PREFACE

This book is the final report of the ICMI study on the Teaching and Learning of Mathematics at University Level. As such it is one of a number of such studies that ICMI has commissioned. The other Study Volumes cover assessment in mathematics education, gender equity, research in mathematics education, the teaching of geometry, and history in mathematics education.

All of these Study Volumes represent a statement of the state of the art in their respective areas. We hope that this is also the case for the current Study Volume.

The current study on university level mathematics was commissioned for essentially four reasons. First, universities world-wide are accepting a much larger and more diverse group of students than has been the case. Consequently, universities have begun to adopt a role more like that of the school system and less like the elite institutions of the past. As a result the educational and pedagogical issues facing universities have changed.

Second, although university student numbers have increased significantly, there has not been a corresponding increase in the number of mathematics majors. Hence mathematics departments have to be more aware of their students' needs in order to retain the students they have and to attract future students. As part of this awareness, departments of mathematics have to take the teaching and learning of mathematics more seriously than perhaps they have in the past.

As a consequence, university mathematicians are more likely to take an interest in mathematics education and what it has to offer. In the past the contact between mathematics educators and practising university teachers had been poor. Thus there is a need to bridge the gap that exists in many countries, between mathematics educators and university mathematicians.

Finally, university mathematicians tend to teach as they were themselves taught. Unless they have a particular interest in teaching they are unlikely to make changes in their teaching or to exchange views, experiences or knowledge with their colleagues at other institutions. Hence this Study was commissioned to provide a forum for discussing, disseminating and interchanging, educational and pedagogical ideas between and among, mathematicians and mathematics educators.

As in every study, an International Programme Committee was appointed by the ICMI Executive Committee to oversee our Study's development. The members of the IPC were

Derek Holton (Ed.), The Teaching and Learning of Mathematics at University Level: An ICMI Study, v—viii.

Nestor Aguilera, Argentina
Michèle Artigue, France
Frank Barrington, Australia
Mohamed E.A. El Tom, Quatar
Joel Hillel, Canada
Derek Holton, New Zealand
Urs Kirchgraber, Switzerland
Lee Peng Yee, Singapore
Mogens Niss, Denmark
Alan Schoenfeld, USA
Hans Wallin, Sweden
Ye Qi-xiao, PRC.

The progress of ICMI Studies takes the following pattern. Once the IPC is appointed they produce a Discussion Document that contains a discussion of the key issues of the Study. This is widely circulated along with a call for reactions by way of abstracts of papers, proposals, the raising of other issues, etc. The Discussion Document for this Study appeared in the ICMI Bulletin, No. 43, December 1997.

As a result of the submissions, participants were invited to attend the Study conference that took place in Singapore in December 1998. This working conference included plenary sessions, submitted papers, panel discussions and working groups. The conference and the ideas and material developed at the conference forms the basis for this Study Volume. Extra material has been assembled since the conference by a number of authors.

One publication related to this Study, which is not in the general pattern of ICMI Studies, was the publication in February, 2000, of a special issue of the International Journal of Mathematics Education in Science and Technology. Papers produced for this issue were expanded versions of papers given at the Singapore conference.

As I said above, the Study conference was a working conference. It consisted of Plenary Sessions, Panel Discussions and Working Groups. The Plenary Sessions were as follows:

Claudi Alsina: Why the Professor should be a stimulating teacher: Towards a new paradigm of teaching mathematics at university level.

Michèle Artigue: What can we learn from didactic research carried out at university level?

Hyman Bass: Research on university-level mathematics education: (Some of) what is needed and why

Bernard Hodgson: Teaching and learning mathematics at the university level: a personal perspective.

Lynn Arthur Steen: Redefining university mathematics: the stealth campaign.

There were three Panel Discussions. The titles of these and the panel members are listed below.

Secondary/Tertiary Transition
Frank Barrington, Myriam Dechamps, Francine Gransard
Mass Education
Garth Gaudry, Gilah Leder
Technology
Ed Dubinsky, Celia Hoyles, Richard Noss

Finally there were eleven working groups. The Titles and Chairs of these Working Groups are listed below. As the titles alone do not necessarily give a clear view of the area covered we have added some explanation.

Secondary-Tertiary Interface, Leigh Wood and Sol Garfunkel
the interface between secondary and tertiary mathematics learning and teaching; interactions between secondary and tertiary teachers.
Mathematics and Other Subjects, Jean-Pierre Bourguigon
what mathematics is needed in other disciplines; which department should undertake this teaching?
Preparation of University Teachers, Harvey Keynes
what is the role of technology in mathematics education at the tertiary level; what should that role be; what programmes exist that use technology?
Assessing Undergradute Mathematics Students, Ken Houston
principles and purposes of assessment; methods of assessment; obstacles to change.
Trends in Curriculum, Joel Hillel
what topics are common to many curricula; what changes have occurred in the recent past; what changes are anticipated in the future?
Practice of University Teaching, John Mason
some principles of teaching; examples of innovative practice.
Mass Education, Nestor Aguilera and Hans Wallin
mathematics as a service course; what mathematics do students need; what is a good model for teaching students with a range of abilities and interests?
Preparation of Primary and Secondary Mathematics Teachers, Honor Williams
what is the current state of preparation; how might this change in the future; what is the role of academic mathematicians in teacher preparations?
Policy Issues, Hyman Bass
what are the different means of policy development? how do these affect practice? in what ways can policy be effected?
The Future of Research in Tertiary Mathematics Education, Annie Selden and John Selden.
what research is being and has been undertaken; how can this be translated into practice; what new directions should be explored?

I would like to thank the participants of the various working groups for their input to the Study. In particular, I would like to thank those who made contributions to the working group reports that appear in this volume. Unfortunately there has not been space in this book to mention them all individually.

As the result of the Study conference and reflecting on the issues raised in the working groups and in the more formal sessions, the Study seemed to naturally fall into seven parts, the seven sections of this book. These are an Introduction, Trends in Curriculum and Teaching Practice, Research, Mathematics and Other Disciplines, Technology, Assessment in Tertiary Mathematics Education, and Teacher Education. Each section has been edited by the people named at the start of that section.

Finally, I should like to thank the following people. First, there are the other members of the IPC. Without their considerable help the Study would never have reached the conference stage. They also provided an invaluable initial refereeing of papers for the special issue of the iJMEST

Second, I would like to thank Lee Peng Yee and his Local Organising Committee. They worked extremely hard to produce a conference that ran like clockwork but that still had a friendly personal touch.

Third, I would like to thank the conference participants and contributors to this Study Volume. It is their expertise that enabled us to produce a book that provides the latest thinking in a range of aspects of university-level mathematics education.

Then fourthly I am extremely grateful for the contribution of the editors of this Volume. Their knowledge and ability have carried this volume over a wide range of areas to present a thorough overview of the topic, and their individual knowledge and skills have enabled the volume to extend to great depths in all areas of the Study.

Next I would like to thank Leanne Kirk, Lenette Grant and Irene Goodwin for their considerable secretarial help throughout my period of engagement with this Study.

Sixth, I would like to thank the two people who were Executive Secretaries of ICMI during the period of the Study, Bernard Hodgson and Mogens Niss. Bernard shepherded through the Study to its final published form; Mogens was indispensable to me throughout and was always available with wise counsel from the beginning to the end of the project. So much that happened could not have happened without his support and guidance.

Finally I want to thank my wife Marilyn for supporting me through this and many other endeavours.

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CLAUDI ALSINA

WHY THE PROFESSOR MUST BE A STIMULATING TEACHER

Towards a new paradigm of teaching mathematics at University level

1. INTRODUCTION

Mathematics at the University level is a complex field to explore. The diversity of institutions and social and cultural contexts, the variety of curricula and courses, the reforms taking place at present, etc., may induce us to believe that perhaps it makes no sense to talk about general or common aspects of our academic activities. But after many years of observing our own profession, of visiting so many places around the world and interacting with so many colleagues I have identified some problems and some challenges that may be of interest for mathematicians who love mathematics and love teaching. The aim of this presentation is to share some critical thoughts and to point out some constructive ideas on the educational goals of teaching mathematics at the university level.

2. SOME CRITICAL VIEWS ON EXISTING MYTHS AND PRACTICES IN UNIVERSITY TEACHING OF MATHEMATICS

In this section I would like to unmask some very general existing 'myths' (Kirwan, 1991) and practices in the teaching of mathematics at the undergraduate level that have a negative influence (Lewis, 1975) on the quality of mathematics teaching.

The researchers-always-make-good-teachers myth. This university myth says that 'researchers are *ipso facto* good teachers ... therefore the key criteria for selection and promotion must be high quality research'. Following Kline (1977) we quote the statement that:

> Hence appointment, promotion, tenure and salary are based entirely on status in research... but for most of the teaching that the universities are, or should be, offering, the research professor is useless.

Derek Holton (Ed.), The Teaching and Learning of Mathematics at University Level: An ICMI Study, 3—12.

