Preface

The theory of Eisenstein series, in the general form given to it by Robert Langlands some forty years ago, has been an important and incredibly useful tool in the fields of automorphic forms, representation theory, number theory and arithmetic geometry. For example, the theory of automorphic L-functions arises out of the calculation of the constant terms of Eisenstein series along parabolic subgroups. Not surprisingly, the two primary approaches to the analytic properties of automorphic L-functions, namely the Langlands–Shahidi method and the Rankin–Selberg method, both rely on the theory of Eisenstein series. In representation theory, Eisenstein series were originally studied by Langlands in order to give the spectral decomposition of the space of L^2-functions of locally symmetric spaces attached to adelic groups. This spectral theory has been used to prove the unitarity of certain local representations. Finally, on the more arithmetic side, the Fourier coefficients of Eisenstein series contain a wealth of arithmetic information which is far from being completely understood. The p-divisibility properties of these coefficients, for example, are instrumental in the construction of p-adic L-functions.

In short, the theory of Eisenstein series seems to have, hidden within it, an inexhaustible number of treasures waiting to be discovered and mined.

With such diverse applications, it is not easy even for the conscientious researcher to keep abreast of current developments. Indeed, different users of Eisenstein series often focus on different aspects of the theory. With this in mind, the workshop "Eisenstein Series and Applications" was held at the American Institute of Mathematics (Palo Alto) from August 15 to 19, 2005. The goal of the workshop was to bring together users of Eisenstein series from different areas who do not normally interact with each other, with the hope that such a juxtaposition of perspectives would provide deeper insight into the arithmetic of Eisenstein series and foster fruitful new collaborations.

This volume contains a collection of articles related to the theme of the workshop. Some, but not all of them, are based on lectures given in the workshop. We hope that the articles assembled here will be useful to a diverse audience and especially to students who are just entering the field.

We would like to take this opportunity to thank all the participants of the workshop for their enthusiastic participation, and the authors who contributed articles to this volume for their efforts and timely submissions, as well as, all the referees who gave the articles their thoughtful considerations. We are grateful to the American Institute of Mathematics and the National Science Foundation for providing generous support, and especially to Brian Conrey, David Farmer and Helen Moore of AIM for their invaluable assistance in the organization of the workshop.

We find it appropriate to dedicate this volume to Robert Langlands, who started it all, on the occasion of his seventieth birthday.

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A Topological Model for Some Summand of the Eisenstein Cohomology of Congruence Subgroups

Jens Franke

Universität Bonn, Beringstraße 1, 53115 Bonn, Germany
E-mail: franke@rhein.iam.uni-bonn.de

Summary. We construct a topological model for certain Eisenstein cohomology of congruence subgroups.

1 Introduction

Let $\mathcal{G}$ be a semisimple algebraic group over $\mathbb{Q}$, let $\mathcal{G}(\mathbb{Q})$ and $\mathcal{G}(\mathbb{A})$ be its rational and adelic groups, and let $\boldsymbol{K} \subset \mathcal{G}(\mathbb{A})$ be a good maximal compact subgroup. Let $\boldsymbol{K} = \boldsymbol{K}_f \boldsymbol{K}_\infty$ with $\boldsymbol{K}_\infty \subset \mathcal{G}(\mathbb{R})$ and $\boldsymbol{K}_f \subset \mathcal{G}(\mathbb{A}_f)$, where $\mathcal{G}(\mathbb{A}_f)$ is the finite adelic group and $\mathcal{G}(\mathbb{R})$ is the group of real points. By our assumption on $\mathcal{G}$, we know that $\mathcal{G}(\mathbb{R})$ and $\boldsymbol{K}_\infty$ are connected Lie groups (cf. Proposition 2.1 below). Then the cohomology of the congruence subgroup $\Gamma = \mathcal{G}(\mathbb{Q}) \cap \boldsymbol{K}_f$ can be computed by

$$H^*(\Gamma, \mathbb{C}) = H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}_\infty, \mathbb{C}\right)^{\boldsymbol{K}_f}, \tag{1.1}$$

where the superscript $\boldsymbol{K}_f$ stands for the subspace of $\boldsymbol{K}_f$-invariants in the $\mathcal{G}(\mathbb{A}_f)$-module

$$H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}_\infty, \mathbb{C}\right) :\, = \operatorname*{colim}_{\boldsymbol{K}^f} H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}^f\boldsymbol{K}_\infty, \mathbb{C}\right). \tag{1.2}$$

The inductive limit is over all open subgroups $\boldsymbol{K}^f \subseteq \boldsymbol{K}_f$. It is clear from definition 1.1 that the Hecke algebra $\mathfrak{H} = C_c^\infty\left(\boldsymbol{K}_f\backslash\mathcal{G}(\mathbb{A}_f)/\boldsymbol{K}_f\right)$ of compactly supported $\boldsymbol{K}_f$-biinvariant functions on $\mathcal{G}(\mathbb{A}_f)$ acts on $H^*(\Gamma, \mathbb{C})$. Let

$$\mathcal{I} :\, = \left\{ f \in \mathfrak{H} = C_c^\infty(\boldsymbol{K}_f\backslash\mathcal{G}(\mathbb{A}_f)/\boldsymbol{K}_f) \,\middle|\, \int_{\mathcal{G}(\mathbb{A}_f)} f(g)\,dg \right\} = 0$$

be the ideal of elements of $\mathfrak{H}$ which act trivially on the constant representation. Since $H^*(\Gamma, \mathbb{C})$ is a finite dimensional vector space, any element of $H^*(\Gamma, \mathbb{C})$ is annihilated by a finite power of an ideal of finite codimension in $\mathfrak{H}$. Therefore, the subspace

$$H^*(\Gamma,\mathbb{C})_{\mathcal{I}} = \{x \in H^*(\Gamma,\mathbb{C})\,|\mathcal{I}^n x = \{0\} \text{ for some } n > 0\}$$

is a direct summand of $H^*(\Gamma,\mathbb{C})$ which, among other elements, contains the constant cohomology class in dimension zero. One of the aims of this article is to study the space $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$.

Our main result gives a topological model for $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$. We first recall the topological model for the cohomology of the constant representation of $\mathcal{G}(\mathbb{R})$, which maps to $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$. Let $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$ be the algebra of $\mathcal{G}(\mathbb{R})$-invariant differential forms on the symmetric space $\mathcal{G}(\mathbb{R})/\boldsymbol{K}_\infty$. Such forms are closed and give rise to $\mathcal{G}(\mathbb{A}_f)$-invariant elements in $H^*\left(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}_\infty,\mathbb{C}\right)$. We get a map of graded vector spaces

$$I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty} \to H^*(\Gamma,\mathbb{C})_{\mathcal{I}}.$$

Furthermore, $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$ is a $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$-module since multiplication by $\mathcal{G}(\mathbb{A}_f)$-invariant cohomology classes, unlike the rest of the multiplicative structure, commutes with the action of the Hecke algebra. Let $\mathcal{G}^{(c)}(\mathbb{R}) \subset \mathcal{G}(\mathbb{C})$ be a compact form of $\mathcal{G}(\mathbb{R})$ such that $\boldsymbol{K}_\infty \subset \mathcal{G}^{(c)}(\mathbb{R})$. Then the homogeneous space $\boldsymbol{X}^{(c)}_{\mathcal{G}} :\, = \mathcal{G}^{(c)}(\mathbb{R})/\boldsymbol{K}_\infty$ is the compact dual of $\mathcal{G}(\mathbb{R})/\boldsymbol{K}_\infty$. The complexified tangent spaces at the origins of $\boldsymbol{X}^{(c)}_{\mathcal{G}}$ and of $\mathcal{G}(\mathbb{R})/\boldsymbol{K}_\infty$ can be identified, and one gets an identification of $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$ with the space of $\mathcal{G}^{(c)}(\mathbb{R})$-invariant forms on $\boldsymbol{X}^{(c)}_{\mathcal{G}}$. The space of $\mathcal{G}^{(c)}(\mathbb{R})$-invariant forms on $\boldsymbol{X}^{(c)}_{\mathcal{G}}$ is equal to the space of harmonic forms (with respect to a $\mathcal{G}^{(c)}(\mathbb{R})$-invariant metric) on $\boldsymbol{X}^{(c)}_{\mathcal{G}}/\boldsymbol{K}_\infty$, hence it is isomorphic to $H^*(\boldsymbol{X}^{(c)}_{\mathcal{G}},\mathbb{C})$. We obtain a multiplicative isomorphism between $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$ and $H^*(\boldsymbol{X}^{(c)}_{\mathcal{G}},\mathbb{C})^{\pi_0(\boldsymbol{K}_\infty)}$.

