Preface

It is imperative for a manufacturing company to run an efficient operation now-a-days to stay competitive in the world market. The advent of new technologies, a continuous improvement in product quality and changing customer requirements have all lead to shorter production runs, which demand effective methodologies for their execution on the shop floor – the ones that minimize work-in-process and cycle time while meeting customer demands. Due to the batch production nature of such an environment, the use of an appropriate production lot size (or sizes) on the shop floor is central to achieving these objectives. One technique that can effectively influence the flow of a batch (or a lot) of jobs over the machines by appropriately determining the size of production lots (also called *sublots* or *transfer lots*) is lot streaming. By splitting a lot of jobs into smaller-size sublots and processing them in an overlapping fashion over the machines, it tends to achieve the above objective. In this book, we present this technique for the flow shop machine configuration, which constitutes the brunt of its development that, thus, also comprises of the core of the related theoretical contributions made in this field of study.

The material presented in the book has been divided into five chapters, while the last chapter, Chap. 6, contains concluding remarks. Chapter 1 introduces the relevant concepts and definitions that are essential for a clear understanding of the material presented in subsequent chapters. To give the reader an appreciation of the potential benefits of lot streaming, analytical expressions to that end are derived. A historical perspective of this technique is given to put the subject matter on lot streaming in proper perspective and to provide the motivation behind the development of this technique. Some application areas that lend themselves to the use of lot streaming are then presented. A glimpse of the material contained in subsequent chapters is also provided to give the reader an idea of what to expect in these chapters. Chapter 2 presents new and generic mathematical models for the lot streaming problems that contain a variety of relevant features. A mathematical model of a problem, in general, can aid in its analysis, and also, in the development of an appropriate mathematical programming-based methodology for its solution. Chapter 2 is written with this intent in mind. Chapters 3–5 present material in the increasing order of difficulty of the lot streaming problems, namely, for two-machine, three-machine, and m-machine problems. Each of these chapters addresses a variety of problems while presenting for each the requisite analytical development leading up to the algorithm for its solution. These algorithms are illustrated through numerical examples to further aid in their understanding.

The material in this book can be used as a supplement to a course in sequencing and scheduling, production planning and control, production management, supply chain management, or to courses in related areas at graduate or advanced undergraduate levels. As background, it requires mathematical maturity and introductory knowledge of optimization concepts and methodologies. The book provides useful ideas and algorithms for practitioners, and it can serve as a useful research reference.

My first and foremost thanks go to one of my graduate students, Puneet Jaiprakash. For his many direct contributions, I consider him to be a coauthor of this book. I used the first draft of this book in my graduate-level course on sequencing and scheduling taught at Virginia Tech during the 2005 Spring semester. The students from this class provided valuable feedback that assisted in improving the exposition of the material presented in the book. In particular, I would like to recognize the contributions made in this regard by Ming Chen and Liming Yao, two of my doctoral students. I would also like to extend my sincere thanks to the anonymous reviewers for their careful reading of the manuscript and insightful comments.

A project of this magnitude cannot be accomplished without the unconditional support, encouragement, and love of the family. For this, I would like to thank my wife, Veena, and our sons, Sumeet and Shivan. Finally, I would like to thank Sandy Dalton for her help in typing the manuscript.

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Generic Mathematical Models for the Flow Shop Lot Streaming Problem

2.1 Introduction

To comprehend the intricacies of a problem situation, it is best to represent it, if possible, as a mathematical model. A mathematical model can also help in possibly identifying some inherent structural properties of the problem and in devising an appropriate algorithm for its solution. Chapter 1 contains a brief review of work on the flow shop lot streaming problems. This work has focused on addressing two-machine, three-machine, and the general m-machine scenarios. In this chapter, we develop some generic mathematical models for the lot streaming problem that encompass all of these scenarios and also that address various features pertinent to lot streaming. We present these models in Sect. 2.2 and also give their illustrations using simple examples. The key features of the models that are presented in this section are summarized in Table 2.1. In Sect. 2.3, we introduce mathematical models for some special cases of the flow shop lot streaming problem that have been presented in the literature.

2.2 Some Generic Mathematical Models for the Flow Shop Lot Streaming Problem

2.2.1 *Notation*

We define the following notation in addition to that presented in Sect. 1.3.2.

Parameters:

RT_{jk}	Removal time of lot j on machine k
FT_j	Fixed transfer time for lot j
VT_j	Variable transfer time per unit for lot j
τ_{jk}	Sublot-attached setup time for a sublot of lot j on machine k
G	A large positive number used to make a constraint redundant

TABLE 2.1. Key features of the mathematical models presented in Sect. 2.2

Section	Number of machine	Number of lots	Setup	Removal time	Transfer time	Intermingling	Wait–No Wait	Sublot type	Intermediate/ No Intermediate Idle Time	Continuous or Discrete Sublots
2.2.2	m	N	Lot–Attached	Yes	Yes	No	Both	C, E	Both	Both
2.2.3	m	N	Lot–Detached	Yes	Yes	No	Both	C, E	Both	Both
2.2.4	m	N	Sublot–Attached	Yes	Yes	Yes	Both	C, E	Both	Both
2.2.5	m	N	Sublot–Detached	Yes	Yes	Yes	Both	C, E	Both	Both
2.2.6	This section addresses the issue of handling variable sublot sizes in the models presented in the above sections									

Variables:

s_{ijk} Sublot size of the ith sublot of lot j on machine k; a generalization of the definition of sublot sizes presented in Sect. 1.3.2

C_{ijk} Completion time of the ith sublot of lot j on machine k; the subscript j is omitted for problems involving a single lot

$$y_{ij} = \begin{cases} 1, \text{ if lot } i \text{ precedes lot } j \\ 0, \text{ otherwise.} \end{cases}$$

2.2.2 *m/N/{C,E,V}/{II,NI}/{CV,DV}/{Lot-Attached Setup and Removal Times, Sublot Transfer Times, No Intermingling}*

The lot streaming problem involving multiple lots deals with the issue of finding the sublot sizes for each lot and the sequence in which to process the lots in order to optimize a performance measure. Here, we consider the objective of minimizing the makespan. However, other performance measures can be conveniently included in the formulations that we present. We make the following assumptions.

1. The sublot transfer times are variable and comprise of two parts, a fixed component, which remains the same for all the sublots of a particular lot and a variable component, which depends on the size of the sublot and is given by $VT_j \cdot s_{ijk}$.
2. The removal times are attached to the last sublot of each lot and are independent of the sequence in which the lots are processed.
3. The number of sublots for all lots is known in advance.

Generic Model 1 (GM1):
Minimize: $C_{\max}$
Subject to:

1. Makespan Constraint:

$$C_{\max} \geq C_{n_j m} + RT_{jm}, \forall n_j, j = 1, \ldots, N.$$

This constraint captures the makespan $C_{\max}$, which is the largest among the completion times of the last sublots of all the lots on the last machine (m).

2. Item Allocation Constraint:

$$\sum_{u=1}^{n_j} s_{ujk} = U_j, \quad \forall j = 1, \ldots, N, k = 1, \ldots, m.$$

This constraint ensures that the sum of the items in the sublots of a lot, $j(j = 1, \ldots, N)$ that is processed on machine $k(k = 1, \ldots, N)$ must be equal to the total number of items in that lot.

The next two constraints capture the type of sublots involved. Constraint (3) can be used in case we have consistent sublots while Constraints (3) and (4) together, capture the requirement of equal sublot sizes. We consider the case of variable sublot sizes later.

3. **Consistent Sublot Constraint:**

$$s_{ijk} = s_{ij(k+1)}, \quad \forall i = 1, \ldots, n_j, j = 1, \ldots, N, k = 1, \ldots, (m-1).$$

4. **Equal Sublot Constraint:**

$$s_{ijk} = s_{(i+1)jk}, \quad \forall j = 1, \ldots, N, k = 1, \ldots, m.$$

5. **Lot-attached Setup Constraint:**

$$s_{1jk} \geq \Psi, \quad \forall j = 1, \ldots, N, k = 1, \ldots, m,$$

where ψ is the minimum number of items required to perform a setup on any machine. In the presence of lot-attached setups, the setup time is associated with the first sublot of every lot. However, there might be technological constraints on the minimum number of items required to perform a setup. The constraint above ensures that the size of the first sublot of all the lots is greater than ψ, thus ensuring that a setup can always be performed once the first sublot has been transferred to machine k from machine $(k-1)$.