This myth calls for a number of observations.

1. Sound knowledge does not necessarily mean active research;
2. The majority of mathematics courses do not include advanced results reached in recent decades;
3. Research takes place in thousands of different specialities, most of it in very narrow fields, and lines of research are often a matter of free choice and quite unrelated to teaching;
4. Unfortunately, research criteria are closely related to the Department's interests and rarely include research into mathematics education.

Let us remember here the critical words expressed in Kline (1977):

> The mania for research has produced an invidious system of academic promotion, perversion of undergraduate education, and contempt for and flight from teaching.

While for graduate, doctoral and post-doctoral teaching activities there is no doubt that only the most up-to-date and active researchers can introduce students to the latest results, techniques and trends, this does not hold true for most undergraduate programmes (see Carrier et al, 1962).

The self-made-teacher tradition. This is another standard mathematical myth and is based upon the claim that excellence in university teaching does not require any specific training - it is just a matter of accumulated experience, clear presentation skills and a sound knowledge of the subject. This approach leaves room for a lot of creative freedom but at the same time it can lead to quite a lot of anxiety, especially for inexperienced young teachers, who will in general try to reproduce the models that they have been exposed to during their own education. This myth does not make provision for students who are exposed to various styles of teaching simultaneously and it also avoids the issue of critical input from colleagues as well as the positive training that one would expect from the institutions involved.

Some classical references on this topic come from the 70s (e.g. CTUM, 1979, EBLE, 1974, Rogers, 1975, Rosenberg, 1972, Wilson, 1974).

Clearly, teaching may benefit from training and this must be a compulsory activity for those who want to teach.

Context-free universal content. This idea justifies the content of many courses as 'basic skills and results which must be learned by everyone taking the course'. This myth generated classic courses that were given to almost everyone entering science or technological university studies. It is taken for granted that some elements of linear algebra, calculus, differential equations, discrete mathematics, probability, statistics, etc., constitute the 'core' curriculum of university mathematics. In particular this myth justifies the concept that teaching is context-free, i.e., independent of personal interests, of specific professional training, of cultural environment, of social circumstances, and so on. While this situation makes for a more flexible teaching organization (anyone can teach anything), it sacrifices students' interest and kills interdisciplinary approaches. This led to wide and even universal sales for some textbooks. We, however, believe that contents must be related to interest, special needs, context, and the like (see COMAP, 1997, Howson, 1988, Pollack, 1988, Steen, 1989).

Deductive organization. In this case, 'teaching' is thought to be assimilated thanks to representations of deductive thinking. Topics are presented linearly, definitions-theorems-proofs are sequentially stated in their most general form. In particular this presentation leads to the need for constant proofs (the more formal the better) and leaves little room for discussion or historical remarks ... "How?" becomes more important than "Why?". (Freudenthal, 1991). Is deduction more important than induction? Is formal reasoning more important than plausible thinking? Clearly, deduction is only one component of mathematical thinking.

The top-down approach. This approach holds that by teaching mathematical topics in their most general form, students will be able to deal with any particular case, any example, any application. This gets rid of the problem of real data and the main elements of mathematics modelling. Learning is a bottom-up process, so teaching top-down is not an effective way of helping learners (see e.g. Begle, 1979).

The perfect-theory presentation. Mathematics courses present positive results, solved problems, bona fide models. Students become convinced that mathematics is almost complete, that theorem proving is just a deductive game, that errors, false trials, and zig-zag arguments, which play such a crucial role in human life, have no place in the mathematical world. Unfortunately, in some ways many textbooks have inherited the cold research-journal style. This style of presentation kidnaps the 'human nature' of mathematical discoveries, the mistakes that were made, the difficulties and the need for simplifications. In some cases (e.g. statistics) this gives the false idea that the 'real subject' is 'the mathematical model', when we know that mathematics may be a powerful tool but it needs to be used in combination with other disciplines or techniques. In addition, we are presented with the paradox that very often this perfect presentation implies only an instrumental understanding instead of a relational understanding. This perfect-theory presentation turns a living discipline into a dead garden.

The 'master class'/formal lecture paradigm. Teaching has frequently been oriented towards 'communicating' mathematical knowledge. Typically, a class for undergraduates would consist of a large group of students sitting, listening and writing in a classroom where a professor delivers several hours per week of spoken-written presentation before a blackboard, see Bligh (1972). After the lectures, students are supposed to study the delivered content by reading notes, the textbook and by solving *ad hoc* exercises proposed for each chapter-talk. This reduces 'teaching' to lecturing, and 'learning' to an individual after-class activity of assimilating results and practising techniques. In particular, as noted by Clements (1998), students spend a lot of time inefficiently or unproductively:

> ... a considerable part of the time is devoted to the transference from the notes of the lecturer to the notepads of the students of relatively straightforward factual material.

While 'master classes'/formal lectures are fine when truly 'masterly', they could nevertheless be combined with other techniques of communicating and working.

The mature students myth. At the freshman level, this myth assumes that during the few weeks between high school and university registration, students have grown in such a way that their integration into the new university atmosphere does not require any special attention. In particular, students going into scientific or technical courses are assumed to be already motivated and aware of the relevance of mathematics to their training, and students going into other studies are assumed to constitute a low-interest class. The diversity of backgrounds is often ignored. The high school curriculum may often be unknown. Clearly, the transition from secondary schools to universities needs special attention.

The routine individual-written assessment. This presents the final test, or a written examination mixing questions and exercises, as an ideal method of marking, i.e., of gauging how well students master the content delivered in lectures. The method focuses on individual preparation and rarely opens doors to project work, group activities, open questions, etc. In its most rigorous form, this assessment is reduced to a final exam to be marked and rarely integrates other activities or information attained during the course into the student's progress. More flexible assessment resources should be considered (see e.g. Dossey, 1998).

The non-emotional audience. This tries to present students enrolled in a course as an audience at a movie show or a theatre. The main goal 'for all' is simply mathematics. Individual problems, emotional difficulties, personality features do not belong to the teaching and learning of mathematics. Tuition is for solving technical doubts or clarifying previous lectures. Outside the classroom or the scheduled office hours there is no place for further human interaction. The university walls keep human nature out. To sum up, let me quote Krantz (1993):

> I don't think that it is healthy for a mathematics teacher to worry about math anxiety. Your job is to teach mathematics. Go do it.

That's a terrible mistake. The 'audience' is a group of people in which each individual needs attention.

We, as mathematics educators working at university level, need to destroy the above myths, practices and considerations by taking some positive steps towards another way of teaching (see Howson, 1994).

3. TOWARDS A NEW PARADIGM OF TEACHING MATHEMATICS AT UNIVERSITY LEVEL

In this section we will identify some changes to be considered, some questions which need to be faced urgently and some goals for our future as mathematicians and mathematics educators.

There is a need to redefine mathematical research as a university activity, combining it with a soundly based teaching excellence. The critical pressure of research has evolved into a crazy rolling snowball: publishing as many papers as possible, going into citation and impact indices, attending an increasing number of congresses. It is time to sit down and think about what the main goals of universities today are. It is just possible that good teaching, fine multimedia and educational materials, virtual projects, community work, etc. are becoming more relevant to administrators and society than subscriptions to journals, abstract announcements and department reports. This does not mean a change from the research-realm to the teaching-paradigm. The 'either-research-or-teaching' polarity is false. With a little wisdom both activities can be (and should be) combined. Research also means writing expository papers, critiques of trends, historical perspectives, good texts, analyses of pedagogical materials, improvement of proofs, suggestions as to new approaches or interdisciplinary applications. Institutions and authorities should recognise and stimulate scholarship and research. And there is no need to say that the creation of exclusive research institutions is to be welcomed. But universities cannot close their eyes to their teaching ends. It is not just a question of achieving one annual award or medal for academic distinction but rather it is a matter of continuously controlling and stimulating the quality of education. Good teaching is according to a classic definition: "building understanding, communicating, engaging, problem solving, nurturing and organizing for learning", a complete task that merits special attention and preparation (see Krantz, 1993).

Research into mathematics education at tertiary level may be itself an interesting field of research and may give rise to useful results for all teachers for application to their teaching. Research into mathematics education is a growing scientific discipline (see Niss, 1998, Thurston, 1990). Nowadays it involves many researchers focusing on a wide range of topics and levels. However, there is clearly still a rich agenda for research on teaching and learning problems at university level. It would be marvellous if in the years to come this university research attracted well qualified mathematics specialists. If institutions wish or need to pay more attention to their educational goals, then mathematics education may – or indeed is certain to – play an increasingly important role in people's *vitae*. Though non-educational research has been a priority in people's careers until now, it could well be healthier if future mathematics specialists combined research with more educational aims. Moreover, research into mathematics education gives rise to useful results which should be disseminated and used, so that all mathematics teaching staff may benefit from an up-to-date knowledge of this field (see Niss, 1998).