Our topological model for $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$ consists of a canonical isomorphism of $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{G}},\mathbb{C})$-modules from $H^*(\Gamma,\mathbb{C})_{\mathcal{I}}$ onto the invariants of a certain group in $H^*(\boldsymbol{U}_{\mathcal{G}},\mathbb{C})$, where $\boldsymbol{U}_{\mathcal{G}} \subset \boldsymbol{X}^{(c)}_{\mathcal{G}}$ is a certain open subset. To give the definition of $\boldsymbol{U}_{\mathcal{G}}$, we first have to introduce some new notation. Let $\mathcal{P}_o$ be a minimal $\mathbb{Q}$-rational parabolic subgroup of $\mathcal{G}$. We consider standard parabolic subgroups $\mathcal{P} \supseteq \mathcal{P}_o$. Let $\mathcal{N}_{\mathcal{P}} \subset \mathcal{P}$ be the radical of $\mathcal{P}$ and let $\mathcal{L}_{\mathcal{P}} = \mathcal{P}/\mathcal{N}_{\mathcal{P}}$. Let

$$\mathcal{M}_{\mathcal{P}} :\, = \Big(\bigcap_{\chi\in X^*(\mathcal{L}_{\mathcal{P}})} \ker(\chi)\Big)^o \tag{1.3}$$

be the connected component of the intersection of the kernels of all $\mathbb{Q}$-rational characters of $\mathcal{L}_{\mathcal{P}}$. To ensure that our constructions do not depend on such a choice, we will never choose a $\mathbb{Q}$-rational section $\mathcal{L}_{\mathcal{P}} \to \mathcal{P}$ of the canonical projection $\mathcal{P} \to \mathcal{L}_{\mathcal{P}}$. We will, however, use the fact that the projection $\mathcal{P} \cap \theta(\mathcal{P}) \to \mathcal{L}_{\mathcal{P}}$, where θ is the Cartan involution defined by $\boldsymbol{K}_\infty$, is an isomorphism of algebraic groups over $\mathbb{R}$. This identifies $\mathcal{L}_{\mathcal{P}}(\mathbb{R})$ and $\mathcal{L}_{\mathcal{P}}(\mathbb{C})$ with subgroups of $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}(\mathbb{C})$. Using this identification, the compact form of $\mathcal{M}_{\mathcal{P}}$ becomes

$$\mathcal{M}_{\mathcal{P}}^{(c)}(\mathbb{R}) = \mathcal{M}_{\mathcal{P}}(\mathbb{C}) \cap \mathcal{G}^{(c)}(\mathbb{R}),$$

a subgroup of the compact form of $\mathcal{G}$, and the compact dual of the symmetric space defined by $\mathcal{M}_{\mathcal{P}}$ becomes

$$\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)} = \mathcal{M}_{\mathcal{P}}^{(c)}(\mathbb{R})/(K_\infty \cap \mathcal{M}_{\mathcal{P}}(\mathbb{R})) \subset \boldsymbol{X}_{\mathcal{G}}^{(c)},$$

a subset of the compact dual of the symmetric space defined by $\mathcal{G}$. We put

$$\boldsymbol{U}_{\mathcal{G}} \colon = \boldsymbol{X}_{\mathcal{G}}^{(c)} - \bigcup_{\mathcal{P} \supseteq \mathcal{P}_o} \boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}. \tag{1.4}$$

The group $\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})$ acts on $\boldsymbol{X}_{\mathcal{G}}^{(c)}$ by left translations and leaves $\boldsymbol{U}_{\mathcal{G}}$ invariant. The action of $\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})$ on the cohomology of $\boldsymbol{X}_{\mathcal{G}}^{(c)}$ is trivial, the action on the cohomology of $\boldsymbol{U}_{\mathcal{G}}$ factorizes over the finite group of connected components $\pi_0\big(\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})\big)$. With these definitions, we can formulate our main result about $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$.

Theorem 1.1. *There is a canonical isomorphism of* $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty} \cong H^*(\boldsymbol{X}_{\mathcal{G}}^{(c)}, \mathbb{C})$*-modules*

$$H^*(\Gamma, \mathbb{C})_{\mathcal{I}} \cong H^*(\boldsymbol{U}_{\mathcal{G}}, \mathbb{C})^{\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})}. \tag{1.5}$$

Furthermore, elements of $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$*, which by definition are annihilated by some power of* $\mathcal{I} \subset \mathfrak{H}$*, are already annihilated by* $\mathcal{I}$ *itself.*

The map $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty} \to H^*(\Gamma, \mathbb{C})$ was first studied by Borel [2], who proved that it is an isomorphism in low dimension. Since $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$ is a direct summand of $H^*(\Gamma, \mathbb{C})$, the question of noninjectivity of Borel's map (which was studied by Speh [22]), can be understood in terms of restriction of cohomology classes from $\boldsymbol{X}_{\mathcal{G}}^{(c)}$ to $\boldsymbol{U}_{\mathcal{G}}$. Our interest in this particular summand was, however, motivated by the fact that it is an important model case for the effects produced by the singularities of Eisenstein series when one studies the cohomology of congruence subgroups in terms of automorphic forms. Our method of studying $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$ uses the results of [9]. It consists of expressing $H^*(\Gamma, \mathbb{C})_{\mathcal{I}}$ as the $(\mathfrak{g}, K)$-cohomology of a direct summand of the space of automorphic forms and of representing this space in terms of Eisenstein series. The Eisenstein series which are of interest are the Eisenstein series starting from the constant functions on the Levi components of standard parabolic subgroups, evaluated at one half the sum of the positive roots. There are many singular hyperplanes which go through this parameter, and the iterated residue of the Eisenstein series is the constant function on $\mathcal{G}(\mathbb{A})$. The contributions from the Eisenstein series starting from a given parabolic subgroup is therefore no direct summand of the space of automorphic forms, but only a quotient of a suitable filtration on the space of automorphic forms. The problem of understanding these extensions was the main motivation for writing this paper. For GL_2 over algebraic number fields, the summand of the cohomology considered in this paper, has been computed by Harder [13, Theorem 4.2.2.]. There are probably more explicit calculations for rank one cases and also some for rank two cases, for instance in [20]. These authors do not use topological models to describe the Eisenstein cohomology, they arrive at explicit formulas.

We can more generally study the $\mathcal{G}(\mathbb{A}_f)$-module of all elements x in the cohomology $H^*(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}_\infty, \mathbb{C})$ which at all but the finitely many ramified places are annihilated by some power of $\mathcal{I}$. Again it turns out that the first power is sufficient. Let $H^*(\mathcal{G})_\mathcal{I}$ be the space of cohomology classes x with that property. Then $H^*(\mathcal{G})_\mathcal{I}$ can be identified with the $\boldsymbol{K}_\infty \cap \mathcal{P}_o(\mathbb{R})$-invariants in the hypercohomology of a complex of sheaves with $\mathcal{G}(\mathbb{A}_f)$-action on $\boldsymbol{X}_\mathcal{G}^{(c)}$. It turns out that the hypercohomology spectral sequence for this complex degenerates, and that the limit filtration can be described in terms of the $\mathcal{G}(\mathbb{A}_f)$-action. However, Hilbert modular forms and SL_3 over imaginary quadratic fields provide easy examples that the limit filtration will usually not split in the category of $\mathcal{G}(\mathbb{A}_f)$-modules. To get a complete picture of $H^*(\mathcal{G})_\mathcal{I}$ as a $\mathcal{G}(\mathbb{A}_f)$-module, one may be forced to carry out the laborious work of explicit calculations for the various families of algebraic groups. As an example, we carry out explicit calculations for SL_n over imaginary quadratic fields. This example shows that while explicit calculations for the various series of classical groups should be possible, the topological model provides a much more vivid picture of the cohomology.

