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0521604915 - Approaches to the Theory of Optimization  
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**CAMBRIDGE TRACTS IN MATHEMATICS**

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***77. Approaches to the theory of optimization***

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optimization*

CAMBRIDGE UNIVERSITY PRESS  
CAMBRIDGE  
LONDON NEW YORK NEW ROCHELLE  
MELBOURNE SYDNEY

Cambridge University Press  
0521604915 - Approaches to the Theory of Optimization  
J. Ponstein  
Frontmatter  
[More information](#)

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PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE  
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS  
The Edinburgh Building, Cambridge CB2 2RU, UK  
40 West 20th Street, New York NY 10011-4211, USA  
477 Williamstown Road, Port Melbourne, VIC 3207, Australia  
Ruiz de Alarcón 13, 28014 Madrid, Spain  
Dock House, The Waterfront, Cape Town 8001, South Africa

<http://www.cambridge.org>

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First published 1980  
First paperback edition 2004

*A catalogue record for this book is available from the British Library*

ISBN 0 521 23155 8 hardback  
ISBN 0 521 60491 5 paperback

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## *Preface*

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Perhaps the title of this book should have been, somewhat in the style of past times, ‘Some approaches towards the theory of optimization, with an emphasis on the topological aspects, ignoring combinatorial problems and almost ignoring combinatorial tools; not going into the algorithmic and numeric problems of effectively finding solutions to problems, yet meant as a contribution to applied and even practical mathematics.’

Even this does not make it sufficiently clear what is going on in the book and what is not. It is worth saying a few words about the omissions. Convex processes are not treated, problems involving more than one objective to be optimized are only touched on lightly. Of game theory, only the simplest model is considered. Since, as far as Lagrangian duality is concerned, any practitioner wants a nonzero multiplier attached to the objective function, theorems where this multiplier is allowed to be zero do not receive any attention. Similarly the reader will not find anything about regularity conditions, which apart from being sufficient are also necessary for a whole class of problems (rather than for a single one). This is because when we try to solve any one problem out of such a class, those conditions may well be too strong for practical purposes. This is not to say that everything in the book is so ‘practical’: some basic theorems rest on the axiom of choice, for example, and sometimes we are satisfied to establish the equality of an infimum and a supremum rather than that of the corresponding minimum and maximum.

Although in practical situations we can often make do with decision variables which are elements of a Banach space, or even a Euclidean space, part of the theory includes a generalization to locally convex topological vector spaces. This provides us with the important possibility of considering, in Banach spaces, topologies weaker than the strong one.

If we single out for competition the results dealt with in the appendixes, the prize for the most beautiful result should perhaps go to Theorem 6.2.2 on the equality of *inf sup* and *sup inf*, because of its generality and its natural and elegant proof; or perhaps to the general fixed point theorem (6.1.18) because of its combinatorially ingenious proof; or to the theory of conjugate duality (chapter 4) because of its power and symmetry. The reader should judge for himself.

The main text does not contain references to the literature. These, together with comments, have been combined in a separate section entitled 'Comments on the text and related literature'. Internal cross-references are indicated either by two numerals, such as 3.14 referring to section 14 of chapter 3, or by three numerals, such as 3.14.19 referring to an item in 3.14.

The text assumes a basic knowledge of topology as well as functional analysis.

Explicitly, J. W. Nieuwenhuis contributed by generalizing some basic theorems, and implicitly he and W. K. Klein Haneveld contributed through many discussions on all kinds of subjects involved. The number of these discussions is only countable, but their effect was invaluable.



## Symbols

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### Latin

|   |   |
|---|---|
| $A$ usually a matrix  | $N^*$ dual negative cone  |
| $B$ usually a matrix  | $p$ perturbation function   |
| $b$ usually a right-hand side                               | $p^d$ dual perturbation function                                      |
| $C$ usually a subset of $X$                                 | $p^{dd}$ bidual perturbation function                                 |
| cl closure of   | $P$ positive cone   |
| $Dh(z, \tau)$ see (4.4.4)                                   | $P^*$ dual positive cone  |
| dom effective domain  | $q'$ usually Fréchet derivative of $q$                                |
| epi epigraph  | $R$ real axis   |
| $f$ objective function                                      | ri relative interior  |
| $F$ bifunction  | $R_n$ $n$ -dimensional Euclidean space                                |
| $F^d$ dual bifunction                                       | $T$ set defined in (3.5.3) or point-to-set mapping defined by (6.1.7) |
| $F^{dd}$ bidual bifunction                                  | $U$ a set or a neighbourhood  |
| $g$ in $g(x) \leq 0$  | $u$ control variable, or in $u \in U$                                 |
| $g_a$ (includes) the active part of $g$ (in $g(x) \leq 0$ ) | $V$ a set or a neighbourhood, or as defined by (3.5.4)                |
| $G$ set (feasible region)                                   | $W$ a set or a neighbourhood  |
| $h$ in $h(x) = 0$ or a function                             | $X$ decision space  |
| $h^*$ conjugate of $h$                                      | $x$ in $x \in X$ , decision variable                                  |
| $h^{**}$ biconjugate of $h$                                 | $X^*$ dual perturbation space   |
| $h'(z_o, z)$ one-sided directional derivative               | $x^*$ in $x^* \in X^*$ , dual perturbation                            |
| $H$ hyperplane or Hamiltonian                               | $Y$ constraint space = perturbation space                             |
| int interior  |   |
| $K(\gamma)$ cone defined by (3.8.6)                         |   |
| $L$ Lagrangian  |   |
| $l_1, l_\infty$ see Example 3.2.9                           |   |
| $L(V)$ see Definition 3.3.4                                 |   |
| $N$ negative cone or nullspace                              |   |

$y$  in  $y \in Y$ , (primal)  
 perturbation  
 $Y^*$  multiplier space  
 $y^*$  in  $y \in Y^*$ , multiplier  
 $Z$  a (perturbation) space  
 $z$  in  $z \in Z$   
 $Z^*$  dual of  $Z$   
 $z^*$  in  $z^* \in Z^*$

**Greek**

$\alpha$  infimum as in (3.5.1)  
 $\beta$  supremum as in (3.5.2)  
 $\delta$  indicator function  
 $\delta^*$  support function

**Other**

$\nabla$  gradient  
 $\partial$  subgradient