1. Reflection and Refraction of Spherical Waves

Our previous book [1.1] was completely focused on the problem of plane and quasi-plane waves in layered media. In the theory of acoustic wave propagation, however, it is important to take into account that the sound source is located at a finite distance from the receiver as well as from the boundaries. The most simple example of this is the classical problem about the field of a point source in the presence of an interface between two homogeneous media. In other words, it is a problem of spherical wave reflection and refraction. For electromagnetic waves this was first considered by *A. Sommerfeld* [1.2]. Later, fundamental works by *H. Weyl*, *H. Ott*, *V. Fock*, *M. Leontovich*, *A. Baños* [1.3–7] appeared. Below we shall follow mainly our own works [1.8–11] which are the further development of *Weyl*'s idea of the representation of a spherical wave as a superposition of plane waves. Using the same techniques, the more difficult problem of the bounded wave beam reflection can be solved as well.

Below we shall consider reflection and refraction of acoustical waves at an interface of two fluids, including moving ones. Analogous problems for fluid–solid as well as two solid halfspace interfaces have also been considered [Refs. 1.12, Chap. 3; 1.13, Sect. 24; 1.14–21 and others]. The reader can find a more complete bibliography on spherical wave reflection and refraction at solid–solid and fluid–solid interfaces in the monographs [1.12, 15, 22, 23].

1.1 Integral Representation of the Sound Field

The main difficulty of the problem of spherical wave reflection and refraction at a planar interface is due to the difference in the symmetry of the wave and the interface – the latter is planar whereas the wave is spherical. It is natural therefore to solve the problem by representing the spherical wave as a superposition of plane waves, the reflection and refraction of which were discussed thoroughly in our first book [1.1].

The sound pressure in divergent spherical waves with arbitrary time dependence is given by

$$p = R^{-1}F(R/c - t) \, , \quad R = \sqrt{x^2 + y^2 + z^2} \, , \tag{1.1.1}$$

where F is an arbitrary smooth function and R is the distance between the observation point and the origin, where we have temporarily placed the source at an arbitrary location. In the case of a monochromatic wave $F(\tau) = \text{const} \cdot \exp(\mathrm{i}\omega\tau)$, $\tau \equiv R/c - t$. Omitting the arbitrary constant and the factor $\exp(-\mathrm{i}\omega t)$ we obtain the expression for a spherical wave $p = R^{-1}\exp(\mathrm{i}kR)$, where $k = \omega/c$ is the wave number.

At the plane $z = 0$, the field of the spherical wave is $r^{-1}\exp(\mathrm{i}kr)$, $r \equiv (x^2 + y^2)^{1/2}$. Let us represent this field as a two-fold Fourier integral in the coordinates x and y:

$$\frac{\exp(\mathrm{i}kr)}{r} = \iint\limits_{-\infty}^{+\infty} A(\xi_1, \xi_2)\exp[\mathrm{i}(\xi_1 x + \xi_2 y)]d\xi_1 d\xi_2 \ ,$$

where

$$A(\xi_1, \xi_2) = \iint\limits_{-\infty}^{+\infty} \frac{dxdy}{4\pi^2 r}\exp[\mathrm{i}(kr - \xi_1 x - \xi_2 y)] \ . \tag{1.1.2}$$

We introduce the polar coordinates

$$\xi_1 = \xi\cos\psi \ , \ \xi_2 = \xi\sin\psi \ , \ \xi = (\xi_1^2 + \xi_2^2)^{1/2} \ ; \ x = r\cos\varphi \ , \ y = r\sin\varphi \ .$$

Then

$$\begin{aligned}(2\pi)^2 A(\xi_1, \xi_2) &= \int_0^{2\pi} d\varphi \int_0^{\infty}\exp[\mathrm{i}r(k - \xi\cos(\psi - \varphi))]dr \\ &= \mathrm{i}\int_0^{2\pi}\frac{d\psi_1}{k - \xi\cos\psi_1} \ , \quad \psi_1 \equiv \varphi - \psi \ . \end{aligned} \tag{1.1.3}$$

We assume that some (may be infinitely small) absorption exists in the medium, hence $\mathrm{Im}\{k\} > 0$ and $\exp(\mathrm{i}kr) \to 0$ at $r \to \infty$. The integral in (1.1.3) can be found in standard tables which give

$$A(\xi_1, \xi_2) = \mathrm{i}\left(2\pi\sqrt{k^2 - \xi^2}\right)^{-1} \quad \text{and finally :}$$

$$\frac{\exp(\mathrm{i}kr)}{r} = \frac{\mathrm{i}}{2\pi}\iint\limits_{-\infty}^{+\infty}\exp\left[\mathrm{i}\left(\xi_1 x + \xi_2 y\right)\right]\frac{d\xi_1 d\xi_2}{\mu} \ , \tag{1.1.4}$$

$$\mu \equiv \sqrt{k^2 - \xi^2} \ , \quad \mathrm{Im}\left\{\mu\right\} \geq 0 \ .$$