By the work of Moeglin and Waldspurger [17], the residual spectrum of GL_n over a number field is now completely understood. The structure of the residues is quite similar to the case investigated in this paper. Therefore, there is some hope that our methods can be used to completely understand the Eisenstein cohomology of GL_n in terms of the cuspidal cohomological representations. Compared with this paper, one has to expect two difficulties. Firstly, there is the possibility of "overlapping Speh segments". In this case, the structure of the Eisenstein cohomology may depend on whether some automorphic L-function vanishes at the center of the functional equation. This effect was first found by Harder [14, §III] in the case of GL_3 over imaginary fields. As a second complication, the Borel–Serre–Solomon–Tits Theorem 4.2 in this paper will not suffice. One needs a Solomon–Tits type theorem with twisted coefficients, which investigates the cohomology of a complex formed by normalised intertwining operators. I hope that the methods of this paper are flexible enough to extend to this new situation.

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2 Notations

We will study connected reductive linear algebraic groups $\mathcal{G}$ over $\mathbb{Q}$. Let $\boldsymbol{K} = \boldsymbol{K}_f \boldsymbol{K}_\infty$ be a good maximal compact subgroup of $\mathcal{G}(\mathbb{A})$, decomposed into its finite adelic factor $\boldsymbol{K}_f$ and its real factor $\boldsymbol{K}_\infty$. Let θ be the Cartan involution with respect to $\boldsymbol{K}_\infty$, and let $\boldsymbol{K}_\infty^o$ be the connected component of $\boldsymbol{K}_\infty$. We denote by $\mathcal{P}_o$ a fixed minimal $\mathbb{Q}$-rational parabolic subgroup of $\mathcal{G}$. Unless otherwise specified, parabolic subgroups $\mathcal{P}$ will be assumed to be defined over $\mathbb{Q}$ and to be standard with respect to $\mathcal{P}_o$. Let $\mathcal{N}_\mathcal{P}$ be the radical of $\mathcal{P}$, and let $\mathcal{L}_\mathcal{P} = \mathcal{P}/\mathcal{N}_\mathcal{P}$ be the Levi component. Unless $\mathcal{P} = \mathcal{G}$, we will not think of $\mathcal{L}_\mathcal{P}$ as a subgroup of $\mathcal{P}$. We will, however, identify $\mathcal{L} \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{R})$ with the $\mathbb{R}$-rational algebraic subgroup $\mathcal{P} \cap \theta(\mathcal{P})$ of $\mathcal{L}_\mathcal{P}$. Let $\mathcal{A}_\mathcal{P}$ be a maximal $\mathbb{Q}$-split torus in the center of $\mathcal{L}_\mathcal{P}$, and let $\mathcal{M}_\mathcal{P}$ be defined by (1.3), such that $\mathcal{L}_\mathcal{P} = \mathcal{A}_\mathcal{P}\mathcal{M}_\mathcal{P}$ is an isogeny. In the case $\mathcal{P} = \mathcal{P}_o$, we will write $\mathcal{M}_o$, $\mathcal{A}_o$, and $\mathcal{N}_o$ instead of $\mathcal{M}_{\mathcal{P}_o}$, $\mathcal{A}_{\mathcal{P}_o}$, and $\mathcal{N}_{\mathcal{P}_o}$. In the case $\mathcal{P} = \mathcal{G}$, $\mathcal{A}_\mathcal{G}$ is a maximal $\mathbb{Q}$-split torus in the center of $\mathcal{G}$, and $\mathcal{M}_\mathcal{G}$ is generated by the derived group of $\mathcal{G}$ and the $\mathbb{Q}$-anisotropic part of the center of $\mathcal{G}$.

Let $\mathcal{G}(\mathbb{A})$ be the adelic group of $\mathcal{G}$. If S is a subset of the set of valuations of $\mathbb{Q}$, let $\mathcal{G}(\mathbb{A}_S)$ be the restricted product over all places $v \in S$ of the groups $\mathcal{G}(\mathbb{Q}_v)$. In the special case where S is the set of finite primes, this is the finite adelic group $\mathcal{G}(\mathbb{A}_f)$. Let $\boldsymbol{K}_S = \boldsymbol{K} \cap \mathcal{G}(\mathbb{A}_S)$. For a parabolic subgroup $\mathcal{P}$, let $\mathcal{A}_\mathcal{P}(\mathbb{R})^+$ be the connected component of the group of real points $\mathcal{A}_\mathcal{P}(\mathbb{R})$. In the special case $\mathcal{P} = \mathcal{G}$, this is the connected component of the group of real points of a maximal $\mathbb{Q}$-split torus in the center of $\mathcal{G}$.

Let $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}(\mathbb{R})$, $\mathfrak{U}(\mathfrak{g})$ its universal enveloping algebra, and $\mathfrak{Z}(\mathfrak{g})$ the center of $\mathfrak{U}(\mathfrak{g})$. Similar notations will be used for the Lie algebras of other groups.

Let $\mathfrak{a}_\mathcal{P}$ be the Lie algebra of $\mathcal{A}_\mathcal{P}(\mathbb{R})$. We will write $\mathfrak{a}_o$ for $\mathfrak{a}_{\mathcal{P}_o}$. If $\mathcal{P} \subseteq \mathcal{Q}$, then it is possible to choose a section $i_\mathcal{Q} \colon \mathcal{L}_\mathcal{Q} \to \mathcal{Q}$ of the projection $\mathcal{Q} \to \mathcal{L}_\mathcal{Q}$. Then $i_\mathcal{Q}\big(\mathrm{pr}_{\mathcal{Q}\to\mathcal{L}_\mathcal{Q}}(\mathcal{P})\big) \subset \mathcal{P}$. We define an embedding $\mathfrak{a}_\mathcal{Q} \to \mathfrak{a}_\mathcal{P}$ as the restriction to $\mathfrak{a}_\mathcal{Q}$ of the differential of the map

$$\mathrm{pr}_{\mathcal{P}\to\mathcal{L}_\mathcal{P}} i_\mathcal{Q}.$$

This embedding is independent of the choice of $i_\mathcal{Q}$. The dual space $\check{\mathfrak{a}}_\mathcal{P}$ of $\mathfrak{a}_\mathcal{P}$ can be identified with the real vector space $X^*(\mathcal{P}) \otimes_\mathbb{Z} \mathbb{R}$ generated by the group of $\mathbb{Q}$-rational characters of $\mathcal{P}$. The same identification can be made for $\mathcal{Q}$. Then restriction of characters from $\mathcal{Q}$ to $\mathcal{P}$ defines an embedding $\check{\mathfrak{a}}_\mathcal{Q} \to$

$\check{\mathfrak{a}}_{\mathcal{P}}$. The embeddings $\mathfrak{a}_{\mathcal{Q}} \to \mathfrak{a}_{\mathcal{P}}$ and $\check{\mathfrak{a}}_{\mathcal{Q}} \to \check{\mathfrak{a}}_{\mathcal{P}}$ define canonical direct sum decompositions $\mathfrak{a}_{\mathcal{P}} = \mathfrak{a}_{\mathcal{Q}} \oplus \mathfrak{a}_{\mathcal{P}}^{\mathcal{Q}}$ and $\check{\mathfrak{a}}_{\mathcal{P}} = \check{\mathfrak{a}}_{\mathcal{Q}} \oplus \check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}}$.

