CHAPTER 1

Explanations

1.1 SALLIES

> Language is an instrument of Logic, but not an indispensable instrument.
> Boole *1847a*, 118

> We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of exact science are mathematics and logic; the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it sees better with one eye than with two. De Morgan *1868a*, 71

> That which is provable, ought not to be believed in science without proof.
> Dedekind *1888a*, preface

> If I compare arithmetic with a tree that unfolds upwards in a multitude of techniques and theorems whilst the root drives into the depths [...]
> Frege *1893a*, xiii

> Arithmetic must be discovered in just the same sense in which Columbus discovered the West Indies, and we no more create numbers than he created the Indians. Russell *1903a*, 451

1.2 SCOPE AND LIMITS OF THE BOOK

1.2.1 *An outline history.* The story told here from §3 onwards is regarded as well known. It begins with the emergence of set theory in the 1870s under the inspiration of Georg Cantor, and the contemporary development of mathematical logic by Gottlob Frege and (especially) Giuseppe Peano. A cumulation of these and some related movements was achieved in the 1900s with the philosophy of mathematics proposed by Alfred North Whitehead and Bertrand Russell. They claimed that "all" mathematics could be founded on a mathematical logic comprising the propositional and predicate calculi (including a logic of relations), with set theory providing many techniques and various other devices to hand, especially to solve the paradoxes of set theory and logic which Russell discovered or collected. Their position was given a definitive presentation in the three volumes of *Principia mathematica* (1910–1913). The name 'logicism' has become attached to this position; it is due (in this sense of the word) to Abraham Fraenkel (§8.7.6) and especially Rudolf Carnap (§8.9.3) only in the late 1920s, but I shall use it throughout.

Various consequences followed, especially revised conceptions of logic and/or logicism from Russell's followers Ludwig Wittgenstein and Frank Ramsey, and from his own revisions of the mid 1920s. Then many techniques and aims were adopted by the Vienna Circle of philosophers, affirmatively with Carnap but negatively from Kurt Gödel in that his incompletability theorem of 1931 showed that the assumptions of consistency and completeness intuitively made by Russell (and by most mathematicians and logicians of that time) could not be sustained in the form intended. No authoritative position, either within or outside logicism, emerged: after 1931 many of the main questions had to be re-framed, and another epoch began.

The tale is fairly familiar, but mostly for its philosophical content; here the main emphasis is laid on the logical and mathematical sides. The story will now be reviewed in more detail from these points of view.

1.2.2 *Mathematical aspects.* First of all, the most pertinent parts of the prehistory are related in §2. The bulk of the chapter is given over to developments of new algebras in France in the early 19th century and their partial adoption in England; and then follow the contributions of George Boole and Augustus De Morgan (§2.4–5), who each adapted one of these algebras to produce a mathematicised logic. The algebras were not the same, so neither were the resulting logics; together they largely founded the tradition of algebraic logic, with some adoption by others (§2.6). By contrast, the prehistory of mathematical logic lies squarely in mathematical analysis, and its origins in Augustin-Louis Cauchy and extension led by Karl Weierstrass are recalled in §2.7, the concluding section of this chapter, to lead in to the main story which then follows. A common feature of both traditions is that their practitioners handled collections in the traditional way of part-whole theory, where, say, the sub-collection of Englishmen is part of the collection of men, and membership to it is not distinguished from inclusion within it.

The set theory introduced in §3 is the '*Mengenlehre*' of Georg Cantor, both the point set topology and transfinite arithmetic and the general theory of sets. In an important contrast with part-whole theory, an object *was* distinguished from its unit set, and belonged to a set S whereas sub-sets were included in S: for example, object a belongs to the set $\{a, b, c\}$ of objects while sets $\{a\}$ and $\{a, b\}$ are subsets of it. The appearance of both approaches to collections explains the phrase 'set theor*ies*' in the sub-title of this book.