The last expression describing the field at the plane $z = 0$ may be extended into the whole space. Each Fourier component gives rise to a plane wave in the space. Formally, it is sufficient for such a “continuation” to add the term $\pm\mathrm{i}\mu z$ in the exponent in the integrand. The plus (minus) sign corresponds to observation points in the halfspace $z > 0\,(z < 0)$ and to the waves propagating in the positive (negative) z-direction. At $\xi > k$ the plane wave is inhomogeneous. The condition $\mathrm{Im}\{\mu\} \geq 0$ ensures boundedness of the field at $|z| \to \infty$. Hence,

$$\frac{\exp(\mathrm{i}kR)}{R} = \frac{\mathrm{i}}{2\pi} \iint_{-\infty}^{+\infty} \exp\left[\mathrm{i}\left(\xi_1 x + \xi_2 y + \mu|z|\right)\right] \frac{d\xi_1 d\xi_2}{\mu}\ , \ \mathrm{Im}\{\mu\} \geq 0. \quad (1.1.5)$$

The validity of this continuation is based on the fact that the right hand side of the last expression satisfies the wave equation (since it is satisfied by the integrand) and gives the correct value for the field at $z = 0$.

Equation (1.1.5) is the expansion of a spherical wave into plane waves. The exponent in the integrand represents a plane wave propagating in the direction given by the components ξ_1, ξ_2, $\mu\,\mathrm{sgn}\,z$ of the wave vector. Note that the direction of the coordinate axes can be chosen arbitrarily. Hence, a spherical wave can be expanded into plane waves in such a way that the inhomogeneous waves entering into the expansion are attenuated not in the z-direction, but in any other direction desired.

We have considered the case of a harmonic spherical wave. Analogous expansion of a spherical wave of general kind (1.1.1) is given in [1.24]. It appears also that the field of concentrated source in some local regions can be represented as a superposition of only homogeneous plane waves [1.25]. In this case, in the integrand a generalized function of ξ_1 and ξ_2 is present instead of $1/\mu$.

Let a spherical wave be radiated at the point S at a distance z_0 from the interface between two homogeneous fluid halfspaces. We assume that the origin of the rectangular coordinates is located at the interface below the source (Fig. 1.1). The plane wave expansion of the spherical wave incident upon the interface will be written in the form of (1.1.5) where in place of z we now have $z - z_0$. At $z \geq 0$ the total field is the sum of incident and reflected waves:

$$p(r, z, z_0) = R^{-1}\exp(\mathrm{i}kR) + p_{\mathrm{r}}\ , \quad R = [(z - z_0)^2 + r^2]^{1/2}\ . \quad (1.1.6)$$

Now our task is the analysis of the reflected wave p_{r}.

Each of the plane waves in the integrand in (1.1.5) acquires the phase $\xi_1 x + \xi_2 y + \mu(z + z_0)$ when traveling from the source to the interface and then to the point of observation. In addition, the amplitude of each plane wave must be multiplied by the reflection coefficient [Ref. 1.1, Eq. (2.2.13)].

$$V = (m\cos\theta - n\cos\theta_1)/(m\cos\theta + n\cos\theta_1)\ , \quad (1.1.7)$$

where θ and θ_1 are incidence and refraction angles, respectively, so that $\xi = k\sin\theta = k_1\sin\theta_1$; $m = \varrho_1/\varrho$, $n \equiv c/c_1$, where $\varrho(\varrho_1)$ and $c(c_1)$ are the density and sound velocity in the upper (lower) medium. Now we obtain for the reflected wave

$$p_{\mathrm{r}} = \frac{\mathrm{i}}{2\pi} \iint\limits_{-\infty}^{+\infty} \frac{d\xi_1 d\xi_2}{\mu} V(\xi) \exp\left[\mathrm{i}\left(\xi_1 x + \xi_2 y + \mu(z + z_0)\right)\right] , \tag{1.1.8}$$

where the reflection coefficient is represented as a function of ξ. Note that V depends only on the modulus $\xi = |\boldsymbol{\xi}|$. It is reasonable then to use the polar coordinates (1.1.2) and while integrating over ψ to use the identity [Ref. 1.25, Chap. 9]

$$\int_0^{2\pi} \exp[\mathrm{i}u \cos(\varphi - \psi)] d\psi = 2\pi J_0(u) ,$$

where $J_0(u)$ is the zero order Bessel function. Now we obtain from (1.1.8)

$$p_{\mathrm{r}} = \mathrm{i} \int_0^{\infty} \frac{\xi d\xi}{\mu} V(\xi) J_0(\xi r) \exp[\mathrm{i}\mu(z + z_0)] . \tag{1.1.9}$$