6. **Sublot Size Constraint:**

$$s_{ijk} \geq 0, \quad \forall i = 2, \ldots, n_j, j = 1, \ldots, N, k = 1, \ldots, m.$$

This constraint ensures nonnegative sublot sizes. These may also be restricted to take integer or real (continuous) values.

7. **Sequential Processing Constraint:**

(a) **First sublot:**

$$C_{1j(k+1)} - p_{j(k+1)}s_{1j(k+1)} \geq C_{1jk} + t_{j(k+1)} + FT_j + VT_j s_{1jk},$$
$$\forall j = 1, \ldots, N, k = 1, \ldots, (m-1).$$

This constraint ensures that the first sublot begins processing on machine $(k+1)$ only after it has completed processing on machine k, has been transferred to machine $(k+1)$ and the setup on machine $(k+1)$ has been completed.

(b) **For sublots 2,…,n_j:**

$$C_{ij(k+1)} - p_{j(k+1)}s_{ij(k+1)} \geq C_{ijk} + FT_j + VT_j s_{ijk},$$
$$\forall i = 2, \ldots, n_i, j = 1, \ldots, N, k = 1, \ldots, (m-1).$$

This constraint ensures that all the sublots, excluding the first one, begin processing on the $(k+1)$ th machine only after they have finished processing on the kth machine and have been transferred to machine $(k+1)$.

By replacing the above inequalities with equalities, the formulation can be adapted to the no-wait flow shop.

8. No-Intermingling Constraint for Machines 1,. . ., *m*:

(a) (i, j) **precedes** (i', j')

$$\begin{aligned}
&\left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) - \left(C_{ijk} - p_{jk}s_{ijk}\right) + G\left(1 - y_{jj'}\right)\\
&\geq \left(U_j - \sum_{u=1}^{i-1} s_{ujk}\right) p_{jk} + RT_{jk} + t_{j'k} + p_{j'k}\sum_{u=1}^{i'-1} s_{uj'k},\\
&\forall (i, j) \text{ and } (i'j') : j \neq j', i = 1, \ldots, n_j, j = 1, \ldots, N,\\
&i' = 1, \ldots, n_{j'}, j' = 1, \ldots, N, k = 1, \ldots, m.
\end{aligned}$$

(b) (i', j') **precedes** (i, j)

$$\begin{aligned}
&\left(C_{ijk} - p_{jk}s_{ijk}\right) - \left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) + Gy_{jj'}\\
&\geq \left(U_{j'} - \sum_{u=1}^{i'-1} s_{uj'k}\right) p_{j'k} + RT_{j'k} + t_{jk} + p_{jk}\sum_{u=1}^{i-1} s_{ujk},\\
&\forall (i, j) \text{ and } (i', j') : j \neq j', i = 1, \ldots, n_j, j = 1, \ldots, N,\\
&i' = 1, \ldots, n_{j'}, j' = 1, \ldots, N, k = 1, \ldots, m.
\end{aligned}$$

For any two lots j and $j'(j \neq j')$, we have two possibilities, namely, j precedes j' or j' precedes j. Since, either one must hold, these are referred to as *disjunctive constraints*. To model these into the formulation, we define a binary variable $y_{jj'}$ which takes a value of 1 if j precedes j', and 0, otherwise. If it takes a value 1, then (8a) holds true since $G(1 - y_{jj'}) = 0$ and (8b) becomes redundant. On the other hand, if $y_{jj'}$ takes a value of zero, then (8b) is enforced and (8a) becomes redundant. For any pair of sublots (i, j) and $(i', j') : j \neq j'$, the terms on the right hand side of (8a) ensure that the difference between the start times of sublots i and i' is atleast equal to the sum of the processing times of the sublots i to n_j of lot j and 1 to $(i' - 1)$ of lot j', the removal time for lot j and setup time for lot j'. These constraints are enforced for all pairs of sublots belonging to lots j and j', and on all the machines.

By replacing the above inequalities with equalities, the formulation can be adapted to the scenario when no intermittent idling is permitted.

9. Station Capacity Constraint:

(a) **First sublot of any lot on Machine 1:**

$$c_{1j1} - p_{j1}s_{1j1} \geq t_{j1}, \quad \forall j = 1, \ldots, N.$$

This constraint ensures that the processing of the first sublot, of any lot appearing first in the sequence, begins after its setup has been completed.

(b) **Sublots** $2, \ldots, n_j$ **of any lot on Machine 1:**

$$C_{(i+1)j1} - p_{j1}s_{(i+1)j1} = C_{ij1}, \quad \forall i = 1, \ldots, (n_j - 1), j = 1, \ldots, N.$$

This constraint captures the fact that the $(i+1)$th sublot of lot j should begin processing on machine 1 only after the completion of its ith sublot.

(c) **All sublots on machines** $k = 2, \ldots, m$:

$$C_{(i+1)jk} - p_{jk}s_{(i+1)jk} \\ \geq C_{ijk}, \quad \forall i = 1, \ldots, (n_j - 1), j = 1, \ldots, N, k = 2, \ldots, m.$$

This constraint ensures that for all the lots processed on machines $k = 2, \ldots, m$, the $(i + 1)$th sublot of lot j begins processing on machine k only after the completion of $i$ts ith sublot on that machine.

Example 2.1 To illustrate the above model, consider a two-machine, three-lot flow shop with the data shown in Tables 2.2 and 2.3. The sublot sizes are consistent, restricted to take integer values and intermittent idling is permitted. Recall, G is a large positive number; $G = 5,000$ was used in this and subsequent problems.

In lieu of the above data, model **GM1** can be written as follows.
Minimize: $C_{\max}$
Subject to:

Makespan Constraint:

$$C_{\max} \geq C_{n_j j2} + RT_{j2}, \quad \forall n_j, j = 1, 2, 3.$$

Item Allocation Constraint:

$$\sum_{u=1}^{n_j} s_{ujk} = U_j, \quad \forall j = 1, 2, 3, k = 1, 2.$$

TABLE 2.2. Data for the Illustration of lot-attached setup model

	Processing time		Setup time		Removal time	
	M/C 1	M/C 2	M/C 1	M/C 2	M/C 1	M/C 2
Lot 1	2	1	1	2	2	1
Lot 2	2	3	2	1	2	2
Lot 3	1	2	2	2	1	2

TABLE 2.3. Data for the Illustration of lot-attached setup model

	n_j	U_j	r_j	FT_j	VT_j
Lot 1	2	4	0	1	1
Lot 2	4	6	0	2	1
Lot 3	3	5	0	1	1

Consistent Sublot Constraint:

$$s_{ij1} = s_{ij2}, \quad \forall i = 1, \ldots, n_j, \ j = 1, 2, 3.$$

Attached-Setup Constraint:

$$s_{1jk} \geq 1, \quad \forall j = 1, 2, 3, k = 1, 2.$$

Sublot Size Constraint:

$$s_{ijk} \geq 0, \quad \text{integer}, \quad \forall i = 2, \ldots, n_j, \ j = 1, 2, 3, k = 1, 2.$$

Sequential Processing Constraint:

(a) **First sublot:**

$$C_{1j2} - p_{j2}s_{1j2} + FT_j + VT_j s_{1j1} + t_{j2}, \quad \forall j = 1, 2, 3.$$

(b) **For sublots** $2, \ldots, n_j$**:**

$$C_{ij2} - p_{j2}s_{ij2} \geq C_{ij1} + FT_j + VT_j \cdot s_{ij1}, \quad \forall i = 2, \ldots, n_j, \ j = 1, 2, 3.$$

No-Intermingling Constraint for Machines 1 & 2:

(a) (i, j) **precedes** (i', j')

$$\begin{aligned} &\left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) - \left(C_{ijk} - p_{jk}s_{ijk}\right) + G(1 - y_{jj'}) \\ &\geq \left(U_j - \sum_{u=1}^{i=1} s_{ujk}\right) p_{jk} + RT_{jk} + t_{j'k} + p_{j'k} \sum_{u=1}^{i'-1} s_{uj'k}, \\ &\forall (i, j) \text{ and } (i', j') : j \neq j', i = 1, \ldots, n_j, \ j = 1, 2, 3, i' = 1, \ldots, n_{j'}, \\ &j' = 1, 2, 3, k = 1, 2. \end{aligned}$$