Next, §4 treats a sextet of related areas contemporary with the main themes outlined above, largely over the period 1870–1900. Firstly, §4.2 records the splitting in the late 1890s of Cantor's *Mengenlehre* into its general and its topological branches, and briefly describes measure theory and functional analysis. Next, §4.3–4 outlines the extension of algebraic logic by Ernst Schröder and Charles Sanders Peirce, where in particular the contributions of Boole and De Morgan were fused in a Boolean logic of relations; Peirce also introduced quantification theory, which Schröder developed. All this work continued within part-whole theory. §4.5 outlines the creation of a version of mathematical logic by Frege, highly regarded today but (as will be explained) modestly noted in his own time; it included elements of set theory. Then follows §4.6 on the first stages in the development of phenomenological logic by Edmund Husserl. Finally, §4.7 notes the early stages of David Hilbert's proof theory (not yet his metamathematics), and of American work in model theory influenced by E. H. Moore.

Then §5 describes the work of Peano and his followers (who were affectionately known as the 'Peanists'), which gained the greatest attention of mathematicians. Inspired by Weierstrass's analysis and *Mengenlehre*, this 'mathematical logic' (Peano's name) was used to express quite a wide range of mathematical theories in terms of proportional and predicate calculi with quantification (but the latter now construed in terms of members of sets rather than part-whole theory). The period covered runs from 1888 to 1900, when Russell and Whitehead became acquainted with the work of the Peanists and were inspired by it to conceive of logicism.

Russell's career in logic is largely contained within the next two chapters. First, §6 begins with his début in both logic and philosophy in the mid 1890s, and records his progress through a philosophical conversion inspired by G. E. Moore, and the *entrée* of Whitehead into foundational studies in 1898. Next comes Russell's discovery of Peano's work in 1900 and his paradox soon afterwards, followed by the publication in 1903 of *The principles of mathematics*, where his first version of logicism was presented. Then §7 records him formally collaborating with Whitehead, gathering further paradoxes, discovering an axiom of choice in set theory, adopting a theory of definite descriptions, and trying various logical systems before settling on the one which they worked out in detail in *Principia mathematica* (hereafter '*PM*'), published in three volumes between 1910 and 1913. Some contemporary reactions by others are recorded, mainly in §7.5.

In §8 is recorded the reception and use of *PM* and of logicism in many hands of various nationalities from the early 1910s to the late 1920s. Russell's own contributions included applications of logical techniques to philosophy from the 1910s, and a new edition of *PM* in the mid 1920s (§8.2–3). His most prominent successors were Wittgenstein and Ramsey, and interest continued in the U.S.A. (§8.3–5). Considerable concern with foundational studies was shown among German-speaking philosophers and mathematicians (§8.7), including the second stage of Hilbert's 'metamathematics' and the emergence of the 'intuitionistic' philosophy of mathematics, primarily with the Dutchman L. E. J. Brouwer. Two new groups arose: logicians in Poland, led by Jan Łukasiewicz and Stanisław Leśniewski, and soon joined by the young Alfred Tarski (§8.8); and the group of philosophers which became known as the 'Vienna Circle', of whom Moritz Schlick, Carnap and Gödel are the most significant here (§8.9).

In briefer order than before, §9 completes the story by reviewing the work of the 1930s. Starting with Gödel's incompletability theorem of 1931, other contemporary work is surveyed, especially by members of the Vienna Circle and some associates. The returns of both Whitehead and Russell to logicism are described, and some new applications and countries of interest are noted. Finally, with special attention to Russell, the concluding §10 reviews the myriad relationships between logics, set theories and the foundations of mathematics treated in this book; the concluding §10.3 contains a flow chart of the mathematical developments described in the book and stresses the lack of an outright "winner". Ten manuscripts, mostly letters to or from Russell, are transcribed in §11. Then follow the bibliography and index.

1.2.3 *Historical presentation.* This book is intended for mathematicians, logicians, historians, and perhaps philosophers and historians of science who take seriously the concerns of the other disciplines. No knowledge of the history is assumed in the reader, and numerous references are given to both the original and the historical literature. However, it does not serve as a textbook for the mathematics, logic or philosophy discussed: the reader is assumed to be already familiar with these, approximately at the level of an undergraduate in his final academic year.

From now on I shall refer to the 'traditions' of algebraic and of mathematical logic; the two together constitute 'symbolic logic'. Occasionally mention will be made of other traditions, such as syllogistic logic or Kantian philosophy. By contrast of term, logicism will constitute a 'school', in contention with those of metamathematics, intuitionism and phenomenology.