The right hand sides of (1.1.5, 8) are often called Weyl integrals whereas the right hand side of (1.1.9) is referred to as the Sommerfeld integral. When $r \neq 0$ it is reasonable to transform the latter, taking into account the relations $J_0(u) = 0.5[H_0^{(1)}(u) - H_0^{(1)}(\mathrm{e}^{\mathrm{i}\pi} u)]$ [Ref. 1.26, Chap. 9] and $\mu(-\xi) = \mu(\xi)$, $V(-\xi) = V(\xi)$. Combining integrals from $H_0^{(1)}(u)$ and $H_0^{(1)}(-u)$ in (1.1.9) into one integral, we obtain

$$p_{\mathrm{r}} = \frac{\mathrm{i}}{2} \int_{-\infty}^{+\infty} \frac{\xi d\xi}{\mu} V(\xi) H_0^{(1)}(\xi r) \exp[\mathrm{i}\mu(z + z_0)] . \tag{1.1.10}$$

Integral representation of the sound field in the lower medium ($z < 0$) can be constructed analogously. We obtain

$$\begin{aligned} p &= \frac{\mathrm{i}}{2} \int_{-\infty}^{+\infty} H_0^{(1)}(\xi r) \exp[\mathrm{i}(\mu z_0 - \mu_1 z)] W(\xi) \frac{\xi d\xi}{\mu} , \\ \mu_1 &= \sqrt{k_1^2 - \xi^2} , \quad \mathrm{Im}\,\{\mu_1\} \geq 0 , \end{aligned} \tag{1.1.11}$$

where W is the transmission coefficient for a plane wave [Ref. 1.1, Eq. (2.2.18)]:

$$\begin{aligned} W &= 2m \cos\theta / (m \cos\theta + n \cos\theta_1) = 2m \cos\theta / (m \cos\theta \\ &\quad + \sqrt{n^2 - \sin^2\theta}) . \end{aligned} \tag{1.1.12}$$

By using the appropriate reflection coefficient $V(\xi)$, one can apply (1.1.10) for the calculation of a wave reflected from an arbitrarily layered halfspace. Note that according to (1.1.10) p_{r} depends only on the *sum* $z_0 + z$ *of the heights* of the source and receiver over the boundary but not on each of them separately. It is important also that if $V \equiv V_0 = \mathrm{const}$ [for example, at reflection from absolutely rigid ($V = 1$) or pressure release ($V = -1$) boundaries] it follows from (1.1.6, 8) that

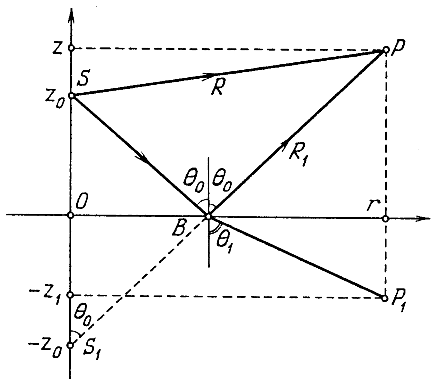

Fig. 1.1 Spherical wave reflection and refraction. S is the source, S_1 is the fictitious source, P and P_1 are observation points

$$p_{\mathrm{r}} = V_0 R_1^{-1} \exp(\mathrm{i}kR_1) \; ; \quad R_1 \equiv [(z+z_0)^2 + r^2]^{1/2} \; . \tag{1.1.13}$$

A reflected wave can be treated, in this case, as a wave emitted by the *fictitious* ("image") *source* S_1 located in the lower medium (Fig. 1.1). The points S and S_1 are located symmetrically with respect to the boundary at $z = 0$.