(b) (i', j') **precedes** (i, j)

$$\begin{aligned} &\left(C_{ijk} - p_{jk}s_{ijk}\right) - \left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) + Gy_{ij'} \\ &\geq \left(U_{j'} - \sum_{u=1}^{i'-1} s_{uj'k}\right) p_{j'k} + RT_{j'k} + t_{jk} + p_{jk} \sum_{u=1}^{i=1} s_{ujk}, \\ &\forall (i, j) \text{ and } (i', j') : j \neq j', i = 1, \ldots, n_j, \ j = 1, 2, 3, i' = 1, \ldots, n_{j'}, \\ &j' = 1, 2, 3, k = 1, 2. \end{aligned}$$

Station Capacity Constraint:

(a) **First sublot of any lot on Machine 1:**

$$C_{1j1} - p_{j1}s_{1j1} \geq t_{j1}, \quad \forall j = 1, 2, 3.$$

TABLE 2.4. Solution for the illustrative Example 2.1

	Lot 1		Lot 2				Lot 3		
Consistent sublot sizes	s_1	s_2	s_1	s_2	s_3	s_4	s_1	s_2	s_3
	2	2	1	1	1	3	1	1	3
Start time on machine 1	33	37	11	13	15	17	2	3	4
Start time on machine 2	42	44	20	23	26	29	7	9	11
Optimal sequence of lots					3–2–1				
Optimal makespan					47				

(b) **Sublots 2, …, n_j of any lot on Machine 1:**

$$C_{(i+1)j1} - p_{j1} \cdot s_{(i+1)j1} = C_{ij1}, \quad \forall i = 2, \ldots, (n_j - 1),\ j = 1, 2, 3.$$

(c) **All sublots on machines 2:**

$$C_{(i+1)j2} - p_{j2} \cdot s_{(i+1)j2} \geq C_{ij2}, \forall i = 1, \ldots, (n_j - 1), \quad j = 1, 2, 3.$$

The above model was coded using AMPL and was solved using the CPLEX optimization software. The optimal sublot sizes and the sequence in which to process the lots are shown in Table 2.4.

2.2.3 *m/N/{C,E,V}/{II,NI}/{CV,DV}/{Lot-Detached Setup and Removal Times, Sublot Transfer Times, No Intermingling}*

The generic formulation above (**GM1**) can easily be adapted to the case of detached setup (designated as model **GM2**) by making the following changes.

1. The lot-attached setup constraint (5) can be relaxed since the setups are detached.
2. The Sequential Processing constraints (7a) and (7b) can be combined to give a single constraint as follows:

$$C_{ij(k+1)} - p_{j(k+1)} s_{ij(k+1)} \geq C_{ijk} + FT_j + VT_j s_{ijk},$$
$$\forall i = 1, \ldots, n_j,\ j = 1, \ldots, N, k = 1, \ldots, (m-1).$$

3. The Station Capacity constraint (9a) now becomes

$$C_{1jk} - p_{jk} s_{1jk} \geq t_{jk}, \quad \forall j = 1, \ldots, N, k = 1, \ldots, m.$$

This constraint ensures that the first sublot of any lot starts after the setup has been completed. It needs to be enforced for machines 2, …, m explicitly and is not implied by the Sequential Processing constraint since the setups are detached.

TABLE 2.5. Data for the illustrative lot-detached setup problem

	Processing time		Setup time		Removal time	
	M/C 1	M/C 2	M/C 1	M/C 2	M/C 1	M/C 2
Lot 1	2	1	1	2	2	1
Lot 2	3	2	2	1	2	2
Lot 3	2	3	2	4	1	2

TABLE 2.6. Data for the illustrative lot-detached setup problem

	n_j	U_j	r_j	FT_j	VT_j
Lot 1	2	4	0	1	1
Lot 2	4	6	0	2	1
Lot 3	3	5	0	1	1

Example 2.2 To illustrate the above model, consider a two-machine, three-lot system with the data shown in Tables 2.5 and 2.6. The sublot sizes are consistent, restricted to take integer values and intermittent idling is permitted.

In lieu of the above data, model **GM2** can be written as follows.
Minimize: $C_{\max}$
Subject to:

Makespan Constraint:

$$C_{\max} \geq C_{n_j j2} + RT_{j2}, \quad \forall n_j, j = 1, 2, 3.$$

Item Allocation Constraint:

$$\sum_{u=1}^{n_j} s_{ujk} = U_j, \quad \forall j = 1, 2, 3, k = 1, 2.$$

Consistent Sublot Constraint:

$$s_{ij1} = s_{ij2}, \quad \forall i = 1, \ldots n_j, j = 1, 2, 3.$$

Sublot Size Constraint:

$$s_{ijk} \geq 0, \quad \text{integer}, \quad \forall i = 2, \ldots, n_j, j = 1, 2, 3, k = 1, 2.$$

Release Time Constraint:

$$\sum_{t=0}^{200} tX_{1j1t} \geq 0, \quad \forall j = 1, 2, 3.$$

Sequential Processing Constraint:

$$C_{ij2} - p_{j2}s_{ij2} \geq C_{ij2} + FT_j + VT_j s_{ij1}, \quad \forall i = 1, \ldots, n_j, j = 1, 2, 3.$$

TABLE 2.7. Solution for the illustrative Example 2.2

	Lot 1		Lot 2				Lot 3		
Consistent sublot sizes	s_1	s_2	s_1	s_2	s_3	s_4	s_1	s_2	s_3
	2	2	1	1	2	2	1	1	3
Start time on machine 1	36	40	15	18	21	27	2	4	6
Start time on machine 2	45	47	28	31	33	37	7	10	16
Optimal sequence of lots					3–2–1				
Optimal makespan					50				

No-Intermingling Constraint for Machines 1 and 2:

(a) (i, j) **precedes** (i', j')

$$\left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) - \left(C_{ijk} - p_{jk}s_{ijk}\right) + G\left(1 - y_{jj'}\right) \geq \left(U_j - \sum_{u=1}^{i-1} s_{ujk}\right) p_{jk} + RT_{jk} + t_{j'k} + p_{j'k}\sum_{u=1}^{i'-1} s_{uj'k}$$

$\forall (i, j)$ and $(i', j') : j \neq j', i = 1, \ldots, n_j, j = 1, 2, 3, i' = 1, \ldots n_j,$
$j' = 1, 2, 3, k = 1, 2.$

(b) (i', j') **precedes** (i, j)

$$\left(C_{ijk} - p_{jk}s_{ijk}\right) - \left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) + Gy_{jj'} \geq \left(U_{j'} - \sum_{u=1}^{i'-1} s_{uj'k}\right) p_{j'k} + RT_{j'k} + t_{jk} + p_{jk}\sum_{u=1}^{i-1} s_{ujk}$$

$\forall (i, j)$ and $(i', j') : j \neq j', i = 1, \ldots, n_j, j = 1, 2, 3, i' = 1, \ldots, n_{j'},$
$j' = 1, 2, 3, k = 1, 2.$

Station Capacity Constraint:

(a) $C_{1j1} - p_{j1}s_{1j1} \geq t_{j1}, \quad \forall j = 1, 2, 3, \forall k = 1, 2.$
(b) $C_{(i+1)j1} - p_{j1}s_{(i+1)j1} = C_{1j1}, \quad \forall i = 1, \ldots, n_{j-1}, \forall j = 1, 2, 3.$
(c) $C_{(i+1)j2} - p_{j2}s_{(i+1)j2} \geq C_{ij2}, \quad \forall i = 1, \ldots, n_{j-1}, \forall j = 1, 2, 3.$

The optimal sublot sizes and the sequence in which to process the lots are shown in Table 2.7.

2.2.4 *m/N/{C,E,V}/{II,NI}/{CV,DV}/{Sublot-Attached Setup and Removal Times, Sublot Transfer Times, Intermingling}*

This problem is identical to the one discussed in Sect. 2.2.2 except for the fact that the setup involved is sublot attached instead of lot attached considered earlier, and also, we permit intermingling of the sublots. The formulation for this problem (designated as Generic Model **GM3**) follows from that presented in Sect. 2.2.2. The constraints (1), (2), (3), (4), and (6) are the same for this problem as well. Constraint (5) is no longer relevant as we now have sublot-attached setups.

The Sequential Processing Constraint for this case is as follows:

$$C_{ij(k+1)} - p_{j(k+1)}s_{ij(k+1)} \geq C_{ijk} + \tau_{j(k+1)} + FT_j + VT_j s_{ijk}$$
$$\forall i = 1, \ldots, n_j,\ j = 1, \ldots, N,\ k = 1, \ldots, (m-1).$$

This constraint ensures that any sublot i begins processing on machine $(k+1)$ only after it has completed processing on machine k has been transferred to machine $(k+1)$ and the setup on machine $(k+1)$ for sublot i has been completed.