JOEL HILLEL

TRENDS IN CURRICULUM

A Working Group Report

1. INTRODUCTION

The Working Group: Trends in Curriculum, examined the various forces which act on a mathematics curriculum, and on curriculum trends, both at local and national levels. 'Curriculum' was considered in its widest sense to mean "matters pertaining to the purposes, goals and content of mathematics education" (Discussion Document for this ICMI Study, 1997), as well as the means for achieving curricular goals. Hence, the discussions in the Working Group touched on undergraduate programmes[1], specific courses, mathematical content, degree of rigour, modes of delivery and interaction, and assessment schemes. Inevitably, the issues discussed in this Working Group overlapped substantially with the other Working Groups of the conference.

2. BACKGROUND

2.1 Who are mathematics students?

Curricular issues are inextricably tied to the question: a mathematics curriculum for whom? The teaching of mathematics at universities and colleges is quite diverse in its organization hence there is a wide range of students populating mathematics courses.

Among those enrolling in mathematics courses, there are students for whom mathematics is the primary subject of their undergraduate studies, possibly coupled with another discipline such as statistics, physics, computer science, or economics. We will refer to this group as '(maths) programme students' so as to distinguish them from 'client students'. The latter come from client departments, traditionally the physical sciences and engineering departments, though nowadays, computer science has become a prominent client, replacing physics in many countries. Other client students come from departments such as social sciences, commerce and

[1] Any international gathering immediately points to the different senses attributed to words such as 'programme', 'course', 'module', or 'paper'. In this document we have adhered to the North American usage of 'programme' and 'course' whereby a programme is made up of a collection of compulsory and optional courses, and a (one semester) course constitutes about 40 hours of instruction.

Derek Holton (Ed.), The Teaching and Learning of Mathematics at University Level: An ICMI Study, 59—69.

economics, and psychology, who are increasingly requiring their students to take some mathematics. Thus, a somewhat facile distinction between the two groups is that programme students want to study mathematics while client students have to. In any case, it is usually expected that client students terminate their mathematics studies after a year or two of their undergraduate studies.

Future school mathematics teachers, particularly at the secondary level can be considered as either programme or client students depending on national criteria for the training of teachers. For example, in certain European countries, prospective secondary teachers have to complete a full 5-year undergraduate mathematics curriculum and hence are, in our terms, programme students. On the other hand, future teachers in North America generally take only a certain number of mathematics courses rather than a full mathematics programme.

2.2 Organization of undergraduate teaching

Many departments of Mathematics are responsible for teaching all programme and client students. In fact, the teaching of client students (the 'service role') is often the bread-and-butter component of departments' teaching and it justifies having a large department of mathematics. Other institutions have 'mini-departments' of mathematics housed in engineering (Polytechniques), finance, economics, or education and teach exclusively the students of their discipline.

Mathematics departments who teach both programme and client students do so in different ways. Some require all their students to take the same courses, say, calculus, differential equations, or linear algebra, resulting in classes with a heterogeneous group of students whose background preparation, career ambitions, and interest in mathematics are quite varied. This 'one-curriculum-for-all' approach inevitably raises a range of issues as to the appropriate level and emphasis, and as to the type and depth of applications. Other departments offer a variety of courses that are specifically geared for one client group or another, viz. 'calculus for engineers', 'calculus for chemists', or 'algebra for teachers'.

Focusing on the mathematics curriculum specifically targeted for programme students, there are also wide variations depending on the traditions of the universities involved. These traditions have to do with: admission standards, the juncture at which a student can choose a mathematics option, the length of study, the number of courses required, course choices (both in mathematics and outside the discipline), and whether or not there is a requirement for studying mathematics together with a cognate discipline. The intended goal(s) of a mathematics programme (even if not explicitly articulated) are also dissimilar. To take but one example, in most Canadian universities, programme students complete either a major or honours in some field of mathematics (e.g. pure, applied, statistics). A major programme is comprised of a certain concentration of core and elective mathematics courses (which can amount to as little as a third of the total number of courses necessary to obtain a bachelor's degree). Except for the first year of the major (in a 4-year programme) where there are several compulsory courses in the other sciences and computer science, students have nearly complete freedom to choose courses complementary to the major. The honours programme, on the other hand, tends to be a more selective programme with a substantially greater number of advanced courses, and may take possibly an extra year to complete. The goal of the honours programme is to train highly qualified persons who can continue doing graduate studies and research or be employed in demanding mathematical fields. On the other hand, the goals for the major programme are more modest, namely to graduate students who are mathematically literate, and who can function comfortably in work situations requiring quantitative, analytic, and mathematical problem solving skills. (For more details on a major and honours programme, see Hillel, this volume, pp. 179-184.)

3. FACTORS INFLUENCING CURRICULUM

3.1 Changes within mathematics

The undergraduate mathematical landscape is always in some state of flux mirroring the organic nature of mathematics. New theories and mathematical tools, sometimes supported by powerful computers, are being developed within mathematics or as applications in cognate disciplines such as physics, computer science, and engineering. Certain new subjects become highly visible (e.g., dynamical systems, computer algebra), others experience a renaissance (e.g., geometry, number theory, numerical analysis), and yet others become more marginalized (e.g., category theory). In fact, Steen has written that "strong departments find that they replace or change significantly half of their courses approximately once a decade" and "as new mathematics is continually created, so mathematics courses must be continually renewed" (Steen, 1992). These on-going updates to the curriculum can be regarded, in a sense, as 'deterministic' aspects of curriculum change, ones that do not put into question the purpose, goals, and means of undergraduate education.

3.2 Changes in the pre-university math curriculum

Secondary school[2] mathematics curricula have undergone tremendous changes in the past 20 years. One most visible change in many countries is the reduction in the number of hours devoted to mathematics and science. For example, in France, up to 1994, secondary school (lycée) students had 15 hours of science teaching per week, of which 9 were in mathematics and physics. By 1999, there are only 8 hours a week in which to teach mathematics, physical sciences, biology and technology. Also in France, traditionally taught subjects like set theory and algebraic structures have been dropped, as well as the emphasis on definitions and proofs. Reports from other countries also allude to a de-emphasis on formal mathematics and on complicated

[2] Secondary school terminates after 11 years of schooling in some countries but lasts for up to 13 in others.

manipulations, the increased use of calculators with computer algebra systems capabilities, and the teaching of synthetic geometry, as well as a teaching approach which relies more on investigative project-oriented work. There are also increasing attempts to introduce quantitative, statistical and probabilistic reasoning in the secondary curriculum.

One should mention here that, quite often, changes in the secondary curriculum are brought about without coordination with the very universities and colleges that the students subsequently attend. Consequently, many university mathematicians are not always aware of the nature and extent of these changes nor of the pressures and constraints on the pre-university system that could explain, for example, why the number of hours devoted to mathematics is being reduced.

3.3 Changing clientele

Most countries have wisely abandoned the elitist view of university education in favour of a more open policy that makes university education accessible to a larger segment of the population. This policy has resulted in a great influx of students to universities (estimated to have increased 6-fold in the past 30 years, Steen, this volume, pp. 303-312), including a massive increase in the number of client students who are enrolled in agriculture, commerce, finance, social sciences, etc. These students tend to be heterogeneous in terms of their mathematical preparation, probably would rather not take mathematics at all if given the choice, and are not very interested in mathematical rigour and abstraction (nor even always convinced about the relevance of mathematics to their careers). At the same time, they constitute, in some universities, the main clientele of a mathematics department. Also, open immigration policies in some countries has resulted in an influx of students whose first language is not the language of instruction.

One general feature of incoming students is that they enter university having logged less hours of mathematics lessons because of the reduction of the number of hours devoted to mathematics at the secondary level. But even when a choice exists for taking more mathematics at the pre-university level, the trend is for students to forego this choice. For example, in England, there has been a significant drop in the number of students completing A-level mathematics. And, among those who do complete their A-level, the number of university mathematics candidates who have completed two A-levels in mathematics has dropped to about 1 in 10 in the 1990s whereas it was 1 in 3 in the 1960s (Simpson, 1998). A recent article indicates a drop of over 30% in students entering mathematics programmes in Germany (Jackson, 2000). Thus, the overall effect is that students' background preparation is not sufficient for meeting the rigours of traditional entry-level university courses in linear algebra and calculus, even for students who are relatively successful in their pre-university courses. (This point was also made by participants from Australia, Brazil, Canada, England, Japan, Malaysia, and the USA.)

Students' attitudes towards education, their study habits, and their expectations, are influenced by the traditions and values of the prevailing culture in which they live. There is also a sense that students nowadays are more career-oriented and thus interested in getting skills that lead directly to jobs. This explains why in some countries, the number of mathematics programme students has been dwindling dramatically and the ablest students are drawn to such fields as computer science, engineering, and finance, where career opportunities are more evident. Within mathematics, there is sharp increase in enrolment in actuarial mathematics, when this option is available (for example, in Australia, Canada - see Hillel, this volume, pp. 179-184, - and Switzerland - see Kirchgraber, this volume, pp. 185-190). There is also an increasing tendency for students to combine study and work, thus taking on part-time jobs to supplement their income.

