$, has area 1. Of course the empty region, $H \wedge \neg H$ $(= G \wedge \neg G)$, has area 0. It is useful to denote those two extreme regions in ways independent of any particular hypotheses H, G. Let's call them $\top$ and $\bot$:

Logical Truth. $\top = H \vee \neg H = G \vee \neg G$

Logical Falsehood. $\bot = H \wedge \neg H = G \wedge \neg G$

We can now verify some further probability laws informally, in terms of areas of diagrams.

Not: $pr(\neg H) = 1 - pr(H)$

Verification. The non-overlapping regions H and $\neg H$ exhaust the whole rectangle, which has area 1. Then $pr(H) + pr(\neg H) = 1$, so $pr(\neg H) = 1 - pr(H)$.

Or: $pr(G \vee H) = pr(G) + pr(H) - pr(G \wedge H)$

Verification. The $G \vee H$ area is the G area plus the H area, except that when you simply add $pr(G) + pr(H)$ you count the $G \wedge H$ part twice. So subtract it on the right-hand side.

The word "but"—a synonym for "and"—may be used when the conjunction may be seen as a contrast, as in "it's green but not healthy", $G \wedge \neg H$:

But Not: $pr(G \wedge \neg H) = pr(G) - pr(G \wedge H)$

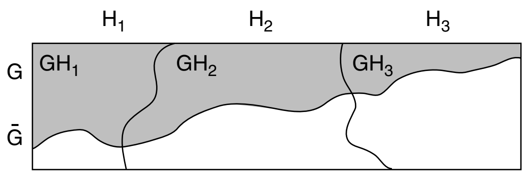
Verification. The $G \wedge \bar{H}$ region is what remains of the G region after the $G \wedge H$ part is deleted.

Dyadic Analysis: $pr(G) = pr(G \wedge H) + pr(G \wedge \neg H)$

Verification. See the diagram for De Morgan (1). The G region is the union of the nonoverlapping $G \wedge H$ and $G \wedge \neg H$ regions.

In general, there is a rule of n-adic analysis for each n, e.g., for n=3:

Triadic Analysis: If H_1, H_2, H_3 *partition* $\top$,[4] then
$pr(G) = pr(G \wedge H_1) + pr(G \wedge H_2) + pr(G \wedge H_3)$.



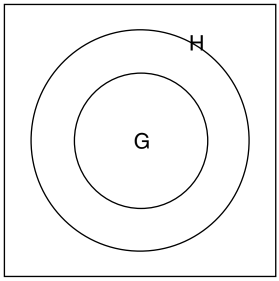
The next rule follows immediately from the fact that logically equivalent hypotheses are always represented by the same region of the diagram—in view of which we use the sign "=" of identity to indicate logical equivalence.

Equivalence: If $H = G$, then $pr(H) = pr(G)$.
(Logically equivalent hypotheses are equiprobable.)

Finally: To be implied by G, the hypothesis H must be true in every case in which G is true. Diagrammatically, this means that the G region is entirely included in the H region. In the figure below, G is represented by the small disk, and H by the large disk; H is sitting on the rim to indicate that H comprises both the annulus and the little disk. Then if

[4] This means that, as a matter of logic, the H's are mutually exclusive ($H_1 \wedge H_2 = H_1 \wedge H_3 = H_2 \wedge H_3 = \bot$) and collectively exhaustive ($H_1 \vee H_2 \vee H_3 = \top$). The equation also holds if the H's merely pr-partition $\top$ in the sense that $pr(H_i \wedge H_j) = 0$ whenever $i \neq j$ and $pr(H_1 \wedge H_2 \wedge H_3) = 1$.

G implies H, the G region can have no larger an area than the H region.



Implication: If G implies H, then $pr(G) \leq pr(H)$.

1.4 Conditional Probability

We identified your ordinary (unconditional) probability for H as the price representing your valuation of the following ticket:

Worth \$1 if *H* is true.	Price \$*pr(H)*

Now we identify your conditional probability for H given D as the price representing your valuation of this ticket:

Worth \$1 if *D*∧*H* is true, Worth \$*pr(H/D)* if *D* is false.	Price \$*pr(H|D)*

The old ticket represented a simple bet on H; the new one represents a conditional bet on H—a bet that is called off (the price of the ticket is refunded) in case the condition D fails. If D and H are both true, the bet is on and you win, the ticket is worth \$1. If D is true but H is false, the bet is on and you lose, the ticket is worthless. And if D is false, the bet is off, you get your \$$pr(H|D)$ back as a refund.

With that understanding we can construct a Dutch book argument for the following rule, which connects conditional and unconditional probabilities:

Product Rule: $pr(H \wedge D) = pr(H|D)pr(D)$

Dutch Book Argument for the Product Rule.[5] Imagine that you own three tickets, which you can sell at prices representing your valuations. The first is ticket (1) above. The second and third are the following two, which represent unconditional bets of \$1 on HD and of \$$pr(H|D)$ *against* D,

(2)	`Worth $1 if` $H \wedge D$ `is true.`	Price \$$pr(H \wedge D)$
(3)	`Worth` $pr(H/D)$ `if` D `is false.`	Price \$$pr(H|D)pr(\neg D)$

Bet (3) has a peculiar payoff: not a whole dollar, but only \$$pr(H|D)$. That is why its price is not the full \$$pr(\neg D)$ but only the fraction $pr(\neg D)$ of the \$$pr(H|D)$ that you stand to win. This payoff was chosen to equal the price of the first ticket, so that the three fit together into a neat book.