By replacing the above inequalities with equalities, the formulation can be adapted to the no-wait flow shop scenario.

The intermingling constraint for machines $1, \ldots, m$ can be expressed as follows:

(a) (i, j) precedes (i', j')

$$C_{i'j'k} - p_{j'k}s_{i'j'k} - C_{ijk} + G(1 - y_{iji'j'}) \geq RT_{jk} + \tau_{j'k},$$
$$\forall (i, j) \text{ and } (i', j'),\ i = 1, \ldots, n_j,\ j = 1, \ldots, N,\ i' = 1, \ldots, n_{j'}$$
$$j' = 1, \ldots, N,\ k = 1, \ldots, m: \ \text{if} j = j', \text{then } i \neq i'.$$

(b) (i', j') precedes (i, j)

$$\left(C_{ijk} - p_{jk}s_{ijk}\right) - C_{i'j'k} + G y_{iji'j'} \geq RT_{j'k} + \tau_{jk}$$
$$\forall (i, j) \text{ and } (i', j'),\ i = 1, \ldots, n_j,\ j = 1, \ldots, N,\ i' = 1, \ldots, n_{j'},$$
$$j' = 1, \ldots, N,\ k = 1, \ldots, m: \ \text{if } j = j', \text{then } i \neq i'.$$

These disjunctive constraints are identical to those presented in Sect. 2.1.2, except that now, since intermingling is allowed, we define a new binary variable $y_{iji'j'}$, which takes a value of 1 if sublot (i, j) precedes (i', j'), and a value of 0 if (i', j') precedes (i, j). For any pair of sublots (i, j) and (i', j') if $j = j'$ then $i \neq i'$ The terms on the right hand side in (a) above ensure that the difference between the start times of sublots (i, j) and (i', j') is at least equal to the sum of processing times of sublot (i, j), the removal time for sublot (i, j), and setup time for(i', j'). These constraints are enforced for all pairs of sublots scheduled on any machine k.

By replacing the inequalities with equalities in the above expressions, the formulation can be adapted to the case of no-intermittent idling.

The station capacity constraint for this case is as follows:

$$C_{ij1} - p_{jk}s_{ij1} \geq \tau_{j1}, \quad \forall i = 1, \ldots n_j,\ j = 1, \ldots, N.$$

This constraint ensures that any sublot i of any job j begins processing on machine 1 only after its setup has been completed.

Example 2.3 To illustrate the above model, consider a two-machine, two-lot system with data as shown in Tables 2.8 and 2.9. The sublot sizes are consistent, restricted to take integer values and intermittent idling is permitted.

TABLE 2.8. Data for the illustrative sublot-attached setup problem

	Processing time		Setup time		Removal time	
	M/C 1	M/C 2	M/C 1	M/C 2	M/C 1	M/C 2
Lot 1	2	1	1	1	1	1
Lot 2	2	1	1	1	1	1

TABLE 2.9. Data for the illustrative sublot-attached setup problem

	n_j	U_j	r_j	FT_j	VT_j
Lot 1	2	4	0	1	1
Lot 2	3	5	0	1	1

In lieu of the above data, model **GM3** can be written as follows.
Minimize: $C_{\max}$
Subject to:

Makespan Constraint:

$$C_{\max} \geq C_{n_j j2} + RT_{j2}, \quad \forall i = 1, \ldots n_j, \ j = 1, 2.$$

Item Allocation Constraint:

$$\sum_{u=1}^{n_j} s_{ujk} = U_j, \quad \forall j = 1, 2, k = 1, 2.$$

Consistent Sublot Constraint:

$$s_{ij1} = s_{ij2}, \quad \forall i = 1, \ldots, n_j, \ j = 1, 2.$$

Sublot Size Constraint:

$$s_{ijk} \geq 0, \text{ integer}, \quad \forall i = 2, \ldots, n_j, \ j = 1, 2, k = 1, 2.$$

Sequential Processing Constraint:

$$C_{ij2} - p_{j2}s_{ij2} \geq c_{ij1} + \tau_{j2} + FT_j + VT_j s_{ij1}, \quad \forall i = 1, \ldots, n_j, \ j = 1, 2.$$

Intermingling Constraint for Machines 1 and 2:

(a) (i, j) **precedes** (i', j')

$$\left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) - \left(C_{ijk} - p_{jk}s_{ijk}\right) + G(1 - y_{iji'j'}) \geq RT_{jk} + \tau_{j'k},$$
$$\forall (i, j) \text{ and } (i', j'), i = 1, \ldots, n_j, \ j = 1, 2, i' = 1, \ldots, n_{j'},$$
$$j' = 1, 2, k = 1, 2 : \text{if } j = j', \text{then } i \neq i'.$$

TABLE 2.10. Solution for the illustrative Example 2.3

	Lot 1		Lot 2		
Consistent sublot sizes	s_1	s_2	s_1	s_2	s_3
	1	3	1	2	2
Start time on machine 1	21	1	25	15	9
Start time on machine 2	27	12	30	23	17
Optimal sequence of lots (s_{ij})			21–32–22–11–12		
Optimal makespan			32		

(b) (i', j') **precedes** (i, j)

$$\left(C_{ijk} - p_{jk}s_{ijk}\right) - \left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) + Gy_{iji'j'}$$
$$\geq RT_{j'k} + \tau_{jk}, \forall (i, j) \text{ and } (i', j'), i = 1, \ldots, n_j, j = 1, 2,$$
$$i' = 1, \ldots, n_{j'} j' = 1, 2, k = 1, 2 : \text{if} j = j', \text{then } i \neq i'.$$

Station Capacity Constraint:

$$C_{ij1} - p_{j1}s_{ij1} \geq \tau_{j1}, \quad \forall i = 1, \ldots, n_j, j = 1, 2.$$

The optimal solution is shown in Table 2.10. Note that in this solution, the sublots of lot 1 are not processed continuously. The third and the second sublots of lot 2 are processed in between the second and the first sublots of lot 1. Note that the numbering of the sublots of a lot is arbitrary.

2.2.5 *m/N/{C,E,V}/{II,NI}/{CV,DV}/{Sublot-Detached Setup and Sublot-Attached Removal Times, Sublot Transfer Times, Intermingling}*

The generic formulation of the sublot-attached setup (**GM3**) problem can be adapted to the case when detached setups are present. We designate the resulting model as **GM4**. The changes that need to be incorporated are as follows.

1. The sequential processing constraint can be replaced with the following constraint.

$$C_{ij(k+1)} - p_{(k+1)}s_{ij(k+1)} \geq C_{ijk} + FT_j + VT_j s_{ijk},$$
$$\forall i = 1, \ldots, n_j, j = 1, \ldots, N, k = 1, \ldots, (m-1).$$

   This constraint is similar to that for the case of sublot-attached setups, except that the setup time for sublot i on machine $(k+1)$ is not considered since the setup is detached.
2. The station capacity constraint is modified as follows.

$$C_{ijk} - p_{jk}s_{ijk} \geq \tau_{jk}, \quad \forall i = 1, \ldots, n_j, j = 1, \ldots, N, k = 1, \ldots, m.$$

This constraint ensures that if any sublot i of lot j is scheduled first on machine k, then it can begin processing only after the setup has been completed.

Example 2.4 If the setup in Example 2.3 were detached, then the model **GM4** can be written as follows.

