INTRODUCTION TO BANACH ALGEBRAS, OPERATORS, AND HARMONIC ANALYSIS

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Definitions and examples

1.1 Definitions

A Banach algebra is first of all an algebra. We start with an algebra A and put a topology on A to make the algebraic operations continuous – in fact, the topology is given by a norm.

Definition 1.1.1 *Let E be a linear space. A* norm *on E is a map $\|\cdot\| : E \to \mathbb{R}$ such that:*

(i) $\|x\| \geq 0 \ \ (x \in E)$*;* $\|x\| = 0$ *if and only if* $x = 0$*;*
(ii) $\|\alpha x\| = |\alpha| \, \|x\| \quad (\alpha \in \mathbb{C}, \ x \in E)$;
(iii) $\|x + y\| \leq \|x\| + \|y\| \quad (x, y \in E)$.

Then $(E, \|\cdot\|)$ is a normed space. *It is a* Banach space *if every Cauchy sequence converges, i.e., if $\|\cdot\|$ is complete.*

Definition 1.1.2 *Let A be an algebra. An* algebra norm *on A is a map $\|\cdot\| : A \to \mathbb{R}$ such that $(A, \|\cdot\|)$ is a normed space, and, further:*

(iv) $\|ab\| \leq \|a\| \, \|b\| \quad (a, b \in A)$.

The normed algebra $(A, \|\cdot\|)$ is a Banach algebra *if $\|\cdot\|$ is a complete norm.*

In Chapters 1–7, we shall usually suppose that a Banach algebra A is *unital*: this means that A has an identity e_A and that $\|e_A\| = 1$. Let A be a Banach algebra with identity. Then, by moving to an equivalent norm, we may suppose that A is unital. It is easy to check that, for each normed algebra A, the map $(a, b) \mapsto ab, \quad A \times A \to A$, is continuous.

H. G. Dales, P. Aiena, J. Eschmeier, K. B. Laursen, and G. A. Willis, *Introduction to Banach Algebras, Operators, and Harmonic Analysis*. Published by Cambridge University Press.

1.2 Examples

Let us give some elementary examples.

(i) Let S be any non-empty set. Then $\mathbb{C}^S$ is the set of functions from S into $\mathbb{C}$. Define *pointwise* algebraic operations by

$$(\alpha f + \beta g)(s) = \alpha f(s) + \beta g(s)\,,$$
$$(fg)(s) = f(s)g(s)\,,$$
$$1(s) = 1\,,$$

for each $s \in S$, each $f, g \in \mathbb{C}^S$, and each $\alpha, \beta \in \mathbb{C}$. Then $\mathbb{C}^S$ is a commutative, unital algebra. We write $\ell^\infty(S)$ for the subset of bounded functions on S, and define the *uniform norm* $|\cdot|_S$ on S by

$$|f|_S = \sup\{|f(s)| : s \in S\} \quad (f \in \ell^\infty(S))\,.$$

Check that $(\ell^\infty(S), |\cdot|_S)$ is a unital Banach algebra.

(ii) Let X be a topological space (e.g., think of $X = \mathbb{R}$). We write $C(X)$ for the algebra of all continuous functions on X, and $C^b(X)$ for the algebra of bounded, continuous functions on X. *Check* that $(C^b(X), |\cdot|_X)$ is a unital Banach algebra.

Now take Ω to be a compact space (e.g., $\Omega = \mathbb{I} = [0, 1]$). Then we have $C^b(\Omega) = C(\Omega)$, and so $(C(\Omega), |\cdot|_\Omega)$ is a unital Banach algebra. This is a very important example.

(iii) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc. The *disc algebra* is

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}\,.$$

Check that $A(\overline{\mathbb{D}})$ is a unital Banach algebra. (You just have to show that $A(\overline{\mathbb{D}})$ is closed in $C(\overline{\mathbb{D}})$: why is this?)

Each $f \in A(\overline{\mathbb{D}})$ has a Taylor expansion about the origin:

$$f = \sum_{n=0}^{\infty} \alpha_n Z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)Z^n}{n!}\,.$$

Here Z is the *coordinate functional*, so that $Z : z \mapsto z$ on $\mathbb{C}$. Some functions in $A(\overline{\mathbb{D}})$ have the further property that

$$\sum_{n=0}^{\infty} |\alpha_n| < \infty\,.$$

(Are there any functions f in $A(\overline{\mathbb{D}})$ without this property?) The subset of functions with this extra property is called $A^+(\overline{\mathbb{D}})$. *Check* that $A^+(\overline{\mathbb{D}})$ is a unital Banach algebra for the norm $\|\cdot\|_1$, where

$$\|f\|_1 = \sum_{n=0}^{\infty} |\alpha_n| \quad \left(f = \sum_{n=0}^{\infty} \alpha_n Z^n \right).$$

(iv) Let X be a compact set in the space $\mathbb{C}^n$. Then $P(X)$ is the family of functions that are the uniform limits on X of the restrictions to X of the polynomials (in n-variables). *Check* that $(P(X), |\cdot|_X)$ is a unital Banach algebra. In fact, $A(\overline{\mathbb{D}}) = P(\overline{\mathbb{D}})$. We shall also be interested in $P(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle.