Let $\Delta_o \subset \check{\mathfrak{a}}_o^{\mathcal{G}}$ be the set of simple positive (with respect to $\mathcal{P}_o$) roots of $\mathcal{A}_o$. The subset $\Delta_o^{\mathcal{P}}$ of simple positive roots which occur in the Lie algebra of $\mathcal{M}_{\mathcal{P}}$ is contained in $\check{\mathfrak{a}}_o^{\mathcal{P}}$. Of course, both definitions require the choice of sections $\mathcal{L}_o \to \mathcal{L}_{\mathcal{P}} \to \mathcal{P}$, but the result does not depend on such a choice. Let $\Delta_{\mathcal{P}}$ be the projection of $\Delta_o - \Delta_o^{\mathcal{P}}$ to $\check{\mathfrak{a}}_{\mathcal{P}}$, and let $\Delta_{\mathcal{P}}^{\mathcal{Q}}$ for $\mathcal{P} \subseteq \mathcal{Q}$ be the projection of $\Delta_o^{\mathcal{Q}} - \Delta_o^{\mathcal{P}}$ to $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}}$. Let $\rho_o \in \check{\mathfrak{a}}_o$ be one half the sum of the positive roots of $\mathcal{A}_o$, and let $\rho_{\mathcal{P}}$ and $\rho_{\mathcal{P}}^{\mathcal{Q}}$ be the projections of ρ_o to $\mathfrak{a}_{\mathcal{P}}$ and to $\mathfrak{a}_{\mathcal{P}}^{\mathcal{Q}}$.

Our notion of a $(\mathfrak{g}, K)$-module is the same as in [26, §6.1]. A $\mathcal{G}(\mathbb{A}_f)$-module is a vector space on which $\mathcal{G}(\mathbb{A}_f)$ acts with open stabilizers. If $\mathbb{K}$ is a field, let $C_c^\infty(\mathcal{G}(\mathbb{A}_f), \mathbb{K})$ be the $\mathcal{G}(\mathbb{A}_f)$-module of compactly supported locally constant $\mathbb{K}$-valued functions on $\mathcal{G}(\mathbb{A}_f)$. If no field is given, it is assumed that $\mathbb{K} = \mathbb{C}$. A similar notation is used for quotients of the adelic group. For quotients of the full adelic group like $C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$ or similar quotients of partial adelic groups which contain $\mathcal{G}(\mathbb{R})$, we adopt the conditions that C^∞-functions have to be locally constant with respect to the finite adelic part and $\boldsymbol{K}_\infty$-finite and infinitely often differentiable with respect to $\mathcal{G}(\mathbb{R})$.

2.1 Connected components of real groups

Let us recall the following fact:

Proposition 2.1. *Let $\mathcal{G}$ be a reductive connected algebraic group over $\mathbb{R}$ and let $\boldsymbol{K}_\infty$ be a maximal compact subgroup of $\mathcal{G}(\mathbb{R})$.*

1. *Then $\pi_0\big(\boldsymbol{K}_\infty\big) \cong \pi_0\big(\mathcal{G}(\mathbb{R})\big)$.*
2. *If $\mathcal{R} \subset \mathcal{Q}$ are parabolic subgroups defined over $\mathbb{R}$, then the map*
$$\pi_0\big(\mathcal{R}(\mathbb{R}) \cap \boldsymbol{K}_\infty\big) \to \pi_0\big(\mathcal{Q}(\mathbb{R}) \cap \boldsymbol{K}_\infty\big)$$
is surjective.
3. *If $\mathcal{G}$ is $\mathbb{R}$-anisotropic or if it is semisimple and simply connected, then $\mathcal{G}(\mathbb{R})$ is connected.*

Proof. The first two assertions are consequences of the Iwasawa decomposition $\mathcal{G}(\mathbb{R}) \cong \mathcal{P}(\mathbb{R})^o \times \boldsymbol{K}_\infty$, where $\mathcal{P}$ is a minimal $\mathbb{R}$-parabolic subgroup (cf. [23, Proposition 5.15]). The third fact is established in [6, Corollaire 4.7] for semisimple simply connected groups and in [5, Corollaire 14.5] for anisotropic groups. □

3 Formulation of the main results

Let $H^*(\mathcal{G})$ be the inductive limit

$$H^*(\mathcal{G}) \colon = \operatorname*{colim}_{\boldsymbol{K}^f} H^*\left(\mathcal{G}(\mathbb{Q})\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\backslash\mathcal{G}(\mathbb{A})/\boldsymbol{K}^f\boldsymbol{K}_\infty^o, \mathbb{C}\right) \tag{3.1}$$

over all sufficiently small compact open subgroups $\boldsymbol{K}^f \subset \mathcal{G}(\mathbb{A}_f)$. This is a $\mathcal{G}(\mathbb{A}_f)$-module. Let $H_c^*(\mathcal{G}, \mathbb{C})$ be the same inductive limit over the cohomology with compact support. For any set of finite primes S, the Hecke algebra $\mathfrak{H}_S = C_c^\infty(\boldsymbol{K}_S \backslash \mathcal{G}(\mathbb{A}_S)/\boldsymbol{K}_S)$ of $\boldsymbol{K}_S$-bi-invariant compactly supported functions on $\mathcal{G}(\mathbb{A}_S)$ acts on $H^*(\mathcal{G}, \mathbb{C})$ and $H_c^*(\mathcal{G}, \mathbb{C})$. Let $\mathcal{I}_S$ be the ideal

$$\mathcal{I}_S \colon = \left\{ f \in \mathfrak{H}_S \;\middle|\; \int_{\mathcal{G}(\mathbb{A}_S)} f(g)\, dg = 0 \right\},$$

and let

$$\begin{aligned} &H^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}} \colon = \\ &\{x \in H^*(\mathcal{G}, \mathbb{C}) \,|\text{for any set } S \text{ of finite primes, } \mathcal{I}_S^m x = \{0\} \text{ for } m \gg 0\} \\ &H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}} \colon = \\ &\{x \in H_c^*(\mathcal{G}, \mathbb{C}) \,|\text{for any set } S \text{ of finite primes, } \mathcal{I}_S^m x = \{0\} \text{ for } m \gg 0\}. \end{aligned}$$

These are direct summands of $H^*(\mathcal{G}, \mathbb{C})$ and $H_c^*(\mathcal{G}, \mathbb{C})$. Our main result describes them as the space of $\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R})$-invariants in the hypercohomology of a complex of sheaves of $\mathcal{G}(\mathbb{A}_f)$-modules on the compact dual.

The construction of these complexes of sheaves follows a general pattern, which associates a chain complex to a functor with values in an abelian category on the poset $\mathfrak{P}_{\mathcal{G}}$ of standard parabolic subgroups. Note that $\mathcal{G}$ is a maximal element of $\mathfrak{P}_{\mathcal{G}}$. Let $\prec$ be a total order on Δ_o. We order successors $\mathcal{Q}$ of $\mathcal{P}$ in $\mathfrak{P}_{\mathcal{G}}$ by the order $\prec$ of the unique element of $\Delta_o^{\mathcal{Q}} - \Delta_o^{\mathcal{P}}$ and denote the i-th successor $(0 \le i < \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}})$ of $\mathcal{P}$ by $\mathcal{P}_i$. Let $\boldsymbol{F}^{\mathcal{P}}$ be a contravariant functor on $\mathfrak{P}_{\mathcal{G}}$. For $\mathcal{P} \subseteq \mathcal{Q}$, let

$$\boldsymbol{F}^{\mathcal{P} \subseteq \mathcal{Q}} \colon \boldsymbol{F}^{\mathcal{Q}} \to \boldsymbol{F}^{\mathcal{P}}$$

be the transition map. We define the chain complex $C^*(\boldsymbol{F}^\bullet)$ by

$$C^k(\boldsymbol{F}^\bullet) = \bigoplus_{\substack{\mathcal{P} \in \mathfrak{P} \\ \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}} = k}} \boldsymbol{F}^{\mathcal{P}}$$

with the differential

$$d\left((f_{\mathcal{P}})_{\substack{\mathcal{P} \in \mathfrak{P} \\ \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}} = k}} \right) = \left(\sum_{i=0}^{k} (-1)^i \boldsymbol{F}^{\mathcal{Q} \subset \mathcal{Q}_i}(f_{\mathcal{Q}_i}) \right)_{\substack{\mathcal{Q} \in \mathfrak{P} \\ \dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} = k+1}}. \tag{3.2}$$

Similarly, let $\boldsymbol{F}_{\mathcal{P}}$ be covariant, with transition maps

$$\boldsymbol{F}_{\mathcal{P} \subseteq \mathcal{Q}} \colon \boldsymbol{F}_{\mathcal{P}} \to \boldsymbol{F}_{\mathcal{Q}}.$$