Departments of mathematics are thus faced with the challenge of having to teach students whose background preparation, learning styles, study habits, and career ambitions are more and more at odds with the traditional lecture-style mathematical training with its Bourbaki-like curriculum, particularly, in pure mathematics. Furthermore, many departments are facing an increase in the number of client students and a decline in the number of programme students.

3.4 Resources

Certain countries have never had adequate resources for higher education; others are experiencing political and economic upheavals which greatly affect education, as well as all other aspects of life. Even more affluent and politically stable countries, are witnessing a continual erosion of the levels of government support for universities. Diminishing resources usually translate into less staff, larger classes, and pressures to be more efficient and financially accountable. In such instances, mathematics departments are finding themselves less able to offer specialized courses to a small number of students and so have redefined an appropriate core of an undergraduate mathematics programme.

3.5 Technology

Computers have impacted on the methods and results of several mathematical domains. Coupled with graphing software, computer algebra systems, dynamic geometry, or differential equations packages, computers and calculators pose interesting challenges to mathematics departments. They have led to questioning what mathematics content is central and what is redundant, as well as, how present-day learning, teaching and assessment practices, can be and ought to be changed.

3.6 External influences: Governments, Research Agencies and Business

There is a prevalent sentiment among mathematicians that they, as the professionals, are in the best position to define the undergraduate curriculum for their students. They view attempts to influence curricular choices by bodies external to the department as an unwarranted intrusion. However, governments who, in most cases, foot the universities' bills, have, in recent years, been much more vocal and

MICHÈLE ARTIGUE

WHAT CAN WE LEARN FROM EDUCATIONAL RESEARCH AT THE UNIVERSITY LEVEL?[1]

1. INTRODUCTION

For more than 20 years, educational research has dealt with mathematics learning and teaching processes at the university level. It has tried to improve our understanding of the difficulties encountered by students and the dysfunction of the educational system; it has also tried to find ways to overcome these problems. What can such research offer to an international study? This is the issue I will address in this article, but first I would like to stress that it is not an easy question to answer, for several reasons including at least the following:

1. Educational research is far from being a unified field. This characteristic was clearly shown in the recent ICMI study entitled "What is research in mathematics education and what are its results?" (See Sierpinska and Kilpatrick, 1996.) The diversity of existing paradigms certainly contributes to the richness of the field but, at the same time, it makes the use and synthesis of research findings more difficult.
2. Learning and teaching processes depend partly on the cultural and social environments in which they develop. Up to a certain point, results obtained are thus time- and space- dependent, their field of validity is necessarily limited. However, these limits are not generally easy to identify.
3. Finally, research-based knowledge is not easily transformed into effective educational policies.

I will come back to this last point later on. Nevertheless, I am convinced that existing research can greatly help us today, if we make its results accessible to a large audience and make the necessary efforts to better link research and practice. I hope that this article will contribute to making this conviction not just a personal one. Before continuing, I would like to point out that the diversity mentioned above does not mean that general tendencies cannot be observed. At the theoretical level, these are indicated, for instance, by the dominating influence of constructivist approaches inspired by Piaget's genetic epistemology, or by the recent move

[1] A shorter version of this paper, Artigue (1999), was published in the *Notices of the American Mathematical Society.*

Derek Holton (Ed.), The Teaching and Learning of Mathematics at University Level: An ICMI Study, 207—220.

attempt to take more account of the social and cultural dimensions of learning and teaching processes (see Sierpinska and Lerman, 1996). But within these general perspectives, researchers have developed a multiplicity of local theoretical frames and methodologies, which differently shape the way research questions are selected and expressed, and the ways they are worked on – thus affecting the kind of results which can be obtained, and the ways they are described. At the cultural level, such general tendencies are also observed. Strong regularities in students' behaviour and difficulties as well as in the teaching problems met by educational institutions, have been observed. These, up to a point, apparently transcend the diversity of cultural environments.

In the following, after characterizing the beginnings of the research enterprise, I will try to overcome some of the above-mentioned difficulties presenting research findings along two main dimensions of learning processes: qualitative changes, reconstructions and breaches on the one hand, cognitive flexibility on the other hand. These dimensions can to some degree, be considered 'transversal' with respect to theoretical and cultural diversities as well as to mathematical domains. No doubt this is a personal choice, induced by my own experience as a university teacher, as a mathematician, and as a education researcher; it shapes the vision I give of research findings, a vision which does not pretend to be objective or exhaustive.

2. FIRST RESEARCH RESULTS: SOME NEGATIVE REPORTS

The first research results obtained at university level can be considered negative ones. Research began by investigating students' knowledge in specific mathematical areas, with particular emphasis on elementary analysis (or calculus in the Anglo-Saxon culture), an area perceived as the main source of failure at the undergraduate level. The results obtained gave statistical evidence of the limitations both of traditional teaching practices and of teaching practices which, reflecting the Bourbaki style, favoured formal and theoretical approaches. The structure and content of the book, *Advanced Mathematical Thinking* (Tall, 1991), gives clear evidence of these facts, noting that:

- by the early eighties, Orton (1980), in his doctoral thesis, showed the reasonable mastery English students had of what can be labelled as 'mere algebraic calculus': calculation of derivatives and primitives (anti-derivatives), but the significant difficulty they had in conceptualizing the limit processes underlying the notions of derivative and integral;
- at about the same time, Tall and Vinner (1981), highlighted the discrepancy between the formal definitions students were able to quote and the criteria they used in order to check properties such as functionality, continuity, derivability. This discrepancy led to the introduction of the notions of concept definition and concept image in order to analyze students' conceptions;

- very early, different authors documented students' difficulties with logical reasoning and proofs, with graphical representations, and especially with connecting analytic and graphical work in flexible ways.

Schoenfeld (1985), also documented the fact that, faced with non-routine tasks, students – even apparently bright students – were unable to efficiently use their mathematical resources.

Research also showed, quite early, that the spontaneous reactions of educational systems to the above-mentioned difficulties were likely to induce vicious circles such as the following. In order to guarantee an acceptable rate of success, an increasingly important issue for political reasons, teachers tended to increase the gap between what was taught and what was assessed. As the content of assessments is considered by students to be what should be learnt, this situation had dramatic effects on their beliefs about mathematics and mathematical activity. This, in turn, did not help them to cope with the complexity of advanced mathematical thinking.

Fortunately, research results are far from being limited to such negative reports. Thanks to an increasing use of qualitative methodologies allowing better explorations of students' thinking and the functioning of didactic institutions (Schoenfeld, 1994), research developed and tested global and local cognitive models. It also organized in coherent structures the many difficulties students encounter with specific mathematical areas, or in the secondary/tertiary transition. It led to research-based teaching designs (or engineering products) which, implemented in experimental environments and progressively refined, were proved to be effective. Without pretending to be exhaustive, let us give some examples, classified according to the two main dimensions given above. (For more details, the reader can refer to the different syntheses in Artigue, 1996, Dorier, 2000, Schoenfeld, 1994, Tall, 1991 and 1996; to the special issues dedicated to advanced mathematical thinking by the journal *Educational Studies in Mathematics* in 1995 edited by Dreyfus; by the journal *Recherches en Didactique des Mathématiques* in 1998 edited by Rogalski; to some of the diverse monographs published by the Mathematical Association of America about calculus reform, innovative teaching practices; and to research about specific undergraduate topics to be found in the MAA Notes on Collegiate Mathematics Education.)

3. QUALITATIVE CHANGES, RECONSTRUCTIONS AND BREACHES IN THE MATHEMATICAL DEVELOPMENT OF KNOWLEDGE AT UNIVERSITY LEVEL

One general and crosscutting finding in mathematics education research is the fact that mathematical learning is a cognitive process that necessarily includes 'discontinuities.' But, depending on the researcher this attention to discontinuities is expressed in different ways. In order to reflect this diversity and the different insights it allows, I will describe three different approaches: the first one, in terms of process/object duality, the second one in terms of epistemological obstacles, the third one in terms of reconstructions of relationships to objects of knowledge.

3.1 Qualitative changes in the transition from processes to objects: APOS theory

As mentioned above, research at the university level is the source of theoretical models. The case of APOS theory, initiated by Dubinsky (see Tall 1991) and progressively refined (see Dubinsky and McDonald, this volume, pp. 275-282), is typical. This theory, which is an adaptation of the Piagetian theory of reflective abstraction, aims at modelling the mental constructions used in advanced mathematical learning. It considers that "understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorized to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organized in schemas" Asiala et al, 1996. Of course, this does not occur all at once and objects, once constructed, can be engaged in new processes and so on. Researchers following this theory use it in order to construct genetic decomposition of concepts taught at university level (in calculus, abstract algebra, etc.) and design teaching processes reflecting the genetic structures they have constructed and tested.

