Chapter One

Introduction

Consider a system of k scalar equations in the form

$$F(\lambda, x) = 0 \in \mathbb{F}^k, \tag{1.1}$$

where $x \in \mathbb{F}^n$ represents the state of a system and $\lambda \in \mathbb{F}^m$ is a vector parameter which controls x. (Here $\mathbb{F}$ denotes the real or complex field.) A solution of (1.1) is a pair $(\lambda, x) \in \mathbb{F}^m \times \mathbb{F}^n$ and the goal is to say as much as possible qualitatively about the solution set.

Since (1.1) is a finite-dimensional nonlinear equation it might seem unnecessarily restrictive or even pointless to distinguish between the λ and x variables. Why not instead write $(\lambda, x) = Z \in \mathbb{F}^{m+n}$ and study the equation $F(Z) = 0$ where singularity theory is all that is needed? For example, when $F : \mathbb{C}^{m+n} \to \mathbb{C}^k$ is given by a power series expansion (that is, F is analytic), a solution Z_0 is called a bifurcation point if, in every neighbourhood of Z_0, the solutions of $F(Z) = 0$ do not form a smooth manifold. Locally the solutions form an analytic variety, a finite union of analytic manifolds of possibly different dimensions. So the qualitative theory of $F(Z) = 0$ in complex finite dimensions is reasonably complete.

However (i) in our applications λ is a parameter and the dependence on λ of the solution set is important; (ii) we are looking for a theory that gives the existence globally (i.e. not only in a neighbourhood of a point) of connected sets of solutions; (iii) we are particularly interested in the infinite-dimensional equation

$$F(\lambda, x) = 0 \tag{1.2}$$

when X and Y are real Banach spaces, $F : \mathbb{R} \times X \to Y$ is real-analytic and

$$F(\lambda, 0) \equiv 0.$$

Let

$$\mathcal{S}_\lambda = \{x \in X : F(\lambda, x) = 0\}.$$

The set $\mathcal{S}_\lambda$ normally depends on the choice of λ and usually varies continuously as λ varies. However, it sometimes happens that there is an abrupt change, a bifurcation, in the solution set, as λ passes through a particular point λ_0. For example, in Figure 1.1 the number of solutions changes from one to two as λ increases through λ_0. For a general treatment of bifurcation theory, see [19].

At this stage it is useful to see an infinite-dimensional example in which the global solution set can be found explicitly.

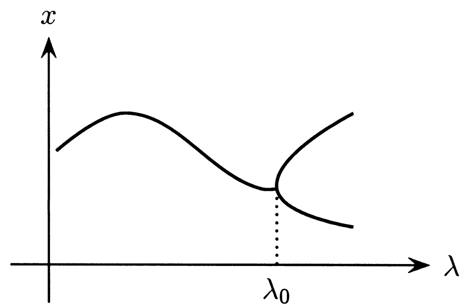

Figure 1.1 The set S_λ splits in two as λ passes through λ_0.

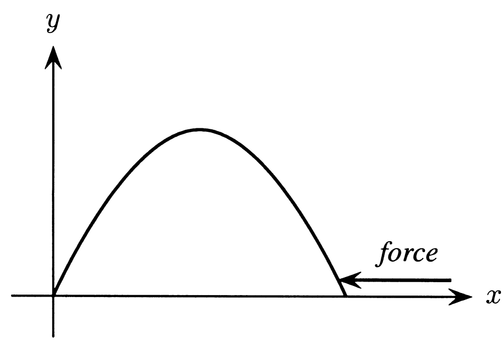

Figure 1.2 The rod bends under the action of a force.

1.1 EXAMPLE: BENDING AN ELASTIC ROD I

Consider an elastic rod of length $L > 0$ with one end fixed at the origin of the (x, y)-plane and with the other free to move on the x-axis under the influence of a force along the x-axis towards the origin. If we suppose that the length of the rod does not change (that it is incompressible) and if the force is big enough, then the rod will bend (see Figure 1.2).