(v) Let X be a compact set in the complex plane $\mathbb{C}$ (or in $\mathbb{C}^n$). Then $A(X)$ is the closed subalgebra of $(C(X), |\cdot|_X)$ consisting of the functions which are analytic on the interior of X, int X. Clearly $A(X) = C(X)$ if and only if int $X = \emptyset$. Also $R(X)$ is the family of functions on X which are the uniform limits on X of the restrictions to X of the rational functions: these are functions of the form p/q, where p and q are polynomials and $0 \notin q(X)$. Clearly we have

$$P(X) \subset R(X) \subset A(X) \subset C(X).$$

The question of the equality of various of these algebras encapsulates much of the classical theory of approximation.

(vi) Let $n \in \mathbb{N}$. Then $C^{(n)}(\mathbb{I})$ consists of the functions f on $\mathbb{I}$ such that f has n derivatives on $\mathbb{I}$ and $f^{(n)} \in C(\mathbb{I})$. *Check* that $C^{(n)}(\mathbb{I})$ is a Banach algebra for the pointwise operations and the norm

$$\|f\|_n = \sum_{k=0}^{n} \frac{1}{k!} |f^{(k)}|_{\mathbb{I}} \quad (f \in C^{(n)}(\mathbb{I})).$$

(vii) Let E and F be linear spaces. Then $\mathcal{L}(E, F)$ is the collection of all linear maps from E to F; it is itself a linear space for the standard operations.

Now let E and F be Banach spaces. Then $\mathcal{B}(E, F)$ is the family of all bounded (i.e., continuous) linear operators from E to F; it is a subspace of $\mathcal{L}(E, F)$ and $\mathcal{B}(E, F)$ is itself a Banach space for the operator norm given by

$$\|T\| = \sup\{\|Tx\| : x \in E, \ \|x\| \le 1\}.$$

We write $\mathcal{L}(E)$ and $\mathcal{B}(E)$ for $\mathcal{L}(E, E)$ and $\mathcal{B}(E, E)$, respectively. The product of two operators S and T in $\mathcal{L}(E)$ is given by composition:

$$(ST)(x) = (S \circ T)(x) = S(Tx) \quad (x \in E).$$

Then trivially $\|ST\| \le \|S\|\,\|T\|$ $(S, T \in \mathcal{B}(E))$, and $(\mathcal{B}(E), \|\cdot\|)$ is a unital Banach algebra; the identity of $\mathcal{B}(E)$ is the identity operator I_E. This is our first non-commutative example.

For example, let E be the finite-dimensional space $\mathbb{C}^n$ (say with the Euclidean norm $\|\cdot\|_2$). Then $\mathcal{L}(E) = \mathcal{B}(E)$ is just the algebra $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$ (with the usual identifications).

(viii) The algebra $\mathbb{C}[[X]]$ of *formal power series in one variable* consists of sequences

$$(\alpha_n : n = 0, 1, 2, \dots),$$

where $\alpha_n \in \mathbb{C}$, with coordinatewise linear operations and the product

$$(\alpha_r)(\beta_s) = (\gamma_n),$$

where $\gamma_n = \sum_{r+s=n} \alpha_r \beta_s$. It helps to think of elements of $\mathbb{C}[[X]]$ as formal series of the form

$$\sum_{n=0}^{\infty} \alpha_n X^n,$$

with the product suggested by the symbolism. This algebra contains as a subalgebra the algebra $\mathbb{C}[X]$ of polynomials in one variable – these polynomials correspond to the sequences (α_n) that are eventually zero.