We order predecessors $\mathcal{Q}$ of $\mathcal{P}$ in $\mathfrak{P}$ according to the order by $\prec$ of the unique element of $\Delta_o^{\mathcal{P}} - \Delta_o^{\mathcal{Q}}$, denote the i-th predecessor $(0 \le i < \dim \mathfrak{a}_o^{\mathcal{P}})$ by ${}_i\mathcal{P}$ and form the chain complex

$$C^k(\boldsymbol{F}_\bullet) = \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}_o^{\mathcal{P}}=k}} \boldsymbol{F}_{\mathcal{P}}$$

with differential

$$d\left((f_{\mathcal{P}})_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}_o^{\mathcal{P}}=k}}\right) = \left(\sum_{i=0}^{k}(-1)^i \boldsymbol{F}_{{}_i\mathcal{Q}\subset\mathcal{Q}}(f_{{}_i\mathcal{Q}})\right)_{\substack{\mathcal{Q}\in\mathfrak{P} \\ \dim \mathfrak{a}_o^{\mathcal{Q}}=k+1}} . \tag{3.3}$$

We apply similar conventions to functors of several variables. For instance, if $\boldsymbol{F}_{\mathcal{P}}^{\mathcal{Q}}$ is covariant with respect to $\mathcal{P}$ and contravariant with respect to $\mathcal{Q}$, then we have the following chain complexes:

- For fixed $\mathcal{P}$, the chain complex $C^*(\boldsymbol{F}_{\mathcal{P}}^{\bullet})$ obtained by applying construction 3.2 to the contravariant variable.
- For fixed $\mathcal{Q}$, the chain complex $C^*(\boldsymbol{F}_{\bullet}^{\mathcal{Q}})$ obtained by applying construction 3.3 to the covariant variable.
- The chain complex $C^*(\boldsymbol{F}_{\bullet}^{\bullet})$, the total complex of the double complex, obtained by applying (3.2) to the contravariant variable and (3.3) to the covariant variable.

Of course, all these complexes depend on the choice of $\prec$. However, they do so only up to unique isomorphism. For instance, let $C^*(\boldsymbol{F}^{\mathcal{P}})_{\prec}$ be formed with respect to $\prec$ and let $C^*(\boldsymbol{F}^{\mathcal{P}})_{\tilde{\prec}}$ be formed with respect to $\tilde{\prec}$. Then we have the isomorphism of complexes

$$\begin{aligned} C^*(\boldsymbol{F}^{\bullet})_{\prec} &\to C^*(\boldsymbol{F}^{\mathcal{P}})_{\tilde{\prec}} \\ (f_{\mathcal{P}})_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=k}} &\to (\varepsilon_{\mathcal{P}} f_{\mathcal{P}})_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=k}} , \end{aligned}$$

where $\varepsilon_{\mathcal{P}}$ is the signature of the permutation of $\Delta_o - \Delta_o^{\mathcal{P}}$ which identifies the total orders $\prec$ and $\tilde{\prec}$ of $\Delta_o - \Delta_o^{\mathcal{P}}$. We will therefore suppress the $\prec$-dependence of $C^*(\boldsymbol{F}^\bullet)$ in our notations. The same applies to $C^*(\boldsymbol{F}_\bullet)$ and the constructions for bifunctors. We will also apply these constructions if $\boldsymbol{F}$ takes values in the category of chain complexes. In this case, $C^*(\boldsymbol{F}^\bullet)$ has the total differential formed by the differential of $\boldsymbol{F}^\bullet$ and (3.2).

Recall the definition of the compact dual $\boldsymbol{X}_{\mathcal{M}_{\mathcal{G}}}^{(c)}$ and of the embeddings $\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)} \to \boldsymbol{X}_{\mathcal{M}_{\mathcal{G}}}^{(c)}$ from the introduction. For a topological space X, a closed subset Y and a vector space V, let V_Y be the constant sheaf with stalk V on Y and let $(i_{Y\subseteq X})_* V_Y$ be its direct image on X. If $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$, let $\boldsymbol{A}(\mathcal{G}, \boldsymbol{K}_\infty^o, \mathbb{K})^{\mathcal{P}}$ be the functor which to $\mathcal{P}\in\mathfrak{P}$ associates the sheaf with $\mathcal{G}(\mathbb{A}_f)$-action

$$\left(i_{\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}\subseteq \boldsymbol{X}_{\mathcal{M}_{\mathcal{G}}}^{(c)}}\right)_* C_c^\infty\left(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{K}\right)_{\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}} .$$

For $\mathcal{P} \subseteq \mathcal{Q}$, $\boldsymbol{A}(\mathcal{G}, \boldsymbol{K}_\infty^o, \mathbb{K})^{\mathcal{P}\subseteq\mathcal{Q}}$ is defined by the inclusion

$$C_c^\infty(\mathcal{Q}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{K}) \subseteq C_c^\infty(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{K}),$$

followed by restriction from $\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{Q}}$ to $\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{P}}$. The group $\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R})$ acts on this complex by left translation, and the resulting action on hypercohomology factorizes over the quotient $\pi_0(\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R}))$. Recall the Borel map $I^*_{\mathcal{M}_\mathcal{G}(\mathbb{R}), \boldsymbol{K}_\infty^o} \to H^*(\mathcal{G}, \mathbb{C})^{\mathcal{G}(\mathbb{A}_f)}$ and the isomorphism $I^*_{\mathcal{M}_\mathcal{G}, \boldsymbol{K}_\infty^o} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}})$ from the introduction.

With this notation, we can formulate our main result as follows:

Theorem 3.1. *There is a canonical isomorphism of $\mathcal{G}(\mathbb{A}_f)$- and $I^*_{\mathcal{M}_\mathcal{G}(\mathbb{R}), \boldsymbol{K}_\infty^o} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}})$-modules between $H^*_c(\mathcal{G}, \mathbb{C})_\mathcal{I}$ and the hypercohomology of the complex associated to the functor $\boldsymbol{A}(\mathcal{G}, \mathbb{C})^\mathcal{P}$*

$$H^*_c(\mathcal{G}, \mathbb{C})_\mathcal{I} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}}, C^*(\boldsymbol{A}(\mathcal{G}, \mathcal{C})))^{\pi_o(\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R}))}. \tag{3.4}$$

This isomorphism identifies the real subspace $H^p_c(\mathcal{G}, \mathbb{R})_\mathcal{I}$ with

$$i^p H^p(\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}}, C^*(\boldsymbol{A}(\mathcal{G}, \mathcal{C})))^{\pi_o(\boldsymbol{K}_\infty^o \cap \mathcal{P}_o(\mathbb{R}))}.$$

The proof of this theorem will occupy most of the remainder of this paper. We will now give some corollaries. Since the sheaf of $\mathcal{G}(\mathbb{A}_f)$-modules $\boldsymbol{A}(\mathcal{G}, \mathbb{C})^\mathcal{P}$ is annihilated by $\mathcal{I}_S$, we have the following result.

Corollary 3.2. *If S is a set of finite places of $\mathbb{Q}$, then $H^*(\mathcal{G}, \mathbb{C})_\mathcal{I}$ and $H^*_c(\mathcal{G}, \mathbb{C})_\mathcal{I}$ are annihilated by $\mathcal{I}_S$ (and not just a power of $\mathcal{I}_S$).*

The assertion about $H^*(\mathcal{G}, \mathbb{C})_\mathcal{I}$ follows from the result about cohomology with compact support by duality.