We suppose that the rod always lies in the (x, y)-plane (there is no twisting out of the plane in the simple model which follows). To describe the rod's configuration let $(x(s), y(s))$ be the coordinates of a point at distance s (measured along the rod) from the end which is fixed at the origin. Since

$$x(s) = \int_0^s \cos\phi(t)dt \text{ and } y(s) = \int_0^s \sin\phi(t)dt,$$

the shape of the rod is given by the angle $\phi(s)$ between the tangent to the rod and the horizontal at the point $(x(s), y(s)),\ s \in [0, L]$.

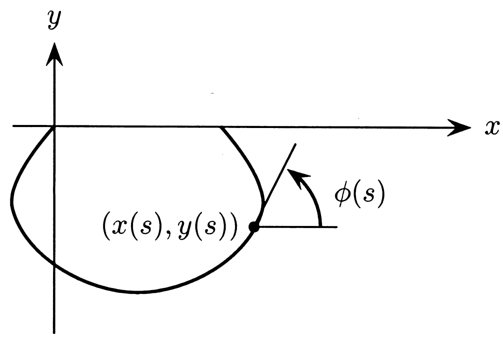

Figure 1.3 The angle between the tangent and the horizontal.

Let P denote the applied force. Then the Euler-Bernoulli theory [5, 6] of bending says that the curvature of the rod at a point is proportional to the moment created by the force. In other words,

$$-k\phi'(s) = Py(s),$$

where k, the constant of proportionality, is determined by the material properties of the rod, $Py(s)$ is the moment of the applied force and $-\phi'(s)$ is the curvature at the point $(x(s), y(s))$. It follows that if $P = 0$ then ϕ must be constant, and that constant must be 0 (mod 2π) since $y(L) = 0$. From now on we consider only the case $P > 0$. Since $y'(s) = \sin\phi(s)$ and $y(0) = y(L) = 0$ this gives

$$\phi''(s) + \lambda \sin\phi(s) = 0,\ s \in [0, L], \quad \phi'(0) = \phi'(L) = 0, \tag{1.3}$$

where $\lambda = P/k > 0$. If ϕ is a solution of (1.3), then so is $2k\pi + \phi$, for any $k \in \mathbb{Z}$. We therefore assume that $\phi(0) \in (-\pi, \pi)$. (If $\phi(0) = \pm\pi$ then ϕ is a constant.)

For all $\lambda > 0$, $(\lambda, \phi) = (\lambda, 0)$ is a solution of (1.3). This means that the mathematical model of bending admits a solution representing a *straight* rod, irrespective of how large the applied force might be. These solutions, $\phi = 0$, $\lambda > 0$ arbitrary, comprise the family of *trivial solutions*. To be realistic the model must also have solution corresponding to a bent rod (such as depicted in Figures 1.2 and 1.3). Note that any solution of (1.3) must satisfy the identity

$$\phi'(s)^2 + 4\lambda \sin^2(\tfrac{1}{2}\phi(s)) = 4\lambda \sin^2(\tfrac{1}{2}\phi_0), \quad s \in [0, L], \tag{1.4}$$

where $\phi_0 = \phi(0)$. This means that $(\phi(s), \phi'(s))$, $s \in [0, L]$, lies on a segment of the curve in (ϕ, ϕ')-phase space (see Figure 1.4) given implicitly by

$$\left\{(\phi, \phi') \in \mathbb{R}^2 : \phi'^2 + 4\lambda \sin^2 \tfrac{1}{2}\phi = 4\lambda \sin^2 \tfrac{1}{2}\phi_0\right\} \subset \mathbb{R}^2.$$

We therefore see that there is a solution joining $(-|\phi_0|, 0)$ to $(|\phi_0|, 0)$ in the half-space $\{(\phi, \phi') \in \mathbb{R}^2,\ \phi' \geq 0\}$ and one joining $(|\phi_0|, 0)$ to $(-|\phi_0|, 0)$ in the