As with any model, the APOS model only gives a partial vision of cognitive development in mathematics, but one cannot deny today that it put to the fore a crucial qualitative discontinuity in the relationships students develop with respect to mathematical concepts. This discontinuity is the transition from a process conception to an object one, the complexity of its acquisition and the dramatic effects of its underestimation by standard teaching practices.[2] Research related to APOS theory also gives experimental evidence of the positive role which can be played by programming activities in adequate languages (such as the language ISETL, cf. Tall, 1991) in order to help students encapsulate processes as objects.

Breaches in the development of mathematical knowledge: Epistemological obstacles. The theory of epistemological obstacles, firstly introduced by Bachelard (1938) and imported into educational research by Brousseau (1997), proposes an approach complementary to cognitive evolution, focussing on its necessary breaches. The fundamental principle of this theory is that scientific knowledge is not built in a continuous process but results from the rejection of previous forms of knowledge: the so-called epistemological obstacles. Researchers following this theory hypothesize that some learning difficulties, often the more resistant ones, result from forms of knowledge which are coherent and have been for a time effective in social and/or educational contexts. They also hypothesize that epistemological obstacles have some kind of universality and thus can be traced in the historical development of the corresponding concepts. At the university level,

[2] Note that a very similar approach was developed independently by Sfard, with more emphasis on the dialectic between the operational and structural dimensions of mathematical concepts in mathematical activity (Sfard, 1991).

such an approach has been fruitfully used in research concerning the concept of limit (cf. Artigue 1998 and Tall 1991 for synthetic views). Researchers such as Sierpinska, (1985), Cornu, (1991) and Schneider, (1991) provide us with historical and experimental evidence of the existence of epistemological obstacles, mainly the following:

- the everyday meaning of the word 'limit', which induces resistant conceptions of the limit as a barrier or as the last term of a process, or tends to restrict convergence to monotonic convergence;
- the overgeneralization of properties of finite processes to infinite processes, following the continuity principle stated by Leibniz;
- the strength of a geometry of forms which prevents students from clearly identifying the objects involved in the limit process and their underlying topology. This makes it difficult for students to appreciate the subtle interaction between the numerical and geometrical settings in the limit process.

Let us give one example (taken from Artigue, 1998) of this last resistance, which occurs even in advanced and bright students. In a research project about differential and integral processes, advanced students were asked the following non-standard question: "How can you explain the following: using the classical decomposition of a sphere into small cylinders in order to find its volume and area, one obtains the expected answer for the volume $\frac{4}{3}\pi R^3$, but $\pi^2 R^2$ for the area instead $4\pi R^2$?" It was observed that, faced with this question, the great majority of advanced students tested got stuck. And, even if they were able to make a correct calculation for the area (which they were not always able to do) they remained unable to resolve the conflict.

As the students eventually said, because the pile of cylinders, geometrically, tends towards the sphere, the magnitudes associated with the cylinders behave in the same way and thus have as a limit the corresponding magnitude for the sphere. Such a resistance may look strange but it appears more normal if we consider the effect produced on mathematicians by the famous Schwarz counterexample showing that, for a surface as simple as a cylinder, limits of areas of triangulations when the size of the triangles tends towards 0, can take any value greater than or equal to the area up to infinity, depending on the choices made in the triangulation process, an effect nicely described by in Lebesgue, (1956). The historical and universal commitments of the theory which leads to such results can be discussed and are presently discussed (see, for instance, Radford, 1997). However, what cannot be negated is the fact that the above-mentioned forms of knowledge constitute resistant difficulties for today's students; moreover, that mathematical learning necessarily implies partial rejection of previous forms of knowledge, which is not easy for students.

LYNN ARTHUR STEEN

REVOLUTION BY STEALTH: REDEFINING UNIVERSITY MATHEMATICS

1. INTRODUCTION

Change, growth, and accountability dominate higher education at the dawn of the twenty-first century. According to delegates at UNESCO's recent World Conference on Higher Education, change is unrelenting, 'civilizational in scope,' affecting everything from the nature of work to the customs of society, from the role of government to the functioning of the economy. Growth in higher education has been equally dramatic, with worldwide enrolment rising from 13 million in 1960 to near 90 million today (World Conference on Higher Education, 1998).

Concomitant with change and growth is pressure from governments everywhere for greater accountability from professors and leaders of higher education, for evidence that, in a world of rapid change, universities are working effectively to address pressing needs of society. Autonomy, the prized possession of universities, presupposes accountability. The escalating pressures of global change, growth, and accountability will create, according to UNESCO Director Federico Mayor, "a radical transformation of the higher education landscape not only more but *different* learning opportunities" (Mayor, 1998).

Much of the upheaval in society and employment is a consequence of the truly revolutionary expansion of worldwide telecommunication, as is the stunning increase in demand for higher education. But this demand is predicated on the belief that universities properly anticipate signals from the changing world of work and create optimal linkages between students' studies and expectations of employers. Unfortunately, few universities have taken up this challenge, at least not unless pushed by external forces.

As the world changes rapidly and higher education grows explosively, universities evolve leisurely. Courses, curricula, and examinations remain steeped in tradition, some centuries old, while autonomy and academic freedom rule in the classroom. Few institutions of higher education readily embrace the culture of assessment that is required to ensure relevance and effectiveness of their curriculum. Under these circumstances it is only natural for political leaders to demand stronger connections between the classroom and the community, between the ivory tower and the industrial park.

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2. UNDERGRADUATE MATHEMATICS

Demands for relevance and accountability are no strangers to undergraduate mathematics. Indeed, post-secondary mathematics can be viewed as higher education in microcosm. Growth in course enrolments has been enormous, paralleling the unprecedented penetration of mathematical methods into new areas of application. These new areas—ranging from biology to finance, from agriculture to neuroscience—have changed profoundly the profile of mathematical practice (see Odom, 1998). Yet for the most part these changes are invisible in the undergraduate mathematics curriculum, which still marches to the drumbeat of topics first developed in the eighteenth and nineteenth centuries.

It is, therefore, not at all surprising that the three themes identified at the UNESCO conference are presaged in the Discussion Document for this ICMI Study: the rapid growth in the number of students at the tertiary level; unprecedented changes in secondary school curricula, in teaching methods, and in technology; and increasing demand for public accountability (see Discussion Document, 1997). Worldwide demands for radical transformation of higher education bear on mathematics as much as on any other discipline.

Post-secondary students study mathematics for many different reasons. Some pursue clear professional goals in careers such as engineering or business where advanced mathematical thinking is directly useful. Some enrol in specialized mathematics courses that are required in programmes that prepare skilled workers such as nurses, automobile mechanics, or electronics technicians. Some study mathematics in order to teach mathematics to children, while others, far more numerous, study mathematics for much the same reason that students study literature or history: for critical thinking, for culture, and for intellectual breadth. Still others enrol in post-secondary courses designed to help older students master parts of secondary mathematics (especially algebra) that they never studied, never learned, or just forgot. (This latter group is especially numerous in countries such as the United States that provide relatively open access to tertiary education, see Phipps, 1998.)

In today's world, the majority of students who enrol in post-secondary education study some type of mathematics. Tomorrow, virtually all will. In the information age, mathematical competence is as essential for self-fulfilment as literacy has been in earlier eras. Both employment and citizenship now require that adults be comfortable with central mathematical notions such as numbers and symbols, graphs and geometry, formulas and equations, measurement and estimation, risks and data. More important, literate adults must be prepared to recognize and interpret mathematics embedded in different contexts, to think mathematically as naturally as they think in their native language (see Steen, 1997).

Since not all of this learning can possibly be accomplished in secondary education, much of it will take place in post-secondary contexts, either in traditional institutions of higher education (such as universities, four- and two-year colleges, polytechnics, or technical institutes) or, increasingly, in non-traditional settings such as the internet, corporate training centres, weekend short-courses, and for-profit universities. This profusion of post-secondary mathematics programmes at the end of the twentieth century contrasts sharply with the very limited forms of university mathematics education at the beginning of this century. The variety of forms, purposes, durations, degrees, and delivery systems of post-secondary mathematics reflects the changing character of society, of careers, and of student needs. Proliferation of choices is without doubt the most significant change that has taken place in tertiary mathematics education in the last one hundred years.