Observe that in every possible case regarding truth and falsity of H and D the tickets (2) and (3) together have the same dollar value as ticket (1). (You can verify that claim with pencil and paper.) Then there is nothing to choose between ticket (1) and tickets (2) and (3) together, and therefore it would be inconsistent to place different values on them. Thus, your price for (1) ought to equal the sum of your prices for (2) and (3):

$$pr(H|D) = pr(H \wedge D) + pr(\neg D)pr(H|D)$$

Now set $pr(\neg D) = 1 - pr(D)$, multiply through, cancel $pr(H|D)$ from both sides and solve for $pr(H \wedge D)$. The result is the product rule. To violate that rule is to place different values on the same commodity bundle in different guises: (1), or the package (2, 3).

The product rule is more familiar in a form where it is solved for the conditional probability $pr(H|G)$:

$$\textbf{Quotient Rule: } pr(H|D) = \frac{pr(H \wedge D)}{pr(D)}, \text{ provided } pr(D) > 0.$$

Graphically, the quotient rule expresses $pr(H|D)$ as the fraction of the D region that lies inside the H region. It is as if calculating $pr(H|D)$

[5] de Finetti (1937, 1980).

were a matter of trimming the whole $D \vee \neg D$ rectangle down to the D part, and using that as the new unit of area.

The quotient rule is often called the definition of conditional probability. It is not. If it were, we could never be in the position we are often in, of making a conditional judgment—say, about how a coin that may or may not be tossed will land—without attributing some particular positive value to the condition that $pr(\text{head}|\text{tossed}) = 1/2$ even though

$$\frac{pr(\text{head} \ \wedge \ \text{tossed})}{pr(\text{tossed})} = \frac{\textit{undefined}}{\textit{undefined}}.$$

Nor—perhaps, less importantly—would we be able to make judgments like the following, about a point (of area 0!) on the Earth's surface:

$$pr(\text{in western hemisphere} \mid \text{on equator}) = 1/2$$

even though

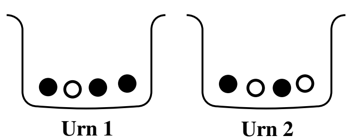
$$\frac{pr(\text{in western hemisphere} \ \wedge \ \text{on equator})}{pr(\text{on equator})} = \frac{0}{0}.$$

The quotient rule merely restates the product rule; and the product rule is no definition but an essential principle relating two distinct sorts of probability.

By applying the product rule to the terms on the right-hand sides of the analysis rules in sec. 1.3 we get the rule of[6]

Total Probability: If the D's partition $\top$
then $pr(H) = \sum_i pr(D_i)pr(H|D_i)$.[7]

EXAMPLE, A ball will be drawn blindly from urn 1 or urn 2, with odds 2:1 of being drawn from urn 2. Is black or white the more probable outcome?



Solution. By the rule of total probability with H = black and D_i = drawn from urn i, we have $pr(H) = pr(H|D_1)P(D_1) + pr(H|D_2)P(D_2) = (\frac{3}{3} \cdot \frac{1}{3}) + (\frac{1}{2} \cdot \frac{2}{3}) = \frac{1}{4} \cdot \frac{1}{3} = \frac{7}{12} > \frac{1}{2}$: Black is the more probable outcome.

[6] Here the sequence of D's is finite or countably infinite.
[7] $= pr(D_1)pr(H|D_1) + pr(D_2)pr(H|D_2) + \ldots$

1.5 Why "|" Cannot Be a Connective

The bar in "$pr(H|D)$" is not a connective that turns pairs H, D of propositions into new, conditional propositions, *H if D*. Rather, it is as if we wrote the conditional probability of H given D as "$pr(H, D)$": The bar is a typographical variant of the comma. Thus we use "*pr*" for a function of one variable as in "$pr(D)$" and "$pr(H \wedge D)$", and also for the corresponding function of two variables as in "$pr(H|D)$". Of course the two are connected—by the product rule.

Then in fact we do not treat the bar as a statement–forming connective, "if"; but why couldn't we? What would go wrong if we did? This question was answered by David Lewis in 1976, pretty much as follows.[8] Consider the simplest special case of the rule of total probability:

$$pr(H) = pr(H|D)pr(D) + pr(H|\neg D)pr(\neg D)$$

Now if "|" is a connective and D and C are propositions, then $D|C$ is a proposition too, and we are entitled to set $H = D|C$ in the rule. Result:

(1) $pr(D|C) = pr[(D|C)|D]pr(D) + pr[(D|C)|\neg D]pr(\neg D)$

So far, so good. But remember: "|" means *if*. Therefore, "$(D|C)|X$" means *If X, then if C then D*. And as we ordinarily use the word "if", this comes to the same as *If X and C, then D*:

(2) $(D|C)|X = D|XC$

(Recall that the identity means the two sides represent the same region, i.e., the two sentences are logically equivalent.) Now by two applications of (2) to (1) we have

(3) $pr(D|C) = pr(D|D \wedge C)pr(D) + pr(D|\neg D \wedge C)pr(\neg D)$

But as $D \wedge C$ and $\neg(D \wedge C)$ respectively imply and contradict D, we have $pr(D|D \wedge C) = 1$ and $pr(D|\neg D \wedge C) = 0$. Therefore, (3) reduces to

(4) $pr(D|C) = pr(D)$

[8] For Lewis's "trivialization" result (1976), see his (1986). For subsequent developments, see the papers Hájek, *Probabilities and Conditionals*, Ellery Eells and Brian Skyrms (eds.), Cambridge University Press (1994), and Hall, *Probabilities and Conditionals*, Ellery Eells and Brian Skyrms (eds.), Cambridge University Press (1994); other papers in this book cover additional developments.