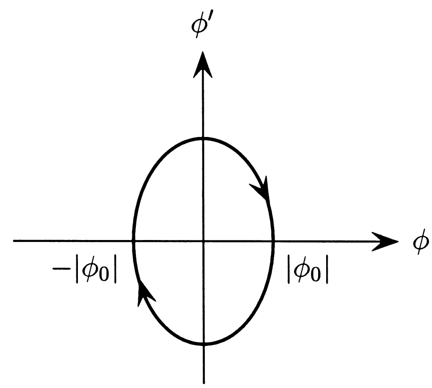

Figure 1.4 The direction of solutions in phase space.

half-space $\{(\phi, \phi') \in \mathbb{R}^2,\ \phi' \leq 0\}$, the corresponding value of L being given by the formula

$$L = \int_0^L \frac{|d\phi|}{|d\phi|/ds} = \int_{-|\phi_0|}^{|\phi_0|} \frac{d\phi}{\sqrt{4\lambda \sin^2 \frac{1}{2}\phi_0 - 4\lambda \sin^2 \frac{1}{2}\phi}}$$
$$= \frac{1}{\sqrt{\lambda}} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2 \frac{1}{2}|\phi_0| \sin^2 \theta}},$$

where $\theta \in [-\pi/2, \pi/2]$ is given by $\sin(\phi/2) = \sin(|\phi_0|/2) \sin\theta$. In fact there are other solutions of (1.4) which in Figure 1.4 go around the curve $\frac{1}{2}K$ times for any positive integer K. For such solutions

$$L = \frac{K}{\sqrt{\lambda}} \int_{-\pi/2}^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2 \frac{1}{2}|\phi_0| \sin^2 \theta}}.$$

This integral increases in $|\phi_0|$ and converges to $+\infty$ as $|\phi_0| \to \pi$.

Since L is the given length of the rod, this relation for each K is an implicit relation between $\phi_0 = \phi(0)$ and λ when (λ, ϕ) is a solution of (1.3). We can best describe the situation with the aid of a bifurcation diagram in which λ is the horizontal axis, ϕ_0 is the vertical axis, and L is fixed, see Figure 1.5.

The different curves correspond to different values of K, and it is easily checked that the K^{th} curve intersects the horizontal axis at $(K\pi/L)^2$.

It is fortunate but unusual that (1.3) can be reduced to (1.4) and that L can be calculated in terms of elliptic integrals. Because of this, solutions to (1.3) of all amplitudes can be found more-or-less explicitly. This is not the case for slightly more complicated problems and almost never for partial differential equations (PDEs).

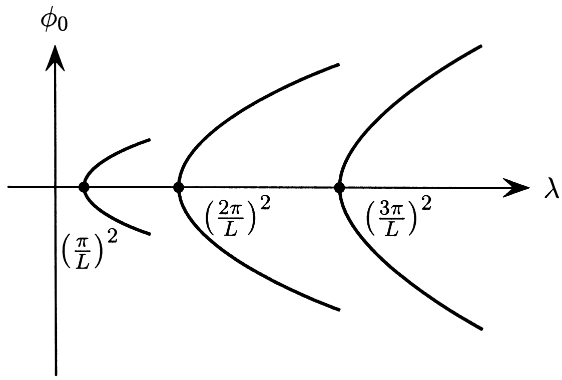

Figure 1.5 Bifurcation diagram.

General methods suitable for PDE applications are based on a study of (1.2). To put (1.3) in such a setting let

$$X = \{\phi \in C^2[0, L] : \phi'(0) = \phi'(L) = 0\},$$
$$Y = C[0, L] \text{ and } F(\lambda, \phi) = \phi'' + \lambda \sin \phi \in Y,$$

for all $(\lambda, \phi) \in \mathbb{R} \times X$. Then $F : \mathbb{R} \times X \to Y$ is smooth (Chapter 3) and real-analytic (Chapter 4). In Chapter 8 we show how equation (1.2) can be reduced locally to a finite-dimensional problem. This means that if (λ_0, x_0) satisfies (1.2) then there is a neighbourhood U of (λ_0, x_0) in $\mathbb{R} \times X$, a neighbourhood V of $(\lambda_0, 0) \in \mathbb{R} \times \mathbb{R}^N$ and an equation