Inter-disciplinary relationships were an important part of the story itself, for symbolic logic was usually seen by mathematicians as too philosophical and by philosophers as too mathematical. De Morgan's remark quoted in §1.1 is especially brilliant, because not only was he both mathematician and logician but also he had only one eye! Thus the title of this book, 'The search for mathematical roots', is a *double entendre*: whether mathematics (or at least some major parts of it) could be founded in something else, such as the mathematical logic of Whitehead and Russell; or the inverse stance, where mathematics itself could serve as the foundation for something else. A third position asserted that mathematics and logic were overlapping disciplines, with set theories occupying some significant place which itself had to be specified; it was upheld by the Peanists, and gained more support after Gödel, especially with W. V. Quine (§9.4.4).

The final clause of the sub-title of this book would read more accurately, but also a little too clumsily, as 'inspired in different ways by Lagrange and

Cauchy, and pursued especially but not only from Cantor and Peano through Whitehead and Russell to Carnap and Gödel', with some important names still missing. Its story differs much from the one in which Frege dominates, the details of the mathematics are at best sketched, and everything is construed in terms of analytic philosophy. For example, the discussion here of *Principia mathematica* does not stop after the first 200 pages but also takes note of the next 1,600, where the formulae are presented. The quality and merits of Whitehead and Russell's logicism should then become clearer, as well as its well-known (and important) confusions and limitations. Again, most histories of these topics are of the 'great man' variety; but here many other people play more minor but significant roles—either as minor figures in the tale or as major ones in some related developments.

Another novelty is that much new information is provided from about 50 archival sources which have been examined. Russell left an enormous *Nachlass*, known as the 'Russell Archives' and cited in this book as 'RA'; so did some other figures (for example, Hilbert, Peirce and Carnap). For several more, valuable collections are available (Boole, Cantor, Dedekind and Gödel); for some, sadly, almost nothing (Peano and Whitehead). Important information has come from the manuscripts of many other figures (including several named earlier), and from some university and publishers' archives. Normally a collection is cited as, say, 'Cantor Papers', followed by an identifying clause or code of a particular document appropriate for its (dis-)organisation. Its location is indicated at the head of the list of his cited works in the bibliography. The main archive locations are recorded in the front matter there, and are also named in the acknowledgements in §1.4.

1.2.4 *Other logics, mathematics and philosophies.* To temper the ambitions just outlined, some modesty is required.

1) A few concurrent developments outside mathematical logic are described, though not in much detail. The limited coverage of algebraic logic was mentioned in §1.2.2: its own relationships with other algebras are treated lightly. An integrated history of post-syllogistic and algebraic logic from the 1820s to the 1920s is *very* desirable.

Again, in §6.2–3 notice is taken of the influential but very non-mathematical neo-Hegelian tradition in logic only in connection with the young Russell, who started out with it but then rejected it at the end of the century.[1] Similarly, phenomenological logic is noted just to the extent of §4.5 on Husserl and §8.7.2,8 on a few followers; and §8.8 and §9.6.7 contain only some of the work of the Polish community of logicians.

[1] Since those kinds of philosophy have fallen out of favour (apart from centres where Germanic influences remain active), the history has become quite mis-remembered. It is thought, even by some historians, that they died very quickly, especially in Anglo-Saxon countries, after the rise of Russell and his associates in the 1900s; however, a different course will be revealed in §9.5.

2) An important neighbour is metamathematics, which in this period was created and dominated by Hilbert with an important school of followers. The story of his search for mathematical roots from Cantor to Gödel is very important; but it is rather different from this one, more involved with the growth of axiomatisation in mathematics and with metamathematics and granting a greater place to geometry, and less concerned with mathematical analysis and the details of Cantor's *Mengenlehre*. So only some portions of it appear here, mainly in §4.4, §8.8 and §9.6.2. Similarly, no attempt is made here to convey other foundational studies undertaken in mathematics at that time, such as the foundations of geometry and of mechanics, or the development of abstract algebras and of quantum mechanics.