To evaluate the cohomology sheaves of the complex $C^*(\boldsymbol{A}(\mathcal{G}, \mathbb{C})^\bullet)$, we have to define some Steinberg-like $\mathcal{G}(\mathbb{A}_f)$-modules. Let

$$\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)} = C^\infty(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{C}) \;/ \sum_{\mathcal{Q}\supset\mathcal{P}} C^\infty(\mathcal{Q}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f), \mathbb{C}), \tag{3.5}$$

and let $\check{\mathfrak{V}}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}$ be the dual of $\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}$. For instance, $\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{G}(\mathbb{A}_f)}$ and $\check{\mathfrak{V}}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{G}(\mathbb{A}_f)}$ are both isomorphic to the constant representation. Recall the definition of the subsets

$$\boldsymbol{U}_{\mathcal{M}_\mathcal{P}} = \boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{P}} - \bigcup_{\mathcal{Q}\subset\mathcal{P}} \boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{Q}}.$$

If V is a sheaf on $\boldsymbol{U}_{\mathcal{M}_\mathcal{P}}$, let $\left(i_{\boldsymbol{U}_{\mathcal{M}_\mathcal{P}} \subseteq \boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{G}}}\right)_! V$ be its continuation by zero.

Corollary 3.3. *The i-th cohomology sheaf of the complex $C^*(\boldsymbol{A}(\mathcal{G}, \boldsymbol{K}_\infty^o, \mathbb{C}))$ is given by*

$$\bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} \left(i_{\boldsymbol{U}_{\mathcal{M}_{\mathcal{P}}}\subseteq\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}}\right)_! \mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}. \tag{3.6}$$

The hypercohomology spectral sequence degenerates, and the limit filtration $\mathrm{Fil}_i\, H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ *has quotients*

$$(\mathrm{Fil}_i/\mathrm{Fil}_{i-1})\, H_c^k(\mathcal{G},\mathbb{C})_{\mathcal{I}} \cong \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} H_c^{k-i}(\boldsymbol{U}_{\mathcal{M}_{\mathcal{P}}})^{\pi_o(\boldsymbol{K}_\infty\cap\mathcal{P}_o(\mathbb{R}))}\otimes\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}, \tag{3.7}$$

where the isomorphism is an isomorphism of modules over $\mathcal{G}(\mathbb{A}_f)$. *This is the only ascending filtration of* $H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ *whose i-th quotient is of the form*

$$\bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} V_{\mathcal{P}}\otimes\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}.$$

Similarly, $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ *has a descending filtration* Fil^i *with quotients*

$$\left(\mathrm{Fil}^i/\mathrm{Fil}^{i+1}\right) H_c^k(\mathcal{G},\mathbb{C})_{\mathcal{I}} \cong \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} H^{k+\dim(\mathfrak{n}_{\mathcal{P}})}(\boldsymbol{U}_{\mathcal{M}_{\mathcal{P}}})^{\pi_o(\boldsymbol{K}_\infty\cap\mathcal{P}_o(\mathbb{R}))}\otimes\check{\mathfrak{V}}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}. \tag{3.8}$$

This is the only descending filtration of $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ *whose i-th quotient is of the form*

$$\bigoplus_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}=i}} V^{\mathcal{P}}\otimes\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}.$$

Proof. By Poincare duality, it suffices to prove the assertions about cohomology with compact support. The formula (3.6) is a consequence of the Solomon–Tits like Theorem 4.2 in the next section, which generalizes [4, §3]. The degeneration of the hypercohomology spectral sequence follows from Hodge theory and the fact that the restriction of an invariant (= harmonic) form on the compact dual of a Levi component of $\mathcal{G}$ to the compact dual of a smaller Levi component is again invariant.

The uniqueness assertion about the filtration of $H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ follows from the next proposition. □

Proposition 3.4. *Let* S *be a set which contains all nonarchimedean primes of* $\mathbb{Q}$ *with finitely many exceptions, and let* $\mathcal{P}\neq\mathcal{Q}$ *be parabolic subgroups of* $\mathcal{G}$. *Then the spaces of S-spherical vectors* ${\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{P}(\mathbb{A}_f)}}^{\boldsymbol{K}_S}$ *and* ${\mathfrak{V}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{Q}(\mathbb{A}_f)}}^{\boldsymbol{K}_S}$ *have finite length as representations of the group*

$$\prod_{\substack{v\ \text{nonarchimedean}\\ v\notin S}} \mathcal{G}(\mathbb{Q}_v),$$

and their Jordan–Hölder series have mutually nonisomorphic quotients.

Proof. This is a consequence of [7, X.4.6.]. □

Unfortunately, Hilbert modular forms and SL_3 over imaginary fields provide examples where the filtration $\mathrm{Fil}_i\, H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ does not split in the category of $\mathcal{G}(\mathbb{A}_f)$-modules.

Since $\boldsymbol{K}_f$ was supposed to be good, we have $\mathcal{P}(\mathbb{A}_f)\boldsymbol{K}_f = \mathcal{G}(\mathbb{A}_f)$ for all parabolic subgroups $\mathcal{P}$. Therefore, $\mathfrak{V}_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)}$ has $\boldsymbol{K}_f$-spherical vectors only if $\mathcal{P} = \mathcal{G}$, and the only quotient of $\mathrm{Fil}_i\, H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ which has a $\boldsymbol{K}_f$-spherical vector is in dimension zero. We get the following corollary.

Corollary 3.5. *The natural maps*

$$H_c^*(\mathcal{G},\mathbb{C})^{\mathcal{G}(\mathbb{A}_f)} \to H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}^{\mathcal{G}(\mathbb{A}_f)} \to H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}^{\boldsymbol{K}_f}$$

are isomorphisms. (The first of these isomorphisms follows from the fact that the constant $\mathcal{G}(\mathbb{A}_f)$-representation is annihilated by $\mathcal{I}$.) Similarly, the maps

$$H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}^{\boldsymbol{K}_f} \to (H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}})_{\mathcal{G}(\mathbb{A}_f)} \to H^*(\mathcal{G},\mathbb{C})_{\mathcal{G}(\mathbb{A}_f)}$$

are isomorphisms, where the subscript $_{\mathcal{G}(\mathbb{A}_f)}$ stands for the space of $\mathcal{G}(\mathbb{A}_f)$-coinvariants. Also, we have isomorphisms of $I^*_{\mathcal{M}_{\mathcal{G}}(\mathbb{R}),\boldsymbol{K}^\circ_\infty} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}})$-modules

$$H^*(\mathcal{G},\mathbb{C})_{\mathcal{G}(\mathbb{A}_f)} \cong H^*(\boldsymbol{U}_{\mathcal{M}_{\mathcal{G}}},\mathbb{C})$$

and

$$H_c^*(\mathcal{G},\mathbb{C})^{\mathcal{G}(\mathbb{A}_f)} \cong H_c^*(\boldsymbol{U}_{\mathcal{M}_{\mathcal{G}}},\mathbb{C}).$$

In particular, this establishes Theorem 1.1 of the introduction.

4 An adelic Borel–Serre–Solomon–Tits theorem

In this section we study the cohomology of the chain complexes associated to certain functors on $\mathfrak{P}$. Let us start with the easiest example. For parabolic subgroups $\mathcal{Q} \subseteq \mathcal{R}$, consider the contravariant functor

$$\boldsymbol{B}(\mathcal{Q},\mathcal{R})^{\mathcal{P}} = \begin{cases} \mathbb{C} & \text{if } \mathcal{Q}\subseteq\mathcal{P}\subseteq\mathcal{R} \\ \{0\} & \text{otherwise} \end{cases}$$

and the covariant functor

$$\boldsymbol{B}(\mathcal{Q},\mathcal{R})_{\mathcal{P}} = \begin{cases} \mathbb{C} & \text{if } \mathcal{Q}\subseteq\mathcal{P}\subseteq\mathcal{R} \\ \{0\} & \text{otherwise} \end{cases}$$

such that $\boldsymbol{B}(\mathcal{Q},\mathcal{R})^{\mathcal{P}\subseteq\tilde{\mathcal{P}}}$ and $\boldsymbol{B}(\mathcal{Q},\mathcal{R})_{\mathcal{P}\subseteq\tilde{\mathcal{P}}}$ are the identities if $\mathcal{Q}\subseteq\mathcal{P}\subseteq\tilde{\mathcal{P}}\subseteq\mathcal{R}$ and zero otherwise.