Minimize: $C_{\max}$
Subject to:
Makespan Constraint:

$$C_{\max} \geq C_{n_j j2} + RT_{j2}, \quad \forall n_j, j = 1, 2.$$

Item Allocation Constraint:

$$\sum_{u=1}^{n_j} s_{ujk} = U_j, \quad \forall j = 1, 2, k = 1, 2.$$

Consistent Sublot Constraint:

$$s_{ij1} = s_{ij2}, \quad \forall i = 1, \ldots, n_j, j = 1, 2.$$

Sublot Size Constraint:

$$s_{ijk} \geq 0, \text{integer}, \quad \forall i = 2, \ldots, n_j, j = 1, 2, k = 1, 2.$$

Sequential Processing Constraint:

$$C_{ij2} - p_{j2}s_{ij2} \geq C_{ij2} + FT_j + VT_j s_{ij1}, \quad \forall i = 1, \ldots, n_j, j = 1, 2.$$

Intermingling Constraint for Machines 1 and 2:

(a) (i, j) precedes (i', j')

$$\begin{aligned}&\left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) - \left(C_{ijk} - p_{jk}s_{ijk}\right) + G(1 - y_{iji'j'})\\ &\quad \geq RT_{jk} + \tau_{j'k}, \forall (i, j) \text{ and } (i', j'), i = 1, \ldots, n_j, j = 1, 2,\\ &\quad i' = 1, \ldots, n_{j'}, j = 1, 2, i' = 1, \ldots, n_{j'}, j' = 1, 2,\\ &\quad k = 1, 2 : \text{if} j = j', \text{then } i \neq i'.\end{aligned}$$

(b) (i', j') precedes (i, j)

$$\begin{aligned}&\left(C_{ijk} - p_{jk}s_{ijk}\right) - \left(C_{i'j'k} - p_{j'k}s_{i'j'k}\right) + Gy_{iji'j'}\\ &\quad \geq RT_{j'k} + \tau_{jk} \forall (i, j) \text{ and } (i', j'), i = 1, \ldots, n_j, j = 1, 2,\\ &\quad i' = 1, \ldots, n_{j'}, j' = 1, 2, k = 1, 2 : \text{if} j = j', \text{then } i \neq i'.\end{aligned}$$

Station Capacity Constraint:

$$C_{ijk} - p_{jk}s_{ijk} \geq \tau_{jk}, \quad \forall i = 1, \ldots, n_j, j = 1, 2, k = 1, 2.$$

The optimal solution is shown in Table 2.11. Note that in the optimal solution, the sublots of lot 1 are not processed continuously. The second sublot of lot 2 is processed in between the first and second sublots of lot 1.

TABLE 2.11. Solution for the illustrative sublot-detached setup problem

	Lot 1		Lot 2		
Consistent sublot sizes	s_1	s_2	s_1	s_2	s_3
	2	2	1	3	1
Start time on machine 1	1	15	25	7	21
Start time on machine 2	10	22	29	17	26
Optimal sequence of lots (s_{ij})			11–22–21–32–12		
Optimal makespan			31		

2.2.6 *The Case of Variable Sublots*

Next, we consider the case of variable sublot sizes as a lot moves from one machine to another. There are the following two ways in which a new sublot can be configured for processing on a machine, after the items constituting that sublot have been processed on the preceding machine.

Case (1). The items constituting a new sublot can be reconfigured to form this sublot only after the completion of the entire sublots to which they belong.

Case (2). The items constituting a new sublot can be reconfigured to form this sublot without the completion of the entire sublots to which they belong.

Consider Case (1) and the scenario of the generic model **GM1**. Constraints (3) and (4) are no longer valid for this case. Also, the sequential processing constraints, are impacted as follows:

(a) **First sublot:**

$$C_{1jk} - p_{jk}s_{1jk} - C_{i'j(k-1)} - FT_j - VT_j s_{i'j(k-1)} - t_{jk} + G(1 - x_{i'1jk}) \geq 0, \forall i' = 1, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m.$$

(b) **For sublot 2**, ..., n_j**:**

$$C_{ijk} - p_{jk}s_{ijk} - C_{i'j(k-1)} - FT_j - VT_j s_{i'j(k-1)} + G(1 - x_{i'ijk}) \geq 0, \forall i = 2, \ldots, n_j, i' = 1, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m.$$

Also, we need to add a new constraint, termed the variable sublot constraint, as follows:

Variable Sublot Constraint:

$$\sum_{h=1}^{i'-1} s_{hj(k-1)} - \sum_{h=1}^{i} s_{hjk} + Gx_{i'ijk} \geq 0,$$
$$\forall i = 1, \ldots, n_j, i' = 2, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m$$
$$x_{1ijk} = 1, \forall i = 1, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m,$$

where $x_{i'ijk} = 1$, if sublot i of lot j on machine k is started no earlier than the completion time of the sublot i' of the same lot on machine $k-1$, and $= 0$, otherwise. Thus, in accordance with Constraint (a) above, if the first sublot of a lot j on machine k starts no earlier than the completion time of sublot i' on machine $k-1$, then, the appropriate relationship between the starting time of this sublot to the completion time of sublot i' and the requisite transfer and setup times must be maintained; otherwise it is relaxed. In a similar manner, Constraint (b) captures this relationship for any other sublot, other than the first sublot. However, if a sublot i on machine k starts earlier than the completion time of a sublot i' on machine $k-1$, then the sum of all the sublots until sublot i on machine k must not exceed the sum of the sublots until sublot $i'-1$ on machine $k-1$. This is captured by the variable sublot constraint.

The above development is applicable for the other generic models as well except that in the case of sublot-attached setup, we need to include a setup time, τ_{jk}, for every sublot rather than just for first sublot. The corresponding constraint is as follows:

$$C_{ijk} - p_{jk}s_{ijk} - C_{i'j(k-1)} - FT_j - VT_j s_{i'j(k-1)} - \tau_{jk}$$
$$+ G \cdot (1 - x_{i'ijk}) \geq 0 \quad \forall i, i = 1, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m.$$

Next, consider Case (2). The sequential processing constraints for this case under the scenario of generic model **GM1** are as follows:

(a) First sublot:

$$C_{1jk} - p_{jk}s_{1jk} - C_{i'j(k-1)} + p_{j(k-1)}s_{i'j(k-1)} - FT_j - VT_j s_{1jk} - t_{jk}$$
$$+ G(1 - x_{i'1jk}) \geq \max\left\{ p_{j(k-1)} \left(s_{1jk} - \sum_{h=1}^{i'-1} s_{hj(k-1)} \right), 0 \right\},$$
$$\forall i' = 1, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m.$$

(b) For sublot 2, ..., n_j:

$$C_{ijk} - p_{jk}s_{ijk} - C_{i'j(k-1)} + p_{j(k-1)}s_{i'j(k-1)} - FT_j - VT_j s_{ijk}$$
$$+ G(1 - x_{i'ijk}) \geq \max\left\{ p_{j(k-1)} \left(\sum_{h=1}^{i} s_{hjk} - \sum_{h=1}^{i'-1} s_{hj(k-1)} \right), 0 \right\},$$
$$\forall i = 2, \ldots, n_j, i' = 1, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m.$$

Variable Sublot Constraint:

$$\sum_{h=1}^{i'-1} s_{hj(k-1)} - \sum_{h=1}^{i} s_{hjk} + Gx_{i'ijk} \geq 0,$$
$$\forall i = 1, \ldots, n_j, i' = 2, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m$$
$$x_{1ijk} = 1, \forall i = 1, \ldots n_j, j = 1, \ldots, N, k = 2, \ldots, m.$$

Note that, in this case, the definition of x is different from that for Case (1). In particular, $x_{i'ijk} = 1$, if sublot i of lot j on machine k is started no earlier than the starting time of sublot i' of the same lot on machine $k - 1$, and $=0$, otherwise. Accordingly, if the first sublot of lot j on machine k starts no earlier than the starting time of sublot i' of the same sublot on machine $k - 1$, then the starting time of sublot i on machine k should be no earlier than the starting time of sublot i' on machine $(k - 1)$ plus the processing time of the jobs from sublot i' to be contained in sublot i on machine k, i.e., $\left(\sum_{h=1}^{i} s_{hjk} - \sum_{h=1}^{i'-1} s_{hj(k-1)}\right) p_{i'j(k-1)}$, along with the transfer and setup times. Note that the maximum operator is needed here since $\sum_{h=1}^{i'-1} s_{hj(k-1)}$ could be larger than $\sum_{h=1}^{i} s_{hjk}$. The corresponding constraints for the first sublot are shown in (a), and for other sublots in (b) above. However, in case the sublot i of lot j on machine k starts earlier than the starting time of i' on machine $k - 1$, then the sum of all the sublots until sublot i on machine k must not exceed the sum of the sublots until sublot $i' - 1$ on machine $k - 1$. This is captured by the variable sublot constraint.

As alluded to earlier for Case (1), the above development is applicable for the other generic models as well except that, in the case of sublot-attached setup, we need to include setup time for all sublots as follows:

$$C_{ijk} - p_{jk}s_{ijk} - C_{i'j(k-1)} + p_{j(k-1)}s_{i'j(k-1)} - FT_j - VT_j s_{ijk} - \tau_{jk}$$
$$+ G(1 - x_{i'ijk}) \geq \max\left\{ p_{j(k-1)}\left(\sum_{h=1}^{i} s_{hjk} - \sum_{h=1}^{i'-1} s_{hj(k-1)}\right), 0\right\},$$
$$\forall i = 1, \ldots, n_j, i' = 1, \ldots, n_j, j = 1, \ldots, N, k = 2, \ldots, m.$$

The above models are illustrated using a three-lot, three-machine problem. The data is given in Table 2.12. The results are depicted in Table 2.13. For the sake of comparison, we have also given results for the case of consistent sublot sizes. As expected, the makespan value obtained for Case (2) of the variable sublots is the smallest, namely, 203, while that for Case (1) of the variable sublots is 208. For the consistent sublots, the makespan value obtained is 213.