3. MATHEMATICAL PRACTICE

The primary purpose of mathematics programmes in higher education is to help students learn whatever mathematics they need, both for their immediate career goals and as preparation for life-long learning. Today's students expect institutions of higher education to offer mathematics courses that support a full range of educational and career goals, including:

- Agriculture
- Biological Sciences
- Business
- Computing
- Elementary Education
- Electronics
- Economics
- Engineering
- Finance
- Geography
- General Education
- Health Sciences
- Law
- Mathematical Research
- Management
- Medical Technology
- Physical Science
- Quantitative Literacy
- Remedial Mathematics
- Secondary Education
- Social Sciences
- Statistics
- Technical Mathematics
- Telecommunications

Even without exploring details of specific curricula or programmes, it should be obvious that the multiplicity of student career interests requires, if you will, multiple mathematics. Consider a few examples of how relatively simple mathematics is used in today's world of high performance work:

- Precision farming relies on satellite imaging data supplemented by soil samples to create terrain maps that reflect soil chemistry and moisture levels. These methods depend on geographic information systems that blend spreadsheet organization with a variety of algorithms for geometric projections (e.g., for rendering onto flat maps oblique satellite images of earth's curved surface).
- Technicians in semiconductor manufacturing plants, analyze real-time data from production processes in order to detect patterns of change that might signal an impending reduction in quality before it actually happens. These methods involve measurement strategies, graphical analyses, and tools of statistical quality control.
- Teams that design new commercial airplanes now engage designers, manufacturing personnel, maintenance workers, and operation managers in joint

planning with the goal of minimizing total costs of construction, maintenance, and operation over the life of the plane. This enterprise involves teamwork among individuals of quite different mathematical training as well as innovative methods of optimization.

- Emergency medical personnel need to interpret quickly and accurately dynamic graphs of heart action that record electrical potential, blood pressure, and other data. With experience, they learn to recognize both regular patterns and common pathologies. With understanding, they can also interpret uncommon signals.

These examples are not primarily about the relation of mathematical theory to applications—the traditional poles of curricular debate—but about something quite different: mathematical practice (see Denning, 1997). Behind each of these situations lurks much good mathematics (e.g., projection operators, optimization algorithms, fluid dynamics, statistical inference) that can be applied in these and many other circumstances. However, most students are not primarily motivated to learn this mathematics, but rather to increase crop yield, minimize manufacturing defects, reduce airplane costs, or stabilize heart patients. Although a mathematician will recognize these as mathematical goals—to increase, minimize, reduce, stabilize—neither students nor their teachers in agriculture, manufacturing, engineering, or medicine would recognize or describe their work in this way. To these individuals, the overwhelming majority of clients of post-secondary mathematics, mathematical methods are merely part of the routine practice of their profession.

Indeed, mathematics in the workplace is often so well hidden as to be invisible to everyone except a discerning observer. In the United States different industries have created skill standards for entry-level employees (e.g., electronics (American Electronics Association, 1994), photonics (Center for Occupational Research and Development, 1995), health care (FarWest Laboratory, 1995), and National Skills Standards Board, 1998). Virtually all of these standards include substantial uses of mathematics, but most such applications are embedded in routine job requirements without any visible hint of the underlying mathematics. Although mathematics is now ubiquitous in business and industry, the mathematics found there is often somewhat different from what students learn in school or college (see Davis, 1996 and Packer, 1997). Similarly, in the wider world of public policy, the gradual incursion of statistics and probability in measuring (and sometimes controlling) personal health, societal habits, and national economies has created whole new territories for students and professors to explore (see Bernstein, 1996, Porter, 1995 and Wise, 1995).

In sharp contrast to this profligate flowering of practical mathematics in diverse post-secondary settings, university mathematics—what mathematicians tend to think of as 'real mathematics'—matured in the last century as a tightly *disciplined* discipline led by professors of world-wide renown who held major chairs in leading universities and research institutes. However, this university mathematics, 'real mathematics' as practiced in real universities, now constitutes only a tiny fraction of post-secondary mathematics. One data point: in the United States, fewer than 15%

of traditional undergraduate mathematics enrolments are in courses above the level of calculus (Loftsgaarden, Rung, and Watkins, 1997). And this does not count non-traditional enrolments, where the variety of offerings is even greater. A realist might well argue that 'real mathematics' is found not in the traditional curriculum inherited from the past but in today's widely dispersed courses, where a multitude of students learn a cornucopia of mathematics in diverse situations for a plethora of purposes.

4. LEARNING MATHEMATICS

Where do students learn mathematics? Some take traditional mathematics courses such as calculus, geometry, and statistics. Some take courses specifically designed for certain professions—mathematics for nurses, statistics for lawyers, calculus for engineers—that are offered either by mathematics departments or by the professional programmes themselves. But many, perhaps even most, pick up mathematics invisibly and indirectly as they take regular courses and internships in their professional fields (e.g., in physiology, geographic information systems, or aircraft design).

Any university dean knows that statistics is more often taught outside of statistics programmes than inside them. The same is true of mathematics, but is not as widely recognized. Every professional programme, from one- and two-year certificates to four- and five-year engineering degrees, offers courses that provide students with mathematics (or statistics) in the context of specific professional practice. This is entirely natural, since most students find that they learn mathematics more readily, and are more likely to be able to use it when needed, if it is taught in a context that fits their career goals and in which the examples resonate with those that appear in their other professional courses.

The appeal of context-based mathematics is no surprise, nor is its widespread presence in university curricula. But what is somewhat new—and growing rapidly—is the extent to which good mathematics is unobtrusively embedded in routine courses in other subjects. Anywhere spreadsheets are used (which is almost everywhere) mathematics is learned. It is also learned in courses that deal with such diverse topics as image processing, environmental policy, and computer-aided manufacturing. From technicians to doctors, from managers to investors, most of the mathematics people use is learned not in a course called mathematics but in the actual practice of their craft. And in today's competitive world, where quantitative skills really count, embedded mathematical tools are often as sophisticated as the techniques of more traditional mathematics.

So tertiary mathematics now appears in three forms: as traditional mathematics courses (both pure and applied) taught primarily in departments of mathematics; as context-based mathematics courses taught in other departments; and as courses in other disciplines that employ significant (albeit often hidden) mathematical methods. I have no data to quantify the 'biomass' of mathematics taught through these three means, but to a first approximation I would conjecture that they are approximately equal.

KAREN KING, JOEL HILLEL AND MICHÈLE ARTIQUE

TECHNOLOGY

A working group report

1. INTRODUCTION

The technology working group focused on the various ways in which technology can impact upon the teaching and learning of mathematics. As was already underlined in the Discussion Document for this ICMI Study, "Worldwide, increasing use is being made of computers and calculators in mathematics instruction. Much mathematical software and many teaching packages are available for a range of curriculum topics. This, of course, raises such issues as what such software and packages offer to the teaching and learning of the subject, and what potential problems for understanding and reasoning they might generate." The Discussion Document proposed to identify and analyze innovative projects and research that are particularly fruitful for advancing our thinking in this domain.

Reflecting on the impact of information technologies on the teaching of mathematics is not new for an ICMI Study – ICMI had already launched a study in 1985 entitled "The influence of Computers and Informatics on Mathematics and its Teaching". That ICMI Study touched all levels of instruction and underlined primarily the impact of computers on several areas, including:

- on mathematics itself; computers have prompted the revisiting of familiar notions such as number and elementary functions, the revitalizing of old problems, and the emergence of new domains. They have extended the range of applications of mathematics, and have blurred the boundaries between pure and applied mathematics;
- on the notion of proof in view of computer-assisted proofs;
- on the practice of mathematicians; computers have led to an increase in experimentation and the use of simulations. They afford new means of communication and accessing information that affect the way mathematicians carry on their professional lives.

This previous ICMI Study also recognized that despite an abundance of interesting experiences, the impact of technology on teaching was still globally weak, and that the introduction of computers in the classroom had not necessarily

Derek Holton (Ed.), The Teaching and Learning of Mathematics at University Level: An ICMI Study, 349—356.

led to any discernible improvements. The working group discussion focused on the present-day role of technology in teaching at the post-secondary level, on the perspectives envisaged for the future and on broader research questions that are affected by the use of technology. It centred mostly on the use of technological tools for supporting students' learning, particularly via visualization; computation, and programming. But, it also recognized the role of such tools for: demonstration by the teacher; presentation of lessons via distance learning; student assessment; and student drill.

2. TECHNOLOGY AS A MEANS FOR SUPPORTING STUDENTS' LEARNING

At the university level in general, and at the collegial level in particular, the introduction of technologies was seen as a means to renew pedagogical practices and to circumvent a style of teaching that was too formal or too algorithmic. It was intended to create better coherence between teaching practice and the constructivist approach to learning. Celia Hoyles, in her description of the potential contribution to post-secondary education of researches carried in the secondary level, has emphasized that:

"There is considerable evidence of the computer's potential to:

- foster more active learning using experimental approaches along with the possibility of helping students to forge connections between different forms of expression, e.g. visual, symbolic ;
- provoke constructionist approaches to learning mathematics where students learn by building, debugging and reflection, with the result that the structure of mathematics and the ways the pieces fit together are open to inspection ;
- motivate explanations in the face of "surprising" feedback : that is, start a process of argumentation which can (with due attention) be connected to formal proof ;
- foster cooperative work, encouraging discussion of different solutions and strategies ; computer work is more visible and more easily "conveyed" between lecturer and students ;
- open a window on to student thought processes : students hold different conceptions of mathematical ideas which are hard to access, even in the case of articulate adults. How students interact with the computer and respond to feedback can give insight into their conceptions and their beliefs about mathematics and the role of computers."