Lemma 4.1. *If $\mathcal{Q} \subset \mathcal{R}$, $C^*(\boldsymbol{B}(\mathcal{Q},\mathcal{R})^\bullet)$ and $C^*(\boldsymbol{B}(\mathcal{Q},\mathcal{R})_\bullet)$ are acyclic. If $\mathcal{Q} = \mathcal{R}$, then the only cohomology group of $C^*(\boldsymbol{B}(\mathcal{Q},\mathcal{R})^\bullet)$ is $\mathbb{C}$ in dimension $\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}$, and the only cohomology group of $C^*(\boldsymbol{B}(\mathcal{Q},\mathcal{R})_\bullet)$ is $\mathbb{C}$ in dimension $\dim \mathfrak{a}_o^{\mathcal{Q}}$.*

This is straightforward.

For a more interesting example, one takes the set of all $\mathbb{C}$-valued functions on $\mathcal{P}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q})$ for $\boldsymbol{F}^{\mathcal{P}}$ together with the obvious inclusions as transition maps. The associated chain complex gives the reduced cohomology of the Tits building of $\mathcal{G}$ shifted by -1; hence by the Solomon–Tits theorem it has cohomology only in degree $\dim \mathfrak{a}_o^{\mathcal{G}}$. The related theorem in which continuous functions on $\mathcal{P}(\mathbb{Q}_v)\backslash\mathcal{G}(\mathbb{Q}_v)$ (with $\mathbb{Q}_v$-rational parabolic subgroups $\mathcal{P}$ which are standard with respect to a minimal $\mathbb{Q}_v$-rational parabolic subgroup) are considered has been proved by Borel and Serre [4, §3]. We need an adelic version of their result.

Theorem 4.2. *Let S be a set of places of $\mathbb{Q}$, and let $\mathcal{R}$ be a standard $\mathbb{Q}$-parabolic subgroup. Let $\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)^\bullet$ be defined by*

$$\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)^{\mathcal{P}} = \begin{cases} C^\infty(\mathcal{P}(\mathbb{A}_S)\backslash\mathcal{G}(\mathbb{A}_S)) & \text{if } \mathcal{P} \subseteq \mathcal{R} \\ \{0\} & \text{otherwise.} \end{cases}$$

(Recall our convention that C^∞-functions are supposed to be $\boldsymbol{K}_\infty$-finite.) Let the transition functions for $\boldsymbol{C}$ be given by the obvious inclusions. Then the complex $C^*(\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)^\bullet)$ is acyclic in dimension $< \dim \mathfrak{a}_o^{\mathcal{G}}$.

Proof. The only difference to the situation considered by Borel and Serre is that we consider quotients of an adelic group by $\mathbb{Q}$-parabolic subgroups, whereas they consider quotients of the v-adic group by $\mathbb{Q}_v$-rational subgroups. Their method is flexible enough to cover our situation. To eliminate any possible doubt, let us give the modified proof.

Since $\boldsymbol{C}(\mathcal{G},\mathcal{P},\mathbb{A}_S)^\bullet$ is the inductive limit of its subfunctors $\boldsymbol{C}(\mathcal{G},\mathcal{P},\mathbb{A}_T)$ for finite T, it suffices to consider the case where S is a finite set of places of $\mathbb{Q}$. We will prove the following proposition.

Proposition 4.3. *Let S be a finite set of places of $\mathbb{Q}$, let B be a Banach space, and let $\mathcal{R} \in \mathfrak{P}$. Let $\tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S)$ be given by spaces of B-valued continuous functions on flag varieties of $\mathcal{G}$*

$$\tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S,B)^{\mathcal{P}} = \begin{cases} C(\mathcal{P}(\mathbb{A}_S)\backslash\mathcal{G}(\mathbb{A}_S),B) & \text{if } \mathcal{P} \subseteq \mathcal{R} \\ \{0\} & \text{otherwise} \end{cases}$$

with the obvious inclusions as transition homomorphisms. Then the complex $C^*(\tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S,B)^\bullet)$ is acyclic in dimension $< \dim \mathfrak{a}_o^{\mathcal{G}}$.

For finite S, the theorem follows from the proposition, since $\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)$ is the inductive limit of its subfunctors $\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)e$ over idempotents e of the convolution algebra $C^\infty(\boldsymbol{K}_S)$. But $\boldsymbol{C}(\mathcal{G},\mathcal{R},\mathbb{A}_S)e = \tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S,\mathbb{C})e$ is a direct summand of $\tilde{\boldsymbol{C}}(\mathcal{R},\mathbb{A}_S,\mathbb{C})$. □

Proof. We proceed by induction on the cardinality of S, starting with the case $S = \emptyset$. For this case, we have $\tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_\emptyset, B)^\bullet \colon = \boldsymbol{B}(\mathcal{P}_o, \mathcal{R})^\bullet \otimes B$ and apply Lemma 4.1.

Let $v \in S$ be such that the proposition has been verified for $S \setminus \{v\}$ and arbitrary $\mathcal{R}$ and B. Let $\mathcal{P}_v \subseteq \mathcal{P}_o$ be a minimal $\mathbb{Q}_v$-parabolic subgroup, and let $\mathcal{A}_v \subset \mathcal{P}_v$ be a maximal $\mathbb{Q}_v$-split torus. Let $w_0,\dots, w_N$ be an enumeration of the elements of the Weyl group $W(\mathcal{A}_v, \mathcal{G}(\mathbb{Q}_v))$ such that $\ell(w_i) \leq \ell(w_j)$ if $i < j$, where $\ell(w)$ is the length of w. Let

$$C(w) = \mathcal{P}_v(\mathbb{Q}_v)\backslash \mathcal{P}_v(\mathbb{Q}_v) w \mathcal{P}_v(\mathbb{Q}_v) \subset \mathcal{P}_v(\mathbb{Q}_v)\backslash \mathcal{G}(\mathbb{Q}_v)$$

be the Schubert cell associated to w, and let $E_i = \bigcup_{j=0}^{i} C(w_j)$. Let Δ_v and $\Delta_v^{\mathcal{P}}$ be defined like Δ_o and $\Delta_o^{\mathcal{P}}$, but with $\mathcal{P}_o$ replaced by $\mathcal{P}_v$. For $\alpha \in \Delta_v$, let s_α be the reflection belonging to α. Let

$$\pi_{\mathcal{P}} \colon \mathcal{P}_v(\mathbb{Q}_v)\backslash \mathcal{G}(\mathbb{Q}_v) \to \mathcal{P}(\mathbb{Q}_v)\backslash \mathcal{G}(\mathbb{Q}_v)$$

be the projection. We have the following consequence of the Bruhat decomposition as given in [4, 2.4.].

Lemma 4.4. *Let $0 \leq i \leq N$. If $\mathcal{P} \supseteq \mathcal{P}_v$ is a $\mathbb{Q}_v$-parabolic subgroup such that $\ell(s_\alpha w_i) > \ell(w_i)$ for all $\alpha \in \Delta_o^{\mathcal{P}}$, then $\pi_{\mathcal{P}}$ induces an isomorphism*

$$C(w_i) \cong \pi_{\mathcal{P}}(C(w_i)) = \pi_{\mathcal{P}}(E_i) - \pi_{\mathcal{P}}(E_{i-1}).$$

Otherwise, we have $\pi_{\mathcal{P}}(E_i) = \pi_{\mathcal{P}}(E_{i-1})$.