TABLE 2.12. Data for the illustration of lot-attached setup

	Processing time			Setup time			Removal time		
	M/C 1	M/C 2	M/C 3	M/C 1	M/C 2	M/C 3	M/C 1	M/C 2	M/C 3
Lot 1	2	1	2	1	2	2	2	1	2
Lot 2	2	4	1	2	1	3	2	2	4
Lot 3	4	2	2	2	2	1	1	2	1

	n_j	U_j	r_j	FT_j	VT_j
Lot 1	5	14	0	4	5
Lot 2	4	16	0	5	4
Lot 3	3	15	0	8	5

Above, we have presented fairly general mathematical models of the m-machine, N-lot streaming problems. There are some mathematical models presented in the literature that are suitable for the special cases of the lot streaming problem that they consider. We present these next.

2.3 Mathematical Models for Special Cases

This section presents mathematical formulations for some special cases of the lot streaming problem, each of which is further analyzed in the following chapters. The key features of these models are summarized in Table 2.14.

2.3.1 *2/1/C/{II,NI}/{CV,DV}/{Lot-Detached Setup, No-Wait}*

This problem addresses the issue of finding the continuous optimal sublot sizes for a single batch in a no-wait flow shop, in the presence of detached setup times [32]. In a no-wait flow shop, idle time can appear before the processing of any sublot i on machine 1 or machine 2. The expression for the makespan in terms of Δ_i (see Fig. 1.10), the idle time on machine 2 immediately preceding the ith sublot, is given as

$$C_{\max} = t_2 + p_2 \cdot U + \Delta_1 + \sum_{i=2}^{n} \Delta_i,$$

where t_1 is the setup time on machine 1, t_2 is the setup time on machine 2, $\Delta_1 = \max\{0, t_1 + p_1 s_1 - t_2\}$, and $\Delta_i = \max\{0, p_1 s_i - p_2 s_{i-1}\}$.

TABLE 2.13. Solutions for the consistent and variable sublot cases

1. Consistent sublot case

		Lot 1					Lot 2				Lot 3		
Consistent sublot sizes		s_1	s_2	s_3	s_4	s_5	s_1	s_2	s_3	s_4	s_1	s_2	s_3
	Sublot size	1	5	2	4	2	3	4	5	4	6	5	4
Machine 1	Start time	64	66	76	80	88	96	102	110	120	2	26	46
	Finish time	66	76	80	88	92	102	110	120	128	26	46	62
	Sublot size	1	5	2	4	2	3	4	5	4	6	5	4
Machine 2	Start time	104	105	110	112	116	120	132	148	168	68	80	90
	Finish time	105	110	112	116	118	132	148	168	184	80	90	98
	Sublot size	1	5	2	4	2	3	4	5	4	6	5	4
Machine 3	Start time	160	162	172	176	184	193	196	200	205	119	131	141
	Finish time	162	172	176	184	188	196	200	205	209	131	141	149
Optimal sequence of lots							3–1–2						
Optimal makespan							213						

2. Sublot availability case

		Lot 1					Lot 2				Lot 3		
Variable sublot sizes		s_1	s_2	s_3	s_4	s_5	s_1	s_2	s_3	s_4	s_1	s_2	s_3
	Sublot size	3	4	0	4	3	3	2	5	6	6	5	4
Machine 1	Start time	64	70	78	78	86	96	102	106	116	2	26	46
	Finish time	70	78	78	86	92	102	106	116	128	26	46	62
	Sublot size	7	0	0	0	7	5	5	4	2	5	6	4
Machine 2	Start time	104	111	111	111	111	121	141	161	177	69	79	91
	Finish time	111	111	111	111	118	141	161	177	185	79	91	99
Machine 3	Sublot size	1	0	0	0	13	5	5	6	0	5	0	10
	Start time	155	157	157	157	157	188	193	198	204	119	129	129
	Finish time	157	157	157	157	183	193	198	204	204	129	129	149
Optimal sequence of lots							3–1–2						
Optimal makespan							208						

3. Item availability case

		Lot 1					Lot 2				Lot 3		
Variable sublot sizes		s_1	s_2	s_3	s_4	s_5	s_1	s_2	s_3	s_4	s_1	s_2	s_3
	Sublot size	1	0	0	0	13	1	0	0	15	1	0	14
Machine 1	Start time	64	66	66	66	66	96	98	98	98	2	6	6
	Finish time	66	66	66	66	92	98	98	98	128	6	6	62
	Sublot size	4	4	3	0	3	3	3	5	5	6	5	4
Machine 2	Start time	102	106	110	113	113	120	132	144	164	68	80	90
	Finish time	106	110	113	113	116	132	144	164	184	80	90	98
	Sublot size	1	0	0	5	8	6	5	3	2	6	5	5
Machine 3	Start time	148	150	150	150	160	183	189	194	197	112	122	132
	Finish time	150	150	150	160	176	189	194	197	199	122	132	142
Optimal sequence of lots							3–1–2						
Optimal makespan							203						

TABLE 2.14. Key features of the mathematical models presented in Sect. 2.3

Section	Number of machines	Number of lots	Sublot type	Inter./No Inter. Idle Time	Continuous/Discrete Sublot Sizes	Setup	Removal time	Transfer time	Interminging	Wait/no wait	Objective function
2.3.1	2	1	C	Both	Both	Lot–Detached	No	No	N/A	No-wait	Makespan
2.3.2	2	N	C	Both	Both	Lot–Attached and lot Detached	No	Yes	No	Wait	Makespan
2.3.3	2	1	C	II	CV	None	No	No	N/A	Wait	Total Sublot Completion Times
2.3.4	2	1	C	II	CV	Sublot–Attached	No	No	N/A	Wait	Total Sublot Completion Times
2.3.5	3	1	C	Both	Both	None	No	No	N/A	Wait	Makespan
2.3.6	m	1	C (two sublots)	II	CS	None	No	No	N/A	Wait	Total Sublot Completion Times
2.3.7	m	1	E	II	CV	Sublot–Attached	No	Yes	N/A	Wait	Unified Cost Function

Let I represent the total idle time on machine 2. In order to minimize the makespan $C_{\max}$, it is sufficient to minimize the total idle time I on machine 2. This problem can be formulated as a linear program.

$$\text{Minimize}: \quad I = \sum_{i=1}^{n} \Delta_i.$$

Subject to :

$$\Delta_1 \geq t_1 + p_1 s_1 - t_2$$

$$\Delta_i \geq p_1 s_i - p_2 s_{i-1}, \quad \forall i, i = 2, \ldots, n$$

$$\sum_{i=1}^{n} s_i = U$$

$$\Delta_i \geq 0, \quad \forall i, i = 1, \ldots, n$$

$$s_i \geq 0, \quad \forall i, i = 1, \ldots, n.$$

A solution to the above model will give the desired sublot sizes and the order of their processing on the machines.

2.3.2 *2/N/C/{II,NI}/{CV,DV}/{Lot-Attached/Detached Setup, Sublot Transfer Times}*

This problem addresses the issue of finding the continuous, optimal sublot sizes and the sequence in which to process the lots in the presence of lot-attached/detached setup times and variable sublot transfer times [36]. These transfer times are made up of a fixed component FT_j and a variable component VT_j, which depends on the size of a sublot.