Hoyles hastened to add that a successful integration of computers necessitates the rethinking of "the content and sequence of the mathematics courses given that students and mathematics have (or should have) changed in the light of the new technology [...] teaching approaches to take into account the broad range of response inevitable in interacting with computers [...] and the relationship of 'computer maths' to paper and pencil maths" (Hoyles, 1999).

The question of what constitutes 'successful integration' of technology to the teaching and learning process was central to the working group discussion. Several presentations by participants on the way in which they have used technology to teach mathematics at the undergraduate level, helped to focus the discussion. These included presentations by: Karen King, on teaching differential equations; Ed Dubinsky, on programming using ISETL; Joel Hillel, on using Maple in teaching linear algebra; and, Rosalind Phang, on using statistical software.

2.1 Changes in the Nature of the Mathematics Taught

King's example illustrated the nature of the changes in teaching differential equations made possible by using a technology that graphs slope fields and direction fields. These enable students to engage in qualitative analyses of previously inaccessible differential equations rather than use traditional analytic techniques. Thus, the focus of a differential equations' course could shift from just finding the solution functions, to graphically organizing the space of solution functions using slope fields and bifurcation diagrams, and to examining the nature of the solution functions (see Rasmussen, 1999).

If one considers, for example, the differential equation

$$dy/dt = 0.3y(1-y/8)(y/3-1),$$

one could attempt to solve this using separation of variables but would not deduce a closed-form general solution. However, with a slope field as shown in Figure 1 derived from a TI-92 program written by King, a student can examine the types of solution functions and their general behaviours, given different initial conditions.

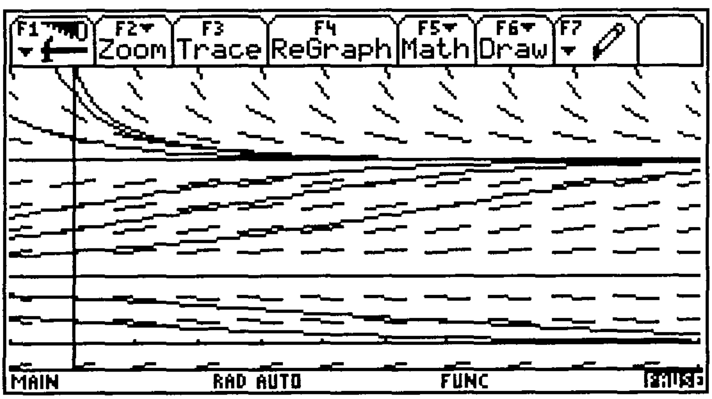

Figure 1 slope field program with slopes and several approximations $dy/dt=0.3y(1-y/8)(y/3-1)$, $y(0)=1, 2, 3, 4, 5, 6, 8, 9, 11, 12$

This example provides an instance where changes to an entire course can be made, including the order in which topics are taught and the mathematics with which the students engage (see Artigue, 1992, and Rasmussen and King, 1999). Such changes, in turn, lead to other changes in the curriculum. For example, the study of dynamical systems has been greatly impacted by the availability of computing technology and has resulted in an early focus on systems of differential equations in many courses. This is but one example where a particular mathematical discipline is changed by technology which, in turns, affects changes in the nature of how it is taught.

2.2 The role of professional tools

Secondary schools tend, by and large, to use software products and calculators that have been specifically conceived for teaching. In contrast, universities mostly tend to use professional tools be they general symbolic manipulators (e.g. Maple, Mathematica, MuPad, Matlab, SciLab) or tools for specific domains such as Statistics (APSS, SASS), though some specific educational software such as Geometer's Sketchpad and Cabri are also relevant for instruction at the tertiary level. Faculty members are familiar with these professional tools since they use them in their own mathematical work and, consequently, they tend to be widely available on campus. There is an ever increasing number of texts that integrate the use of a software package, for example, "Calculus and Mathematica" (Uhl, 1999) or "Ordinary Differential Equations using MATLAB" (Polking, 1995). Individual universities have also written primers that bridge between the particular program or technology that they use and the mathematics texts in use in the department (see Colgan, 1999 for a discussion of such a primer from Australia).

Professional software tools are particularly powerful and, at first sight, seem to take full charge of what traditionally has been the mathematics work expected of students. They embody a tremendous amount of mathematical knowledge that, nevertheless, remains invisible and inaccessible to the users. The availability of these powerful tools raised the inevitable question in the working group regarding the necessary mathematical knowledge of users if they are to become efficient and in reasonable control of such tools. These tools also force us to both question and redefine the content of mathematical training, notably in sectors where mathematics is a service course. It prompts us to ask under what conditions can they become means for students to construct mathematical knowledge, over and above their role as powerful computational tools.

In response to the question regarding the necessary skills/concepts that students must possess before they can use a powerful CAS tool, Hillel suggested seizing the 'black box' feature as a learning opportunity. He presented an example on teaching the Cayley-Hamilton Theorem in linear algebra, where students are first asked to use Maple to build inductive evidence for that theorem. By using the software, students can compute the characteristic polynomial $f(x)$ of a given matrix A and then compute $f(A)$. Among other things, it becomes apparent that the result of the computation is a square matrix, not a number, to which students must therefore attend and about which they must be explicit. Students also can explore these computations for several matrices, focusing the process of computing $f(x)$ rather than on the actual computations. Such activities can take place prior to introducing the theorem and its proof in class. This is a pedagogical choice, a kind of 'didactical inversion' which is made within the larger context of the instructor's course design (see Kent and Noss, 1999, and Noss, 1999).

In a slightly different vein, participants also recognized that there are computer applications designed for other purposes that could be mathematically exploited (e.g., Excel). This raised the questions of how one would characterize the difference (for instructors, for users, and for the types of tasks and interactions) in using educational mathematical tools, professional mathematical tools, and the mathematical usage of tools designed for other purposes.

2.3 The role of programming

In the analysis of the potential of computers for mathematics learning, programming has always played an important role. In the early days of computers when tools for scientific calculations were very different from those of today, programming was essential. But even if software packages have evolved, programming can be seen as a means to change students' relation to algorithmic work, so important in mathematics, by putting the accent on the construction of algorithms rather than on their execution. This shift is seen as a way to give sense to both the algorithms and to the underlying concepts.

Dubinsky presented to the working group the use of programming in a function-based program language (ISETL) to facilitate students' learning about functions (see Dubinsky, 1999). Instead of having students use conventional programs, the students write their own. His work illustrates particularly well the conceptual gains that students make when they have to write mathematical constructions as programs. His approach is built on a theoretical model that looks at learning in terms of actions, processes, and objects. ISETL is particularly well adapted for mathematics, since it favours transforming actions into processes and encapsulation of processes as mathematical objects (see Dubinsky and MacDonald, this volume. pp. 275-282).

Programming activities could also be implemented via scripting which is an automatic execution of an often used sequence of commands. Scripting capabilities are now built into many applications, such as Excel. Whether one uses a programming or scripting language, it is important to pay attention to the kinds of instructional tasks that fit well with the language. Tasks that are appropriate to a function-based language such as ISETL, would not be so in other languages that do not operate the same way.

Finally, it was noted that programming can also play a large role in introducing students to the world of algorithms and the concomitant notions of complexity, validity, and efficiency.

KEN HOUSTON

ASSESSING UNDERGRADUATE MATHEMATICS STUDENTS

1. INTRODUCTION

Any discussion of assessment must necessarily include a discussion of the curriculum, how it is designed and organised, and what it contains. It must examine the aims of the course that students are taking, and the objectives set for that course and the individual modules that comprise the course. (Here I am using terminology common in the UK. The 'course' students take is 'the whole thing', the 'programme'. A course in this sense consists of 'modules' or 'units', commonly called 'courses' in the USA, so beware of confusion!) The discussion must consider who is doing the assessing, why they are doing it, what they are doing and how it is being done. It must consider how assessors become assessors and how those assessed are prepared for assessment. And it must consider if the assessment is valid and consistent, and if it is seen to be so.

It might also be useful at this stage to define what we mean by a 'mathematician'. There is a real sense in which almost everyone could be described as a mathematician in that they make use of some aspect of mathematics – be it only arithmetic or other things learnt at primary/elementary school. The term could be used of those who have taken a first degree in mathematics and who use it in their employment. Or it could be reserved only for those who have a PhD and who are doing research in pure mathematics or an application of mathematics. We will use the middle of the road term. In other words, a mathematician will be one who has studied the subject at least to bachelors degree standard (and of course that varies across the world!), and who is using some aspect of advanced mathematics in their work. Such people could join a professional or learned society such as the UK based Institute of Mathematics and its Applications. So we are primarily concerned with the higher education of these people who can rightly be considered to be professional mathematicians. But also there are many disciplines wherein mathematics is an extensive and substantial component of study. Examples are physics or electronic engineering. The mathematical education of professionals in such fields as these could also come under the remit of this article in that many of the suggestions made could enhance the teaching, learning and assessment of students in these fields.