A *weight on* $\mathbb{Z}^+$ is a function $\omega : \mathbb{Z}^+ \to \mathbb{R}^+ \setminus \{0\}$ such that $\omega(0) = 1$ and

$$\omega(m+n) \le \omega(m)\omega(n) \quad (m, n \in \mathbb{Z}^+).$$

Check that $\omega_n = e^{-n}$ and $\omega_n = e^{-n^2}$ define weights on $\mathbb{Z}^+$. For such a weight ω, define

$$\ell^1(\omega) = \left\{ (\alpha_n) \in \mathbb{C}[[X]] : \|\alpha\|_\omega = \sum_{n=0}^{\infty} |\alpha_n|\,\omega_n < \infty \right\}.$$

Check that $\ell^1(\omega)$ is a subalgebra of $\mathbb{C}[[X]]$, and that $(\ell^1(\omega), \|\cdot\|_\omega)$ is a commutative, unital Banach algebra.

(viii) Let G be a group, and let

$$\ell^1(G) = \left\{ f \in \mathbb{C}^G : \|f\|_1 = \sum_{s \in G} |f(s)| < \infty \right\}.$$

Then $(\ell^1(G), \|\cdot\|_1)$ is a Banach space. We can think of an element of $\ell^1(G)$ as

$$\sum_{s\in G} \alpha_s \delta_s ,$$

where $\sum |\alpha_s| < \infty$; here $\delta_s(s) = 1$ and $\delta_s(t) = 0 \ (t \neq s)$.

We define a product on $\ell^1(G)$ that is not the pointwise product; it is denoted by $\star$ and is sometimes called *convolution multiplication*. In this multiplication,

$$\delta_s \star \delta_t = \delta_{st} \quad (s, t \in G) ,$$

where st is the product in G. (Actually this formula defines the product.) Thus

$$(f \star g)(t) = \sum \{ f(r)g(s) : rs = t \} \quad (t \in G) . \tag{1.2.1}$$

Check that $\ell^1(G)$ is a unital Banach algebra for this product and the norm $\|\cdot\|_1$. It is commutative if and only if G is an abelian group. Special case: take $G = \mathbb{Z}$, a group with respect to addition.

(ix) (Strictly, this example needs the theory of the Lebesgue integral on $\mathbb{R}$.) The Banach space $L^1(\mathbb{R})$ has the norm $\|\cdot\|_1$ given by

$$\| f \|_1 = \int_{-\infty}^{\infty} | f(t)| \,\mathrm{d}t .$$

For functions $f, g \in L^1(\mathbb{R})$, define their convolution product $f \star g$ by

$$(f \star g)(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\mathrm{d}s \quad (t \in \mathbb{R}) .$$

Integration theory shows that $f \star g$ is defined almost everywhere (a.e.) and that $f \star g$ gives an element of $L^1(\mathbb{R})$; further, $\| f \star g\|_1 \leq \| f\|_1 \|g\|_1$, and so we obtain a commutative Banach algebra (which does not have an identity).

This example is central to the theory of Fourier transforms.

(x) Let U be a non-empty, open set in $\mathbb{C}$ (or in $\mathbb{C}^n$). Then $H(U)$ denotes the set of analytic (or holomorphic) functions on U. Clearly $H(U)$ is an algebra for the pointwise operations. However the algebra $H(U)$ is *not* a Banach algebra. For each compact subset K of U, define

$$p_K(f) = | f |_K \quad (f \in H(U)) .$$

Then each p_K is an algebra seminorm on $H(U)$. The space $H(U)$ is a Fréchet space with respect to the family of these seminorms; in this topology, $f_n \to f$ if and only if (f_n) converges to f uniformly on compact subsets of U. The algebra is a *Fréchet algebra* because $p_K(fg) \leq p_K(f)p_K(g)$ in each case.

A related algebra is $H^\infty(U)$, the algebra of bounded analytic functions on U. *Check* that this algebra is a Banach algebra with respect to the uniform norm $|\cdot|_U$.

1.3 Philosophy of why we study Banach algebras

There are several reasons why we study Banach algebras. They:

- cover many examples;
- have an abstract approach that leads to clear, quick proofs and new insights;
- blend algebra and analysis;
- have beautiful results on intrinsic structure.

1.4 Basic properties

We begin our study of general Banach algebras by considering invertible elements in such algebras.

Definition 1.4.1 *Let A be a unital algebra. An element $a \in A$ is* invertible *if there exists an element $b \in A$ with $ab = ba = e_A$. The element b is unique; it is called the* inverse *of a, and written a^{-1}. The set of invertible elements of A is denoted by* $\operatorname{Inv} A$.

Check that $a, b \in \operatorname{Inv} A \Rightarrow ab \in \operatorname{Inv} A$ and $(ab)^{-1} = b^{-1}a^{-1}$.