$$f(\lambda, z) = 0 \in \mathbb{R}^M, \ (\lambda, z) \in \mathbb{R} \times \mathbb{R}^N, \ \ N, \ M \in \mathbb{N},$$

such that the solutions of the two equations are in one-to-one correspondence. The reduction to finite dimensions in §8.2 is called Lyapunov-Schmidt reduction and leads immediately to a local bifurcation theory based on the implicit function theorem. In particular, it yields a classical relation between a nonlinear problem and its linearization.

1.2 PRINCIPLE OF LINEARIZATION

Roughly speaking, the principle of linearization [39] derives from the feeling that when $F(\lambda, 0) = 0$ for all λ and solutions with $\|x\|$ small are sought, the nonlinear problem $F(\lambda, x) = 0$ might as well be replaced with the linear equation $\partial_x F[(\lambda, 0)]x = 0$, where $\partial_x F[(\lambda, 0)]$ denotes the linearization of F with respect to x at $x = 0$. Since $\sin \phi = \phi + O(|\phi|^3)$ as $\phi \to 0$, the linearization of the elastic-rod problem at $(\lambda_0, 0)$ is

$$\phi''(s) + \lambda_0 \phi(s) = 0, \ \ s \in [0, L], \quad \phi'(0) = \phi'(L) = 0, \ \ \lambda_0 > 0,$$

and this problem has non-trivial solutions if and only if

$$\lambda_0 = (K\pi/L)^2 \text{ with } \phi(s) = \cos(K\pi s/L), \quad K \in \mathbb{N}.$$

The question is, can any inference be drawn from this about the nonlinear problem (1.3)? The answer is that in quite general situations (including equation (1.3) as a special case) λ_0 is a bifurcation point on the line of trivial solutions of (1.2) only if the linearized problem $d_x F[(\lambda_0, 0)]x = 0$ has a non-trivial solution. The fact that this is also sufficient for bifurcation from the line of trivial solutions for (1.3) (but not in general) is a consequence of the theory of bifurcation from a simple eigenvalue, see §8.4 and §8.5.

1.3 GLOBAL THEORY

It is clear from Figure 1.5 that there is more to the solution set of equation (1.3) than is predicted by local theory. Global features of the diagram are not a consequence of finite-dimensional reduction methods alone. We will see in Chapter 9 how local bifurcation theory, the implicit function theorem and some elementary results on real-analytic varieties can be used to piece together a global picture of the solution set of (1.2), without assumptions about the size of the solutions under consideration. Provided some general functional-analytic structure is present and F is real-analytic, the global continuum $\mathcal{C}$ of solutions which bifurcates from the trivial solutions at a simple eigenvalue contains a continuous curve $\mathfrak{R}$ with the following properties.

- $\mathfrak{R} = \{(\Lambda(s), \kappa(s)) : s \in [0, \infty)\} \subset \mathcal{C}$ is either unbounded or forms a closed loop in $\mathbb{R} \times X$.
- For each $s^* \in (0, \infty)$ there exists $\rho^* : (-1, 1) \to \mathbb{R}$ (a re-parameterization) which is continuous, injective, and

  $$\rho^*(0) = s^*, \quad t \mapsto (\Lambda(\rho^*(t)), \kappa(\rho^*(t)), \quad t \in (-1, 1), \text{ is analytic.}$$

  This does not imply that $\mathfrak{R}$ is locally a smooth curve. (The map $\sigma : (-1, 1) \to \mathbb{R}^2$ given by $\sigma(t) = (t^2, t^3)$ is real-analytic and its image is a curve with a cusp.) Nor does it preclude the possibility of secondary bifurcation points on $\mathfrak{R}$. In particular, since $(\Lambda, \kappa) : [0, \infty) \to \mathbb{R} \times X$ is not required to be globally injective; self-intersection of $\mathfrak{R}$ (as in a figure eight) is not ruled out.
- Secondary bifurcation points on the bifurcating branch, if any, are isolated.