1.2 Reflected Wave

Consider the sound field in the upper medium and assume that the distance R_1 from the "image" source S_1 is large compared to the wavelength. We shall begin with the integral expansion (1.1.10) and follow mainly works [1.8, 10, 11, 27]. Let us use the asymptotic expansion of the Hankel function [Ref. 1.26, Chap. 9]

$$H_0^{(1)}(u) = \left(\frac{2}{\pi u}\right)^{1/2} \exp\left[\mathrm{i}\left(u - \frac{\pi}{4}\right)\right]\left[1 - \frac{\mathrm{i}}{8u} + O(u^{-2})\right] \; , \qquad -\pi < \arg u < 2\pi \; . \tag{1.2.1}$$

The integration variable ξ in (1.1.10) we replace with $q = \xi/k = \sin\theta$, where θ is the angle of incidence of the corresponding plane wave. To take into account the energy dissipation in the medium we assume that the wave number k is complex. Now (1.1.10) can be written down as

$$p_{\mathrm{r}} = \left(\frac{k}{2\pi r}\right)^{1/2} \exp\left(\frac{\mathrm{i}\pi}{4}\right) \int_{-\infty\exp(-\mathrm{i}\alpha)}^{+\infty\exp(-\mathrm{i}\alpha)} F(q) \exp[|kR_1| f(q)] dq \; , \tag{1.2.2}$$

where $a \equiv \exp(\mathrm{i}\alpha) \equiv k/|k|$,

$$f(q) = \mathrm{i}a[q \sin\theta_0 + (1-q^2)^{1/2} \cos\theta_0] \; , \quad \theta_0 = \arcsin(r/R_1) \; , \tag{1.2.3}$$

$$F(q) = \sqrt{\frac{q}{1-q^2}} V \left[1 - \frac{\mathrm{i}}{8krq} + O\left(\frac{1}{k^2r^2}\right)\right] = \sqrt{\frac{q}{1-q^2}}$$
$$\times \left[1 - \frac{\mathrm{i}}{8krq} + O\left(\frac{1}{k^2r^2}\right)\right] \frac{m\sqrt{1-q^2} - \sqrt{n^2-q^2}}{m\sqrt{1-q^2} + \sqrt{n^2-q^2}} \, . \tag{1.2.4}$$

In the case of a nonabsorbing medium (k real) $a = 1$. The inequality $0 \leq \alpha < \pi/4$ is valid since the real and imaginary parts of k^2 are positive. In the lower medium $k_1 = nk = an|k|$. Hence we have $0 \leq \arg(an) < \pi/4$ at any value of the refraction index in an absorbing medium.

Since we have assumed $kR_1 \gg 1$, it is reasonable to treat the integral in (1.2.2) by the method of steepest descent (also called the saddle point method or the passage method) described in Appendix A. Equation (A.1.1) for the saddle point q_s has a single solution $q_\mathrm{s} = \sin\theta_0$. At this point we have $f(q_\mathrm{s}) = \mathrm{i}a$, $f''(q_\mathrm{s}) = -\mathrm{i}a/\cos^2\theta_0$. The passage path or path of steepest descent γ_1 is specified by (A.1.3):

$$q\sin\theta_0 + (1-q^2)^{1/2}\cos\theta_0 = 1 + \mathrm{i}s^2/a \, , \quad -\infty < s < \infty \, . \tag{1.2.5}$$

It is easy to verify that at infinity the path γ_1 asymptotically approaches the rays $q = |q|\exp[\mathrm{i}(\theta_0 - \alpha)]$ and $q = |q|\exp[\mathrm{i}(\pi - \theta_0 - \alpha)]$. It crosses the real axis q at two points. The first is the saddle point q_s, the second lies to the right of the point $q = 1/\sin\theta_0$ and approaches it when $\alpha \to 0$ (Fig. 1.2).

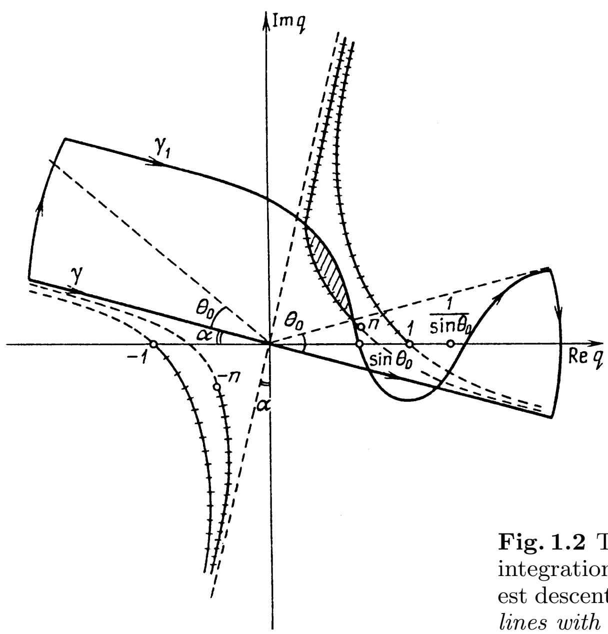

Fig. 1.2 Transformation of an initial integration path to the path of steepest descent γ_1. The cuts are shown by *lines with transverse strokes*