3) Another neighbouring discipline to logic is linguistics, which during our period was concerned not only with grammar and syntax but also with traditional questions such as the origins of language in humans and the classification of languages. One would assume that links to logics, especially mathematicised ones, were strong, in particular through the common link of semiotics, the science of signs, for which common algebra was the supreme case; indeed, we shall note in §2.2.1 that in the 17th century John Locke had used 'semiotics' and 'logic' as synonyms. However, with the exception of Peirce (§4.3.8) the connections were slight—indeed, already so in the 18th century when linguistics was well developed while logic languished. More work is needed on this puzzling situation, which is largely side-stepped here.

4) Almost all of the logics described here were 'finitary'; that is, both formulae and proofs were finitely long. From time to time we shall come across an 'infinitary' logic, usually "horizontal" extensions to infinitely long formulae while in §9.2.5 appears a "vertical" foray to infinitely long proofs; but their main histories lie after our period.

5) A few modern versions of logicism have been proposed in recent years, and also various figures in our story have been invoked in support or criticism of current positions in epistemology and the philosophy of science. I have noted only a few cases in a footnote in §10.2.3, since modernised versions of the older thought are involved. More generally, I have made no attempt to treat the huge literature which comments without originality on the developments described in this book. Logicism has inspired many opinions about logic and the philosophy of mathematics from Russell's time to today, but often offered with little knowledge of the technical details or applications of his logic.

6) The story concentrates upon the research level of work: its (non-)diffusion into education is touched upon only on opportune occasions. The impact upon teaching during the period under consideration seems to have been rather slight, but the matter merits more investigation than it receives here.

1.3 Citations, terminology and notations

1.3.1 *References and the bibliography.* The best source for the original literature is the German reviewing journal *Jahrbuch über die Fortschritte der Mathematik*, where it was categorised in amusingly varied distributions over the years between the sections on 'Philosophie', 'Grundlagen', 'Mengenlehre' and 'Logik'. Among bibliographies, Church *1936a* and *1938a* stand out for logic, and Risse *1979a* and Vega Reñon *1996a* are also useful; for set theory Fraenkel *1953a* is supreme. Toepell *1991a* provides basic data on German mathematicians, including several logicians. My general encyclopaedia *1994a* for the history and philosophy of mathematics has pt. 5 devoted to logics and foundations, and each article has a bibliography of mostly secondary sources; some articles in other parts are also relevant. Among philosophical reference works, note especially Burkhardt and Smith *1991a*.

Most works are cited by dating codes in italics with a letter, such as 'Russell *1906a*'; the full details are given in the bibliography, which also conveys dates of all authors when known. When a manuscript is cited, whether or not it has been published on some later occasion, then the reference is prefaced by '*m*' as in 'Russell *m1906a*', in which case there is no '*1906a*'. Collected or selected editions or translations of works and/or correspondence are cited by words such as '*Works*' or '*Letters*'; if a particular volume is cited in the text, then the volume number is added also in italics, as in 'Husserl *Works 12*'. Different editions of a work are marked by subscript numbers. '*PM*' is cited wherever possible by the asterisk number of the proposition or definition; if page numbers are needed, they are to the second edition. A few works on a figure without named author or editor are cited under his name with a prime attached; for example, 'Couturat *1983a′*' is a volume on his life and work.

This strategy of avoiding page numbers has been followed whenever possible for works which have received multiple publication—original appearance (maybe more than once), re-appearance in an edition of the author's works and/or anthologies, and maybe a translation or two. In such cases, article or even theorem or equation numbers have been used instead. Where a page number is necessary, an accessible and reliable source has been chosen, and its status is indicated in the bibliography entry by the sign '‡'.

Finally, '§' is used to indicate chapters and their sections and sub-sections; no chapter has more than nine sections, and no section has more than nine sub-sections. Equations or expressions are numbered consecutively within a sub-section; for example, (255.3) is the third equation in §2.5.5.

1.3.2 *Translations, quotations and notations.* All non-English texts have been translated into English; usually the translations are my own. Several of our main authors have been translated into English, but not always with happy results—too free, and often not drawing upon the correct philosophical distinctions in the original language (especially German). Occasionally issues of translation are discussed. Apart from in §11, my own insertions into quotations, of any kind, are enclosed within square brackets.