For ease of understanding, the situation on hand is depicted in Fig. 2.1 for $N = 1$. In this figure, F and V represent fixed and variable transfer times; and t_1 and t_2 are lot-detached setup times on machines 1 and 2, respectively. Note that Δ_1, the idle time on machine 2 before the start of sublot 1 on that machine can be

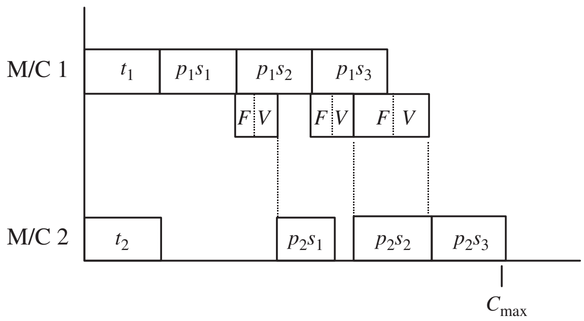

FIGURE 2.1. Graphical depiction of sublot-attached transfer times

expressed as follows:

$$\Delta_1 = \max\{0, t_1 + p_1 \cdot s_1 + VT \cdot s_1 + FT - t_2\}.$$

If we let $t_1' = t_1 + FT, \quad p_1' = p_1 + VT$, and $t_2' = t_2$, then we have,

$$\Delta_1 = \max\left\{0, t_1' + p_1' - t_2'\right\}.$$

Similarly, Δ_i, the idle time on machine 2 before the start of sublot i, can be given as follows:

$$\begin{aligned}\Delta_i &= \max\left\{0, t_1 + p_1 \sum_{u=1}^{i-1} s_u + p_1 s_i + VTs_i + FT - t_2 - p_2 \sum_{u=1}^{i-1} s_u - \sum_{u=1}^{i-1} \Delta_u\right\} \\ &= \max\left\{0, t_1' + p_1 \sum_{u=1}^{i} s_u - t_2' - p_2' \sum_{u=1}^{i-1} s_u - \sum_{u=1}^{i-1} \Delta_u\right\}, \quad \forall i = 2, \ldots, n_j,\end{aligned}$$

where $p_2' = p_2' + VT$.

Now, if we designate by I_j^{DS} the total idle time on machine 2 under sublot-detached setup for lot j, then a formulation for the problem of determining optimal sublot sizes for lot j that minimizes the makespan (or, equivalently I_j^{DS}), can be given as follows.

$$\begin{aligned}&\text{Minimize :} \quad I_j^{\text{DS}}.\\ &\text{Subject to :}\\ &I_j^{\text{DS}} \geq t_{j1}' - t_{j2}' + p_{j1}' s_{1j}\\ &I_j^{\text{DS}} \geq t_{j1}' - t_{j2}' + p_{j1}'(s_{1j} + s_{2j}) - p_{j2}' s_1 j\\ &\vdots\\ &I_j^{\text{DS}} \geq t_{j1}' - t_{j2}' + p_{j1}' \sum_{u=1}^{n_j} s_{uj} - p_{j2}' \sum_{u=1}^{n_j - 1} s_{uj}\\ &\sum_{u=1}^{n_j} s_{uj} = U_j\\ &I_j^{\text{DS}} \geq 0\\ &s_{ij} \geq 0, \quad \forall i, i = 1, \ldots, n_j.\end{aligned}$$

In the case of lot-attached setups, the only change that we need to make is in the determination of Δ_1, which now becomes,

$$\Delta_1 = \max\{0, t_1 + p_1 \cdot s_1 + VTs_1 + FT\}.$$

Accordingly, the formulation for the lot-attached setup is as follows:

$$
\begin{aligned}
&\text{Minimize}: \quad I_j^{\mathrm{AS}}.\\
&\text{Subject to}:\\
&I_j^{\mathrm{AS}} \geq t'_{j1} + p'_{j1}s_{1j}\\
&\vdots\\
&I_j^{\mathrm{AS}} \geq t'_{j1} - t'_{j2} + p'_{j1}\sum_{u=1}^{n_j} s_{uj} - p'_{j2}\sum_{u=1}^{n_j=1} s_{uj}\\
&\sum_{u=1}^{n_j} s_{uj} = U_j\\
&I_j^{\mathrm{AS}} \geq 0\\
&s_{1j} \geq 1\\
&s_{ij} \geq 0, \quad \forall i = 1, \ldots, n_j,
\end{aligned}
$$

where I_j^{AS} is the total idle time on machine 2 under sublot-attached setup for lot j. Once the sublot sizes have been obtained for each lot for either the lot-detached or lot-attached setup case, the lots are sequenced in accordance with the Johnson's rule [19]. This is further explained in Chap. 3.

A slightly different version of the above formulation is presented in [8] for the detached setup case, which includes removal time for each lot, and is based on the concept of run-in and run-out times.

2.3.3 *2/1/C/II/CV/*$\sum_{i=1}^{n} s_i C_{i2}$

This problem can be described as follows: Given a two-machine flow shop with a single lot, determine the continuous and consistent sublot sizes such that the total weighted sublot completion time, i.e., $\sum_{i=1}^{n} s_i C_{i2}$, is minimized [31]. This is essentially a sublot sizing problem, and can be formulated as a linear program as follows.

$$
\text{Minimize}: \quad F(s, C) \equiv \sum_{i=1}^{n} s_i C_{i2}.
$$

Subject to :

$$C_{ik} \geq C_{i-1k} + p_2 s_i, \quad \forall i = 2, \ldots, n, k = 1, 2, \tag{2.1}$$

$$C_{i2} \geq C_{i1} + p_2 s_i, \quad \forall i = 2, \ldots, n, \tag{2.2}$$

$$C_{11} \geq s_1 p_1, \tag{2.3}$$

$$\sum_{i=1}^{n} s_i = U, \tag{2.4}$$

$$s_i \geq 0, \quad \forall i = 1, \ldots, n, \tag{2.5}$$

$$C_{i,k} \geq 0, \quad \forall i = 2, \ldots, n, k = 1, 2. \tag{2.6}$$

As mentioned above, the objective function minimizes the total weighted sublot completion time. Constraint (2.1) ensures that sublots on any machine are processed only after the preceding sublot has finished processing. Constraint (2.2) captures the fact that machine 2 processes sublots only after it has finished processing on machine 1. Constraint (2.3) ensures that the completion time of the first sublot is greater than or equal to its processing time. Constraint (2.4) imposes the requirement that the sublot sizes add up to the lot size. Constraint (2.5) and (2.6) represent the nonnegativity of the sublot sizes and the completion times.

2.3.4 *2/1/C/II/CV/*$\sum_{i=1}^{\bar{n}} s_i C_{i2}$*, Sublot-Attached Setup*

This problem is like the one in Sect. 2.3.3 except that, now, the sublot-attached setups are present and also the number of sublots is not known *a priori* [5]. Let $\bar{n}$ be an upper bound on the number of sublots. A mathematical model of this problem is as follows.

$$\text{Minimize}: F(\text{s}, \text{C}) \equiv \sum_{i=1}^{\bar{n}} s_i C_{i2} \tag{2.7}$$

Subject to :

$$C_{i2} = i \cdot t_2 + p_2 \sum_{j=1}^{i} s_j + I_i, \quad \forall i = 1, \ldots, \bar{n}, \tag{2.8}$$

$$\begin{aligned} I_1 &= t_1 + p_1 s_1 \\ I_i &\geq I_{i-1}, \quad \forall i = 2, \ldots, \bar{n}, \end{aligned} \tag{2.9}$$

$$I_i \geq \left(i t_1 + p_1 \sum_{j=1}^{i} s_j \right) - \left((i-1) t_2 + p_2 \sum_{j=1}^{i-1} s_j \right), \quad \forall i = 2, \ldots, \bar{n}, \tag{2.10}$$

$$\sum_{i=1}^{M} s_i = U, \tag{2.11}$$

$$s_i \geq 0, \quad \forall i = 1, \ldots, \bar{n}. \tag{2.12}$$

The objective function $F(\boldsymbol{s}, \boldsymbol{C})$ seeks to minimize the total weighted sublot completion time of all the $\bar{n}$ possible positive sublots. Constraint (2.8) defines the completion time of any sublot i on machine 2 as the sum of

(i) Setup times of all previous sublots including sublot i on machine 2
(ii) Processing times of all previous sublots including sublot i on machine 2
(iii) Cumulative idle time appearing before sublot i on machine 2

Constraint (2.9) defines the idle time appearing before sublot 1 on machine 2 as the sum of its setup and processing time on machine 1. Constraint (2.10) defines the cumulative idle time on machine 2 for sublots $2, \ldots, \bar{n}$. The following two cases are possible:

(i) The cumulative idle time remains the same i.e., $I_i = I_{i-1}$, implying that sublot $(i-1)$ finishes processing on machine 2 later than the completion of sublot i on machine 1
(ii) The cumulative idle time increases implying that sublot $(i-1)$ finishes processing on machine 2 before the completion of sublot i on machine 1

Constraint (2.11) ensures that the sum of the sublot sizes does not exceed the given lot size. The last constraint restricts the sublot sizes to be nonnegative.