Traditionally assessment in higher education was solely summative and consisted of one or more time-constrained, unseen, written examination papers per module. A typical, and in some places predominant, purpose of assessment was to put students in what was believed to be rank order of ability. Students were, perhaps,

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asked to prove a theorem or to apply a result, or to see if they could solve some previously unseen problem. Generally this method succeeded in putting students in a rank order and in labelling them excellent, above average, below average or fail. But was it rank order of ability in mathematics or rank order of ability to perform well in time-constrained, unseen, written examination papers? Sadly it was the latter, and while the two may coincide, this is not guaranteed. Taking time-constrained, unseen, written examination papers is a rite of passage, which students will never have to do again after graduation and which bears little relationship to the ways in which mathematicians work. While it is true that working mathematicians are sometimes under pressure to produce results to a deadline, the whole concept of time-constrained, unseen, written examinations is somewhat artificial and unrelated to working life.

It is in this context that people started to think about change, change in the way courses are designed and organised, change in the way course and module objectives are specified and change in the way students are assessed and in the way the outcomes of assessment are reported. It is usually the case that 'what you assess is what you get', that is, the assessment instruments used determine the nature of the teaching and the nature of the learning. Learning mathematics for the principal purpose of passing examinations often leads to surface learning, to memory learning alone, to learning that can only see small parts and not the whole of a subject, to learning wherein many of the skills and much of the knowledge required to be a working mathematician are overlooked. In time-constrained, unseen, written examinations no problem can be set that takes longer to solve than the time available for the examination. There are no opportunities for discussion, for research, for reflection or for using computer technology. Since these are important aspects of the working mathematician's life, it seems a pity to ignore them. And it seems a pity to leave out the possibilities for deep learning of the subject, that is, learning which is consolidated, learning which will be retained because it connects with previous learning, learning which develops curiosity and a thirst for more, learning which is demonstrably useful in working life.

This is, of course, a caricature of 'traditional' assessment, but it is not too far from the truth, and it brings out the reasons why some people in some societies became unhappy with university and college education. Consequently those who educate students now pay attention to stating aims and objectives, to redesigning curricula and structures and to devising assessment methods which promote the learning we want to happen and which measure the extent to which it has happened. And they pay attention to the need to convince students and funding bodies that they are getting good value for their investment of time and money.

The discussion on course design and assessment is also tied up with the discussion on 'graduateness'. What is it that characterises college or university graduates and distinguishes them from those who are not? Is it just superior knowledge of a particular topic, or is it more than that? It is, of course, more than that. It is not easy to define or even to describe, but it has to do with an outlook on life, a way of dealing with problems and situations, and a way of interacting with other people. (This is not to denigrate the learning that non-college graduates get from 'the university of life', nor to suggest that they are inferior as people. It is to do with considering the 'added value' of college or university education.) Traditionally graduateness was absorbed, simply through the university experience, but now that we have systems of mass education in many countries of the world, we need to pay attention to the development of graduate attributes in students so that they do, indeed, get value for money. In many instances, and mathematics is no exception, it is the 'more than' that is important when it comes to finding and keeping employment. Subject knowledge is important but so also are personal attributes. It is highly desirable that students develop what have come to be known as 'key skills' while they are undergraduates, and not just because employers are saying that the graduates they employ are weak in this area. Innovative mathematics curricula seek to do this by embedding the development of key skills in their teaching and learning structures. (Key skills are often described as employability skills or transferable skills. They include such skills as written, oral and visual communication, time management, group-work and team-work, critical reflection and self assessment, and computer and IT, and aural skills.)

Who are the stakeholders in an undergraduate's education? First and foremost are the students themselves. They are investing time and effort and they want to know that they are getting a return on this investment. Most of them realise that it is not enough for them to be given a grade; they know that they have to earn it. So they need to know what performance standards are required and they need to be able to recognise within themselves whether they have achieved these standards or not. This raises the question of self-assessment and ways of promoting self-assessment. Giving 'grades that count' is one way of encouraging students to carry out tasks.

The next stakeholder to consider are the teachers. It is their job to enable learning and so they need to know what learning has taken place. Financial sponsors of students are also stakeholders. They, too, want to know if they are getting a good return on their investment. Finally, in the stakeholder debate, there is a demand from society, students themselves, universities, prospective employers, that students be summatively assessed, ranked and labelled in such a way that they may be measured, not just against what they are supposed to have learned, but also against their peers across the world.

This chapter will consider all of these features, but will focus on assessment, as that is its theme. It will look at the purposes and principles of assessment and then it will move on to consider the aims and objectives of courses and modules. Innovative methods of assessment will be reviewed and discussed, and this will be the biggest part of the chapter. Ways of disseminating information about new assessment practices will be discussed, as will obstacles to change. Finally pertinent research issues will be mentioned. The chapter will close with an annotated bibliography of pertinent books and papers dealing with these issues.

2. PRINCIPLES AND PURPOSES OF ASSESSMENT

Perhaps the only principle that should be applied is 'fitness for purpose'. To achieve this, assessment methods should be intimately related to the Aims and Objectives of the Module under consideration. And it should be born in mind that the assessment methods used will influence the learning behaviour of students to a considerable extent.

There are a number of purposes of assessment that should be considered:

1. to inform learners about their own learning.
2. to inform teachers of the strengths and weaknesses of the learners and of themselves so that appropriate teaching strategies can be adopted.
3. to inform other stakeholders – society, funders, employers including the next educational stage.
4. to encourage learners to take a critical-reflective approach to everything that they do, that is, to self assess before submitting.
5. to provide a summative evaluation of achievement.

3. AIMS AND OBJECTIVES

Aims and objectives should be established both for a course and for each of the modules that comprise the course. The aims of a course are statements that identify the broad educational purposes of the course and may refer to the ways in which it addresses the needs of the stakeholders. Here are some examples; there are, of course, many more and each provider must write their own:

1. To provide a broad education in mathematics, statistics and computing for students who have demonstrated that they have the ability or who are considered to have the potential to benefit from the course.
2. To develop knowledge, understanding and experience of the theory, practice and application of selected areas of mathematics, statistics, operations research and computing so that graduates are able to use the skills and techniques of these areas to solve problems arising in industry, commerce and the public sector.
3. To develop key skills.
4. To provide students with an intellectual challenge and the practical skills to respond appropriately to further developments and situations in their careers.
5. To prepare students for the possibility of further study at post graduate level, including a PhD programme or a teacher training programme.

It would be necessary to indicate how each of the modules selected for a course helps to achieve the aims of the course. The aims of the individual modules should 'map' to the overall aims of the course. Objectives are statements of the intended learning outcomes that would demonstrate successful completion of the course or module, and that would warrant progression through the course and the eventual award of a degree. Module objectives should identify the knowledge, skills and attributes developed by a module, and course objectives should identify the knowledge, skills and attributes developed by the totality of modules selected for the course. Objectives may include reference to subject knowledge and understanding, cognitive skills, practical skills and key skills. They should be clearly relevant to fulfilling the aims and, above all, they should be assessable, that is, we should be able to devise assessment instruments that allow students to demonstrate that they have achieved the learning intended, and, if appropriate, to what extent. Here are some examples of course objectives: -

On completion of their studies graduates will have:

1. an understanding of the principles, techniques and applications of selected areas of mathematics, statistics, operations research and computing.
2. the ability and confidence to analyse and solve problems both of a routine and of a less obvious nature.
3. the ability and confidence to construct and use mathematical models of systems or situations, employing a creative and critical approach.
4. effective communication skills using a variety of media.
5. effective teamwork skills.

A course document should demonstrate how the aims and objectives of the constituent modules contribute to the overall course aims and objectives. Here is an example of the aims and objectives of a module, taken from an introductory module on mathematical modelling. (These aims and objectives are those of module MAT112J2, University of Ulster. Full details may be read under 'Syllabus Outline' at http://www.infj.ulst.ac.uk/cdmx23/mat112j2.html.) Note that an indication of the method of assessment of each objective is given.

Aims: The aims of this module are to:

1. enable students to understand the modelling process, to formulate appropriate mathematical models and to appreciate their limitations.
2. develop an understanding of mathematical methods and their role in modelling.
3. study a number of mathematical models.
4. develop in students a range of key skills.

It can be seen how these module aims help to meet the aims of the course listed above. Thus this module contributes to developing mathematical understanding, problem solving, and key skills.

Objectives: On completion of this module, students should be able to:

1. Formulate mathematical models and use them to solve problems of an appropriate level. (Assessed by coursework and written examination.)
2. Solve simple differential equations using calculus and computer algebra systems. (Assessed by written examination.)
3. Describe and criticise some mathematical models. (Assessed by coursework.)
4. Work in groups and report their work in a variety of media. (Assessed by coursework.)
5. Work both independently and in support of one another. (Assessed by coursework.)
6. Demonstrate other key skills. (Assessed by coursework.)