Now let $(A, \|\cdot\|)$ be a unital Banach algebra. *Check* that, for each $a \in A$, we have

$$\lim_{n\to\infty} \|a^n\|^{1/n} = \inf\{\|a^n\|^{1/n} : n \in \mathbb{N}\} \le \|a\| .$$

Theorem 1.4.2 *Let $(A, \|\cdot\|)$ be a unital Banach algebra.*

(i) *Suppose that $a \in A$ and* $\lim \|a^n\|^{1/n} < 1$. *Then* $e_A - a \in \operatorname{Inv} A$.
(ii) $\operatorname{Inv} A \supset \{b \in A : \|e_A - b\| < 1\}$.
(iii) $\operatorname{Inv} A$ *is an open subset of A.*
(iv) *The map* $a \mapsto a^{-1}$, $\operatorname{Inv} A \to \operatorname{Inv} A$, *is continuous.*

Proof (i) The series $e_A + \sum_{n=1}^{\infty} a^n$ converges to $(e_A - a)^{-1}$.

(ii) This is immediate from (i).

(iii) Take $a \in \operatorname{Inv} A$, and then take $b \in A$ with $\|b\| < \|a^{-1}\|^{-1}$. Note that $a - b = a(e_A - a^{-1}b)$ and $\|a^{-1}b\| < 1$. By (i), $e_A - a^{-1}b \in \operatorname{Inv} A$. Hence $a - b \in \operatorname{Inv} A$.

(iv) Exercise: use the inequality that

$$\|b^{-1} - a^{-1}\| \leq 2\|a^{-1}\|^2\|b - a\|$$

whenever $a, b \in \operatorname{Inv} A$ with $\|b - a\| \leq 1/2\|a^{-1}\|$. □

1.5 Exercises

1. Check the details of the examples.

2. Prove Theorem 1.4.2(iv).

3. Identify $\operatorname{Inv} A$ for as many as possible of the examples A given in §1.2. (Easy for $A = C(\Omega)$, $A = A(\overline{\mathbb{D}})$, $A = H(U)$, $A = \mathcal{B}(E)$; harder for the algebra $A = A^+(\overline{\mathbb{D}})$; not possible in general for $\ell^1(G)$.) Show that $\operatorname{Inv} \mathbb{C}[[X]] = \{(\alpha_n) : \alpha_0 \neq 0\}$.

4. For $f \in L^1(\mathbb{T})$ (in particular for $f \in C(\mathbb{T})$), the *Fourier coefficients* are

$$\widehat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{\mathrm{i}\theta}) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta \quad (k \in \mathbb{Z}).$$

Let $s_n(\theta) = \sum_{-n}^{+n} \widehat{f}(k) e^{\mathrm{i}k\theta}$ and set

$$\sigma_n(f) = \frac{1}{n+1}(s_0 + \cdots + s_n).$$

Then *Féjer's theorem* says that: for each $f \in C(\mathbb{T})$, $\sigma_n(f) \to f$ uniformly on $\mathbb{T}$.

Deduce that the following are equivalent for $f \in C(\mathbb{T})$:

(a) $f \in P(\mathbb{T})$;
(b) $f = F \mid \mathbb{T}$ for some $F \in A(\overline{\mathbb{D}})$;
(c) $\widehat{f}(-k) = 0 \ (k \in \mathbb{N})$.

We can now identify $A(\overline{\mathbb{D}})$ with $P(\mathbb{T})$ (why?), and regard $A(\overline{\mathbb{D}})$ as a closed subalgebra of $C(\mathbb{T})$ – if we should wish to do this!

5. Let

$$W(\mathbb{T}) = \{f \in C(\mathbb{T}) : \|f\|_1 = \sum_{k=-\infty}^{\infty} |\widehat{f}(k)| < \infty\}.$$

Check that $(W(\mathbb{T}), \|\cdot\|_1)$ is a commutative, unital Banach algebra (for the pointwise operations). *Check* that the map

$$\sum_{n=-\infty}^{\infty} \alpha_n \delta_n \mapsto \sum_{n=-\infty}^{\infty} \alpha_n Z^n, \quad \ell^1(\mathbb{Z}) \to W(\mathbb{T}),$$

is an isometric isomorphism. (W stands for N. Wiener, who was the first to study these algebras.)

6. Let $L^1(\mathbb{I})$ be the Banach space of (equivalence classes of) integrable functions on $\mathbb{I}$, with the norm

$$\| f \|_1 = \int_0^1 |f(t)| \, \mathrm{d}t \quad (f \in L^1(\mathbb{I})) .$$

For $f, g \in L^1(\mathbb{I})$, define $f \star g$ by

$$(f \star g)(t) = \int_0^t f(t-s)g(s) \, \mathrm{d}s \quad (t \in \mathbb{I}) .$$

Show that $L^1(\mathbb{I})$ is a Banach algebra for this product. It is called the *Volterra algebra*, and is denoted by $\mathcal{V}$.