2.3.5 *3/1/C/{NI,II}/{CV,DV}/{No Setup}*

This problem addresses the sublot sizing problem for a three-machine flow shop by minimizing the completion time of the last sublot on machine 3 when the sublot sizes are consistent [35]. Let C_{ik} denote the completion time of the ith sublot on machine k. Then, we have

$$\text{Minimize}: C_{3n}$$
$$\text{Subject to}:$$
$$C_{11} \geq s_1 p_1 \quad (2.13)$$
$$C_{ik} \geq C_{i,(k-1)} + p_k s_i, \quad \forall i = 1, 2, \ldots, n, k = 2, 3, \quad (2.14)$$
$$C_{ik} \geq C_{(i-1),k} + p_k s_i, \quad \forall i = 2, \ldots, n, k = 1, 2, 3, \quad (2.15)$$
$$\sum_{i=1}^{n} s_i = U, \quad (2.16)$$
$$s_i \geq 0, \quad \forall i = 1, 2, \ldots, n.$$

Constraints (2.14) and (2.15) ensure that any sublot i begins processing on machine k after its completion on the previous machine or the processing of the $(i-1)$th sublot on machine k, whichever is maximum. Constraint (2.16) imposes that the total number of items in all sublots equals U. The no-idling and discrete version can be obtained by replacing the inequalities with equalities and by restricting the sublot sizes to take integer values, respectively.

2.3.6 *m/1/C/II/CV/*$\sum_{i=1}^{2} x_i C_{im}$

We, now, consider the problem of minimizing the total weighted sublot completion time in an m-machine flow shop consisting of a single lot [34]. The number of sublots is restricted to two on each machine, the sublots sizes are consistent and can take real values. Let x_1 and $x_2 = (1 - x_1)$ be the proportion of work allocated to the first and second sublots, respectively. Let $C_{i,k}$ denote the completion time of the ith sublot on machine k and p_k be the processing time per item on machine k.

The mathematical formulation for this problem is as follows.

$$\begin{aligned}
\text{Minimize :}\quad & F(x_1, x_2) = (x_1 C_{1m} + x_2 C_{2m}).\\
\text{Subject to :}\quad & C_{11} \geq x_1 p_1,\\
& C_{2k} \geq C_{1k} + x_2 p_k, \quad \forall k = 1, \ldots, m,\\
& C_{ik+1} \geq C_{ik} + x_i p_{k+1}, \quad \forall i = 1, 2; k = 1, \ldots, (m-1),\\
& x_1 + x_2 = 1,\\
& C_{ik} \geq 0, \quad \forall i = 1, 2,\ k = 1, \ldots, m \text{ and } x_1, x_2 > 0.
\end{aligned}$$

The completion time of the sublots can be written as

$$\begin{aligned}
C_{1m} &= x_1 \sum_{k=1}^{m} p_k \text{ and}\\
C_{2m} &= \max_{1 \leq k \leq m} \left\{ x_1 \sum_{l=1}^{k} p_l + x_2 \sum_{l=k}^{m} p_l \right\}.
\end{aligned}$$

Making the above substitutions along with $x_2 = 1 - x_1$, in the expression for flowtime, we have

$$F(x_1) = x_1^2 \sum_{k=1}^{m} p_k + (1 - x_1) \max_{1 \leq k \leq m} \left\{ x_1 \sum_{l=1}^{k} p_l + (1 - x_1) \sum_{l=k}^{m} p_l \right\}.$$

Simplification of the above expression gives

$$F(x_1) = \max_{1 \leq k \leq m} \left\{ x_1^2 \left(\left(2 \sum_{l=k}^{m} p_l \right) - p_k \right) + x_1 \left(\sum_{l=1}^{k} p_l - 2 \sum_{l=k}^{m} p_l \right) + \sum_{l=k}^{m} p_l \right\}.$$

Let

$$a_k = \left(2 \cdot \sum_{l=k}^{m} p_l \right) - p_k \; b_k = \left(\sum_{l=1}^{k} p_l - 2 \sum_{l=k}^{m} p_l \right) \quad \text{and} \quad c_k = \sum_{l=k}^{m} p_l.$$

Therefore,

$$F(x_1) = \max_{1 \leq k \leq m} \left\{ a_k x_1^2 + b_k x_1 + c_k \right\}.$$

Hence, an equivalent formulation can be written as,

$$\begin{aligned}
& \text{Minimize : } F(x_1)\\
& \text{Subject to :}\\
& F(x_1) \geq a_k x_1^2 + b_k x_1 + c_k, \quad k = 1, \ldots, m
\end{aligned}$$

where

$$a_k = \left(2\sum_{l=k}^{m} p_l\right) - p_k,$$

$$b_k = \left(\sum_{l=1}^{k} p_l - 2\sum_{l=k}^{m} p_l\right), \quad \text{and}$$

$$c_k = \sum_{l=k}^{m} p_l.$$

2.3.7 *m/1/E/II/CV/Sublot-Attached Setup, Transfer Times/Unified Cost Function*

We now consider a hybrid objective function consisting of a weighted sum of the makespan ($C_{\max}$), (sublot) mean flow time (MFT), average work-in-process (WIP), sublot-attached setup (SAS), and transfer time (TT), in an m-machine flow shop with a single lot and continuous and equal sublot sizes [25].

The problem is to determine an optimal number of sublots (n) so as to minimize the above hybrid cost function. This problem can be formulated as an integer program as follows.

Minimize : $\quad Z(n) \equiv c_1 C_{\max}(n) + c_2\text{MFT}(n) + c_3\text{WIP}(n) + c_4 t_k(n) + c_5\text{TT}(n).$

Subject to:

$$C_{\max}(n) = \left\{\frac{U}{n}\sum_{k=1}^{m} p_k + \sum_{k=1}^{m} t_k\right\} + (n-1)\max_{1\le k\le m}\left\{\frac{U}{n}p_k + t_k\right\},$$

$$\text{MFT}(n) = \frac{U}{n}\sum_{k=1}^{m} p_k + \sum_{k=1}^{m} t_k + \frac{n-1}{2}\max_{1\le k\le m}\left\{\frac{U}{n}p_k + t_k\right\},$$

$$\text{WIP}(n) = U\left\{\frac{\frac{U}{n}\sum_{k=1}^{m} p_k + \sum_{k=1}^{m} t_k + \frac{n-1}{2}\max_{1\le k\le m}\left\{\frac{U}{n}p_k + t_k\right\}}{\frac{U}{n}\sum_{k=1}^{m} p_k + \sum_{k=1}^{m} t_k + (n-1)\max_{1\le k\le m}\left\{\frac{U}{n}p_k + t_k\right\}}\right\},$$

$$\text{SAS}(n) = n\sum_{k=1}^{m} t_k,$$

$$\text{TT}(n) = n(m-1)\text{TT},$$

$$1 \le n \le U \quad \text{and} \quad \text{integer}.$$

2.4 Chapter Summary

In this chapter, we have presented some generic mathematical models for the flow shop lot streaming problem. These generic models capture the various important features that may be encountered in practice. These include lot-attached (detached) setup, sublot-attached (detached) setup, lot removal time, and sublot transfer time. The removal time of a lot is assumed to be attached to the last sublot of a lot and is independent of the sequence in which the lots are processed or the size of the last sublot. The sublot transfer time, on the other hand, is assumed to be comprised of two components, one being fixed and identical for all the sublots of a lot while the other depends on the sublot size. The transfer time and removal time differ in that, during the occurrence of the former, the machine from where the transfer occurs is free to process another sublot, while, when the latter is encountered, the machine is occupied and cannot process the next lot. We also consider the situations of equal, consistent, and variable sublot sizes. In the case of variable sublots, as a lot moves from one machine to another, a new sublot can be formed in two ways. According to one of these ways, the jobs constituting a new sublot can be reconfigured to form this sublot only after the completion of the entire sublots from the previous machine to which they belong. The other way is for the jobs constituting a new sublot to be reconfigured to form this sublot without having completed the entire sublots to which they belong. We present models for both of these situations. We also consider situations in which the sublots belonging to different lots may or may not be intermingled.

We have provided illustrations for the use of several of the models that we have developed, which depict optimal sublot sizes and the sequence in which to process the lots to achieve minimum makespan values. These models are integer programs due to the presence of disjunctive constraints (for determining the sequence in which to process the lots) and the requirement of integer sublot sizes. They are solved using the CPLEX solver.

Mathematical models for some special cases of the flow shop lot streaming problem have been discussed in the literature. We have also presented these models in this chapter.