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AXIOMATIC DOMAIN THEORY IN CATEGORIES OF PARTIAL MAPS

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Dedicado
a mis padres, Cristina y Luis, y
a mis hermanos, Verónica y Alejandro.

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Preface

This thesis is an investigation into *axiomatic categorical domain theory* as needed for the denotational semantics of deterministic programming languages.

To provide a direct semantic treatment of non-terminating computations, we make partiality the core of our theory. Thus, we focus on categories of partial maps. We study representability of partial maps and show its equivalence with classifiability. We observe that, once partiality is taken as primitive, a notion of approximation may be derived. In fact, two notions of approximation, contextual approximation and specialisation, based on testing and observing partial maps are considered and shown to coincide. Further we characterise when the approximation relation between partial maps is domain-theoretic in the (technical) sense that the category of partial maps **Cpo**-enriches with respect to it.

Concerning the semantics of type constructors in categories of partial maps, we present a characterisation of colimits of diagrams of total maps; study order-enriched partial cartesian closure; and provide conditions to guarantee the existence of the limits needed to solve recursive type equations. Concerning the semantics of recursive types, we motivate the study of enriched algebraic compactness and make it the central concept when interpreting recursive types. We establish the fundamental property of algebraically compact categories, namely that recursive types on them admit canonical interpretations, and show that in algebraically compact categories recursive types reduce to inductive types. Special attention is paid to **Cpo**-algebraic compactness, leading to the identification of a 2-category of kinds with very strong closure properties.

As an application of the theory developed, enriched categorical models of the metalanguage FPC (a type theory with sums, products, exponentials and recursive types) are defined and two abstract examples of models, including domain-theoretic models, are axiomatised. Further, FPC is considered as a programming language with a call-by-value operational semantics and a denotational semantics defined on top of a categorical model. Operational and denotational semantics are related via a computational soundness result. The interpretation of FPC expressions in domain-theoretic **Poset**-models is observed to be representation-independent. And, to culminate, a computational adequacy result for an axiomatisation of absolute non-trivial domain-theoretic models is proved.

Acknowledgements/Agradecimientos

I will always remain in intellectual debt to my supervisor Gordon Plotkin for having taught me how to do research with his example. Discussing my ideas with him was always —and still is— a pleasure: his suggestions are helpful, and his comments and questions are insightful. This thesis would not have been possible without his stimulating guidance.

I had two second supervisors Barry Jay and John Power from whom I learnt a great deal of category theory and to whom I am grateful for their support and involvement in my work.

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A mis padres Cristina y Luis les estoy agradecido porque siempre confiaron en mi, porque siempre me apoyaron en todo y porque les debo todo lo que soy. A mis hermanos Verónica y Alejandro les agradezco su invaluable amistad.

Declaration

This thesis was composed by myself. The work reported herein, unless otherwise stated, is my own.

Edinburgh, June 1994 M.P.F.

Note Added in Print

This book is the author's Ph.D. thesis. Since this work was completed progress in various directions has been made but these new developments have not been incorporated in the text. For a recent overview of the subject, including directions of research, the reader is referred to the expository article [FJM+96, § Axiomatic Domain Theory].

Edinburgh, March 1996 M.P.F.

"Another word about Category Theory: I actually feel that it is particularly significant and important for the theory and for the whole area of semantics. But it must be approached with great caution, for the sheer number of definitions *and* axioms can try the most patient reader. It seems to me to be especially necessary in discussing applications of abstract mathematical ideas to keep the motivation strongly in mind. This is often hard to do if the categories get too thick, but of course it all depends on the writer. Category Theory is especially useful in stating *general* properties of structures and in *characterizing* constructions uniquely; however, there often is a problem actually justifying the *existence* of certain constructions, and a direct approach can be quicker than quoting lots of theorems. But, man cannot live by construction alone: theorems have to be proved in order to get the proper value out of the work. Domain Theory must also be convenient for demonstrating the soundness of various proof rules for properties of recursively defined objects and recursively defined domains, and I think that Category Theory can be helpful here. A step in the right direction has been made in the LCF system (see [GMW79]), which, however, does not take advantage of general Category Theory; but the whole area needs much more development in my opinion."

Dana Scott[1]

[1] In [Sco82].