Set $u(t) = 1 \ (t \in \mathbb{I})$, so that

$$(u \star f)(t) = \int_0^t f(s) \, \mathrm{d}s .$$

Calculate $u^{\star n}$ and $\|u^{\star n}\|_1$, where $u^{\star n}$ denotes the nth power of u in the algebra $\mathcal{V}$. The map $V : f \mapsto u \star f$ on $L^1(\mathbb{I})$ is the *Volterra operator*, discussed in later chapters.

1.6 Additional notes

1. By an *algebra* A, we always mean a linear space over $\mathbb{C}$ together with a multiplication such that $a(bc) = (ab)c$, $a(b+c) = ab + ac$, $(a+b)c = ac + bc$, and $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for $a, b, c \in A$ and $\alpha \in \mathbb{C}$. The algebra has an *identity* e_A if $e_A a = a e_A = a \ (a \in A)$. Suppose that A does not have an identity. Then $A^\# = \mathbb{C} \odot A$ is a unital algebra for the product

$$(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab) \quad (\alpha, \beta \in \mathbb{C}, \ a, b \in A) ;$$

if A is a Banach algebra, then so is $A^\#$ for the norm $\|(\alpha, a)\| = |\alpha| + \|a\|$.

2. For $f \in \mathbb{C}^S$, define $\overline{f}(s) = \overline{f(s)}$, the complex conjugate of $f(s)$. Then the map $f \mapsto \overline{f}$ is an involution on $\mathbb{C}^S$ and on $C(\Omega)$. *Check* that $|f|_\Omega^2 = |f\overline{f}|_\Omega$ in the latter case. The algebra $C(\Omega)$ with this involution is the canonical example of a commutative, unital C^*-algebra; see §3.5.

3. Let Ω be a locally compact space (e.g., $\mathbb{R}$). For a continuous function f on Ω, supp f, the *support* of f, is the closure of the set $\{x \in \Omega : f(x) \neq 0\}$. We write $C_{00}(\Omega)$ for the algebra of functions of compact support, and $C_0(\Omega)$ for the algebra of functions f that *vanish at infinity*, i.e., $\{x \in \Omega : |f(x)| \geq \varepsilon\}$ is compact for each $\varepsilon > 0$. *Check* that $(C_0(\Omega), |\cdot|_\Omega)$ is a Banach algebra. Is $(C_{00}(\Omega), |\cdot|_\Omega)$ also a Banach algebra? Is it dense in $(C_0(\Omega), |\cdot|_\Omega)$?

4. A closed, unital subalgebra A of an algebra $(C(\Omega), |\cdot|_\Omega)$ such that, for each $x, y \in \Omega$ with $x \neq y$ there exists $f \in A$ with $f(x) \neq f(y)$, is called a *uniform algebra*.

5. In the text, we defined $\ell^1(G)$ for a group G. *Check* that the construction (with the product being defined in (1.2.1)) also works for a semigroup S instead of G – save that $\ell^1(S)$ is unital only if S has an identity.

6. There is a common generalization of $L^1(\mathbb{R})$ and $\ell^1(G)$. Each locally compact group G has a *left Haar measure* m, and $L^1(G)$, consisting of measurable functions f on G with

$$\|f\|_1 = \int_G |f(t)|\,\mathrm{d}m(t) < \infty\,,$$

becomes a Banach algebra for the product

$$(f \star g)(t) = \int_G f(s)g(s^{-1}t)\,\mathrm{d}m(s)\,.$$

This is the *group algebra* of G. Note that G need not be abelian. See Part II

7. There is no norm $\|\cdot\|$ on $H(U)$ such that $(H(U), \|\cdot\|)$ is a Banach algebra: see Dales (2000, 5.2.33(ii)).

8. Most of the above is in Rudin (1973, 10.1–10.7) and Rudin (1996, 18.1–18.4). For uniform algebras, including the disc algebra $A(\overline{\mathbb{D}})$, see Gamelin (1969). The disc algebra is utilized in Part III, Theorem 14.12. All the examples are given in substantial detail in Dales (2000). See, for example, Dales (2000, §2.1). Uniform algebras and group algebras are discussed in §4.3 and §3.3 of Dales (2000), respectively. The group algebras $L^1(G)$ are a main topic of Part II of this book; for the related measure algebra $M(G)$, see Proposition 9.1.2. For the theory of topological algebras, including Fréchet algebras, see Dales (2000, §2.2).