Preface

The notion of amenability has its origins in the beginnings of modern measure theory: Does a finitely additive set function exist which is invariant under a certain group action? Since the 1940s, amenability has become an important concept in abstract harmonic analysis (or rather, more generally, in the theory of semitopological semigroups). In 1972, B. E. Johnson showed that the amenability of a locally compact group G can be characterized in terms of the Hochschild cohomology of its group algebra $L^1(G)$: this initiated the theory of amenable Banach algebras. Since then, amenability has penetrated other branches of mathematics, such as von Neumann algebras, operator spaces, and even differential geometry.

In the summer term of 1999, I taught a course on amenability at the Universität des Saarlandes. My goals were lofty: I wanted to show my students how the concept of amenability orginated from measure theoretic problems — of course, the Banach–Tarski paradox would have to be covered —, how it moved from there to abstract harmonic analysis, how it then ventured into the theory of Banach algebras, and how it impacted areas as diverse as von Neumann algebras and differential geometry. I had also planned to include very recent developments such as C. J. Read's construction of a commutative, radical, amenable Banach algebra or Z.-J. Ruan's notion of operator amenability. On top of all this, I wanted my lectures to be accessible to students who had taken a one-year course in functional analysis (including the basics of Banach and C^*-algebras), but who had not not necessarily any background in operator algebras, homological algebra, or abstract harmonic analysis. Lofty as they were, these goals were unattainable, of course, in a one-semester course ...

The present notes are an attempt to resurrect the original plan of my lectures at least in written form. They are a polished (and greatly expanded) version of the notes I actually used in class. In particular, this is *not* a research monograph that exhaustively presents our current state of knowledge on amenability. These notes are intended to introduce second year graduate students to a fascinating area of modern mathematics and lead them, within a reasonable period of time, to a level from where they can go on to read original research papers on the subject. This has, of course, influenced the exposition: The order in which material is presented is pedagogically rather than systematically motivated, and the style is, in general, chatty and informal.

I am a firm believer in learning by doing: Nobody has ever learned anything — especially not mathematics! — just by passively absorbing a teacher's performance. For this reason, there are numerous exercises interspersed with the main text of these notes (I didn't count them). These exercises vary greatly in their degree of difficulty: Some just ask the reader to make a fairly obvious, but nevertheless important observation, others ask him/her to fill in a tedious detail of a proof, and again others challenge him/her to deepen his/her understanding of the material by actively wrestling with it. My frequent interjections in the text serve a similar purpose: Just because something is claimed to be clear/obvious/immediate, that doesn't mean that one can just believe it without really checking.

The background required from a student working through these notes varies somewhat throughout the text:

– **Chapter 0**: This chapter only requires a very modest background in linear and abstract algebra, along with some mathematical maturity.
– **Chapter 1**: Anyone who has taken a course in measure theory and a first course in functional analysis should be able to read this chapter; the necessary background from abstract harmonic analysis is put together in Appendix A.
– **Chapter 2**: If you know the basics of Banach algebra theory, you're fit for this chapter. In addition, some facts about Banach space tensor products are needed: Appendix B contains a crash course on Banach space tensor products that provides the necessary background.
– **Chapter 3**: As for Chapter 2; additional background material from Banach space theory is collected in Appendix C.
– **Chapter 4**: You need a bit more background from Banach algebra theory than in the previous chapters. Also, a few basic facts on von Neumann algebras and one deep C^*-algebraic result are required. References are provided.
– **Chapter 5**: As for the previous three chapters. Some previous exposure to the language of categories and functors helps, of course, but isn't necessary. If you already know about homological algebra, this chapter should be very easy for you.
– **Chapter 6**: This is the most challenging chapter of these notes. Although references to standard texts on C^*- and von Neumann algebras are provided wherever necessary, anyone who wishes to get something out of this chapter, needs a certain degree of fluency in operator algebras.
– **Chapter 7**: As for Chapters 2 to 5; the necessary background from operator spaces can be found in Appendix D.
– **Chapter 8**: This chapter builds on Chapter 2 (and to a much lesser extent on Chapter 4). It contains the necessary background from infinite-dimensional differential geometry.

Each chapter concludes with a section entitled *Notes and comments*: These sections contain references to the original literature, as well as outlines of results that were not included in the main text, along with suggestions for further reading.

Since these notes are not intended to be a monograph, I have perhaps been a little less accurate when it comes to giving credit for results than I would have been otherwise. Hence, the universal disclaimer applies: Just because I didn't explicitly attribute a result to someone else, that doesn't mean that it's due to me. Furthermore, these notes perfectly reflect my preferences and prejudices towards amenability: It is my (highly subjective) belief that Banach algebras are the natural setting in which to deal with amenability. For this reason, most of these notes are devoted to amenable Banach algebras and their next of kin. Amenable groups are only presented to set the stage for amenable Banach algebras; amenable semigroups, (semi)group actions, representations, groupoids, etc., aren't even mentioned. This does not mean that I consider these topics unimportant or uninteresting: I just don't know enough about these things (yet) to be able to teach a course on them, let alone publish my lecture notes.

Finally, I would like to thank my students who attended the actual *Lectures on Amenability* at Saarbrücken and who made the course a joy to teach: Kim Louis, Matthias Neufang, Stefanie Schmidt, and a fourth one whom everyone only knew as “the algebraist”. I would also like to thank H. Garth Dales of the University of Leeds for his encouragement when he learned of my project of turning my class notes into some sort of book. Thanks are especially due to Matthias Neufang and Ross Stokke who read preliminary versions of the entire manuscript; both prevented an embarassing number of errors from making it into the final version. Of course, all omissions, inaccuracies, and outright errors that remain are my fault alone.

By the way, unless explicitly stated otherwise, all spaces and algebras in these notes are over $\mathbb{C}$.

Edmonton, August 2001 *Volker Runde*