As far as possible, I have followed the terms and symbols used by the historical figures, and in quotations they are preserved or translated exactly. But several ordinary words, in any language, were used as technical terms (for example, 'concept' and 'number'). Quite often I have used quotation marks or quoted the original word alongside the translation; and I use 'notion' as a neutral all-purpose word to cover concepts and general ideas. In addition, a variety of terms, or changes in terms, has occurred over time, and the most modern version is often *not* adopted here. In particular, I use 'set' when in Cantor's *Mengenlehre* but follow Russell in speaking of 'classes', which was his technical term with 'sets' as informal talk. Some further terms in Russell are explained in §6.1.1.

From 1904 the word 'logistic' was adopted to denote the new mathematical logic (§7.5.1), but it covered both the position of the Peanists and that of Russell. I try to make clear its sense in each context, and use 'Peanism' or 'logicism' where possible.

Related problems arise from our custom of distinguishing a theory, language or logic from its metatheory, metalanguage or metalogic; for it clearly emerged only during the early 1930s (§9.2–3, §9.6.7). Apart from some tantalising partial anticipations in the 1920s, especially in the U.S.A. (§8.5), earlier it was either explicitly avoided (by Russell, for example) or observed only in certain special cases, such as distinguishing a descriptive phrase from its possible referent. In particular, the conditional connective ('if . . . then') between propositions was muddled with implication between their names, and propositions themselves with (well-formed) sentences in languages. I have tried to *follow* these kinds of conflation, in order to reconstruct the muddles of the story; the logic is worse, but the history much better. So I have not distinguished name-forming single quotation marks from quasi-quotes; however, I use double quotation marks as scare-quotes for special uses of terms. Lastly, the reader should bear in mind that often I *mention* an historical figure *using* some quoted term or notation.

In quotations from and explanations of original work, the original symbols are used or at least described. However, for my own text I have had to make choices, since various notations have been entertained in logic and set theories over the decades. Several of them have their origins in Peano or in Whitehead and Russell, and they serve as my basic lexicon here (including some conflations discussed above):

$\sim$ *or* $\bar{\ }$	not $\vee$ (inclusive) or	$\cdot$	and $\vdash$ assertion
$\supset$	if . . . then *or* implication	$\equiv$	if and only if *or* equivalence
(x) *or* $_x$	. . . for all x . . .	$(\exists x)$	there is an x such that . . .
$=$	identity *or* equality	$:=$ *or* $=$ Df.[2]	equality by definition
$\ni$	such that	$\in$ *or* ε	is a member of
$\cup$	union of classes	$\cap$	intersection of classes
$\subseteq$	improper inclusion of classes	$\subset$ *or* $\supset$	proper inclusion of classes
$-$	difference of classes	ι‘	unit class of
V	universal class *or* tautology	Λ	empty class *or* contradiction
$\{a, b, \ldots\}$	unordered class	$(a, b, \ldots)$	ordered class
$\hat{x}\phi(x)$	the x's such that (class abstraction)	$(\imath x)(\phi x)$	the x such that (definite description)

In addition, to reduce the density of brackets I have made some use of Peano's systems of dots: the larger their number at a location, the greater their scope. Dots indicating logical conjunction take the highest priority, and there the scope lies in both directions; then come dots following expressions which use brackets for quantifiers; and finally there are dots around connectives joining propositions.

I use the usual Roman or Greek letters for mathematical and for logical functions, distinguishing the two types by enclosing the argument variable of a mathematical function within brackets (such as '$f(x)$'). Relations are normally represented by upper case Roman letters. Further explanations, such as Russell's enthusiastic use of '!', are made in context.

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[2] '$:=$ ' has become popular in recent years: De Morgan had used it to define 'singular identity' between individual members of classes (*1862a*, 307). '$=$ Df.' belongs to Russell: according to Chwistek *1992a*, 242, the variant '$=_{\mathrm{Df}}$ ' (not employed here) was introduced by W. Wilcosz; but it was already presented in the form '$=_{\mathrm{Def}}$ ' in Burali-Forti *1894b*, 26 (§5.3.7).

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All efforts have been made to locate copyright holders of a few other quoted texts.

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