See Theorem 9.1.1 for a complete statement and §9.3 for an application to the elastic-rod problem. This result about real-analytic global bifurcation from a simple eigenvalue is a sharpened version of a theorem due to Dancer. His general results [24, 26] deal with bifurcation from eigenvalues of higher multiplicity and give the path-connectedness of solutions sets that are not essentially one-dimensional. Since his hypotheses are less restrictive, his conclusions are necessarily somewhat less precise. The topological theory of global bifurcation without analyticity assumptions was developed slightly earlier, first for nonlinear Sturm-Liouville problems (such as (1.3)) by Crandall & Rabinowitz [21], then for partial differential equations by Rabinowitz [50], and for problems with a positivity structure by Dancer [25] and Turner [64]. Their basic tool was an infinite-dimensional topological degree function and the outcome was the existence of a global connected (but not always path-connected) set of solutions. Although it is sometimes possible to argue from the implicit function theorem that the connected set given by topological methods is a smooth curve in $\mathbb{R} \times X$, this approach fails if there is a secondary bifurcation point on the bifurcating branch.

What is important is that in the analytic case a one-dimensional branch can be followed unambiguously through a secondary bifurcation point. In fact a one-dimensional branch is uniquely determined globally by its behaviour in an open set and can be parameterized globally, even when it intersects manifolds of solutions of different dimensions (see §7.5).

1.4 LAYOUT

We begin in Chapter 2 with a review, without proofs, of the linear functional analysis needed for nonlinear theory. Chapter 3 introduces the main results from nonlinear analysis, including the inverse and implicit function theorems for functions of limited differentiability in Banach spaces. Chapter 4 covers similar ground for analytic operators and operator equations in Banach spaces. In Chapters 5, 6 and 7 we consider finite-dimensional analyticity with particular regard to analyticity over the field $\mathbb{R}$. We prove the classical theorems of Weierstrass on the reduction of an analytic equation to a canonical form which involves a polynomial equation for one variable in which the coefficients are analytic functions of the other variables.

Chapter 8 deals with the finite-dimensional reduction of infinite dimensional problems. When the infinite-dimensional problem involves analytic operators, so does the finite-dimensional reduction and the mapping from solutions of the latter to solutions of the former is also analytic. This chapter is the link between the theory of finite-dimensional analytic varieties and infinite-dimensional problems in Banach spaces. Chapter 9 considers what conclusions can then be drawn about global one-dimensional branches of solutions of real-analytic operator equations. This concludes the abstract theory.

Chapter 10 illustrates our discussion of global real-analytic bifurcation theory with a substantial example from mathematical hydrodynamics: the existence question for steady two-dimensional periodic waves on an infinitely deep ocean. There is only one real parameter λ in the problem, the square of the Froude number which represents the speed of the wave.

In his 1847 paper [56] Stokes discussed nonlinear waves with small amplitudes using power series. At the time the proof of convergence was very difficult and only in the 1920s did Nekrasov [47] and Levi-Civita [42], independently, settle the question. Nowadays the existence of small-amplitude water waves can be recognised as nothing more complicated than bifurcation from a simple eigenvalue.

In an 1880 note, Stokes [57] conjectured the existence of a large amplitude periodic wave with a stagnation point and a corner containing an angle of 120° at its highest point. He further speculated that this wave of extreme form marks the limit of steady periodic waves in terms of amplitude (*the Stokes wave of greatest height*).

In Chapter 10 we show how real-analytic global bifurcation theory can account for the existence of waves of all amplitudes from zero up to that of Stokes' highest wave. See [60] for an account of topological methods applied to the same problem; the conclusions there are, in general, weaker.

Almost all the material here is to be found in the literature. The novelty is in the selection and organization of the material with bifurcation theory in mind. Each chapter ends with notes on sources.