Let $\mathrm{Fil}^i\, \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^{\mathcal{P}}$ be the set of all $f \in \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^{\mathcal{P}}$ which vanish on

$$\big(\mathcal{P}(\mathbb{A}_{S\setminus\{v\}})\backslash \mathcal{G}(\mathbb{A}_{S\setminus\{v\}})\big) \times \pi_{\mathcal{P}}(E_i). \tag{4.1}$$

This is a subfunctor of $\tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^\bullet$. Let us consider $0 \leq i \leq N$. If there exists no $\mathbb{Q}$-parabolic subgroup $\mathcal{Q} \supseteq \mathcal{P}_v$ such that $\ell(s_\alpha w_i) > \ell(w_i)$ for all $\alpha \in \Delta_v^{\mathcal{Q}}$, then Lemma 4.4 implies $\mathrm{Fil}^{i-1}\, \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^\bullet = \mathrm{Fil}^{i}\, \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^\bullet$. Otherwise, let $\mathcal{Q}_i$ be the largest $\mathbb{Q}$-parabolic subgroup with this property. Let $C(C(w), B)$ be the Banach space of continuous B-valued functions on $C(w)$, and let $\overline{C_c(C(w), B)} \subset C(C(w), B)$ be the closure of the set of compactly supported functions. Identifying B-valued continuous functions on (4.1) with $C(\pi_{\mathcal{P}}(E_i), B)$-valued continuous functions on

$$\mathcal{P}(\mathbb{A}_{S\setminus\{v\}})\backslash \mathcal{G}(\mathbb{A}_{S\setminus\{v\}})$$

and using the isomorphism

$$\ker\big(C(\pi_{\mathcal{P}}(E_i), B) \to C(\pi_{\mathcal{P}}(E_{i-1}), B)\big) = \begin{cases} \{0\} & \text{if } \mathcal{P} \not\subset \mathcal{Q}_i \\ \overline{C_c(C(w_i), B)} & \text{if } \mathcal{P} \subset \mathcal{Q}_i, \end{cases}$$

we get an isomorphism

$$\big(\mathrm{Fil}^{i-1}/\mathrm{Fil}^{i}\big)\, \tilde{\boldsymbol{C}}(\mathcal{R}, \mathbb{A}_S, B)^{\mathcal{P}} \cong \tilde{\boldsymbol{C}}(\mathcal{R} \cap \mathcal{Q}_i, \mathbb{A}_{S-\{v\}}, \overline{C_c(C(w_i), B)}),$$

and the induction argument is complete. □

Generalising the definition of $\mathfrak{V}_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)}$ in the third section, we define

$$\mathfrak{V}_{\mathcal{P}(\mathbb{A}_S)}^{\mathcal{G}(\mathbb{A}_S)} = C^\infty\left(\mathcal{P}(\mathbb{A}_S)\backslash\mathcal{G}(\mathbb{A}_S)\right) \Big/ \sum_{\mathcal{Q}\supset\mathcal{P}} C^\infty\left(\mathcal{Q}(\mathbb{A}_S)\backslash\mathcal{G}(\mathbb{A}_S)\right) \tag{4.2}$$

where it is understood that if S contains the archimedean place, then induction at this place is $(\mathfrak{g}, \boldsymbol{K})$-module induction. Let $\check{\mathfrak{V}}_{\mathcal{P}(\mathbb{A}_S)}^{\mathcal{G}(\mathbb{A}_S)}$ be the $\boldsymbol{K}_S$-finite dual of $\mathfrak{V}_{\mathcal{P}(\mathbb{A}_S)}^{\mathcal{G}(\mathbb{A}_S)}$. We put $\mathfrak{S}t_{\mathcal{G}(\mathbb{A}_S)} = \mathfrak{V}_{\mathcal{G}(\mathbb{A}_S)}^{\mathcal{P}_o(\mathbb{A}_S)}$ and $\check{\mathfrak{S}}t_{\mathcal{G}(\mathbb{A}_S)} = \check{\mathfrak{V}}_{\mathcal{G}(\mathbb{A}_S)}^{\mathcal{P}_o(\mathbb{A}_S)}$. These can be considered as Steinberg-like modules, although they are highly non-irreducible unless S consists of a single place v at which $\mathcal{P}_o$ is also a minimal $\mathbb{Q}_v$-parabolic subgroup.

If we choose Haar measures on $\mathcal{G}(\mathbb{A})$ and $\mathcal{P}_o(\mathbb{A})$, then the dual of

$$C^\infty(\mathbb{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) = \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}$$

can be identified with $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}_{2\rho_o}$. This allows us to view

$$\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}$$

as a submodule of $\operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C}_{2\rho_o}$. It is the orthogonal complement of

$$\sum_{\mathcal{Q}\subset\mathcal{P}} C^\infty(\mathcal{Q}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})),$$

hence it decreases if $\mathcal{P}$ increases. This allows us to define

$$\boldsymbol{D}(\mathcal{G})^{\mathcal{P}} = \left\{ \begin{array}{ll} \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} \neq \mathcal{P}_o \\ \sum_{\mathcal{P}\in\mathfrak{P}} \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} = \mathcal{P}_o \end{array} \right\} \subset \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C}_{2\rho_o}. \tag{4.3}$$

Theorem 4.5. *If* $\dim \mathfrak{a}_o^{\mathcal{G}} > 0$, *then* $C^*(\boldsymbol{D}(\mathcal{G})^\bullet)$ *is acyclic.*

Proof. $\boldsymbol{D}(\mathcal{G})^\bullet \subset \boldsymbol{B}(\mathcal{P}_o, \mathcal{G})^\bullet \otimes \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C}_{2\rho_o}$ is the orthogonal complement of

$$\boldsymbol{M}_\bullet \subset \boldsymbol{B}(\mathcal{P}_o, \mathcal{G})_\bullet \otimes \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C},$$

where

$$\boldsymbol{M}_{\mathcal{P}} = \left\{ \begin{array}{ll} \sum_{\substack{\mathcal{R}\subseteq\mathcal{P} \\ \dim \mathfrak{a}_o^{\mathcal{R}}=1}} C^\infty(\mathcal{R}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) & \text{if } \mathcal{P} \neq \mathcal{P}_o \\ \mathbb{C} = \bigcap_{\substack{\mathcal{R}\in\mathfrak{P} \\ \dim \mathfrak{a}_o^{\mathcal{R}}=1}} C^\infty(\mathcal{R}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) & \text{otherwise} \end{array} \right\} \subset C^\infty(\mathcal{P}_o(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})).$$

By Lemma 4.1, it suffices to show that $C^*(\boldsymbol{M}_\bullet)$ is acyclic. Let

$$\tilde{\boldsymbol{M}}_{\mathcal{P}} = \begin{cases} \boldsymbol{M}_{\mathcal{P}} & \text{if } \mathcal{P} \neq \mathcal{P}_o \\ \{0\} & \text{if } \mathcal{P} = \mathcal{P}_o. \end{cases}$$

Since $\mathbb{C} \subseteq \boldsymbol{M}_{\mathcal{P}_o} \subseteq H^1(C^*(\tilde{\boldsymbol{M}}_\bullet))$, the acyclicity of $C^*(\boldsymbol{M}_\bullet)$ and the theorem will follow if we show that $C^*(\tilde{\boldsymbol{M}}_\bullet)$ has only one one-dimensional cohomology space in dimension one.

We will reduce this to Theorem 4.2 by introducing a functor of two variables $\boldsymbol{N}_\bullet^\bullet$ and using the spectral sequence for its double complex. We define $\boldsymbol{N}_\bullet^\bullet$ by

$$\boldsymbol{N}_{\mathcal{P}}^{\mathcal{Q}} = \begin{cases} \{0\} & \text{if } \mathcal{Q} \not\subseteq \mathcal{P} \text{ or if } \mathcal{Q} = \mathcal{P}_o \\ C^\infty(\mathcal{Q}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) & \text{otherwise.} \end{cases}$$

It is a consequence of Theorem 4.2 (applied to $\boldsymbol{C}(\mathcal{G}, \mathcal{L}_{\mathcal{G}}, \mathbb{A})$) that

$$H^k(C^*(\boldsymbol{N}_{\mathcal{P}}^\bullet)) = \begin{cases} \{0\} & \text{if } k \neq \dim \mathfrak{a}_o^{\mathcal{G}} - 1 \\ \tilde{\boldsymbol{M}}_{\mathcal{P}} & \text{if } k = \dim \mathfrak{a}_o^{\mathcal{G}} - 1. \end{cases}$$

Since

$$\boldsymbol{N}_\bullet^{\mathcal{Q}} = \begin{cases} \{0\} & \text{if } \mathcal{Q} = \mathcal{P}_o \\ \boldsymbol{B}(\mathcal{Q}, \mathcal{G})_\bullet \otimes C^\infty(\mathcal{Q}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) & \text{if } \mathcal{Q} \supset \mathcal{P}_o, \end{cases}$$

Lemma 4.1 implies

$$H^l\left(C^*(\boldsymbol{N}_\bullet^{\mathcal{Q}})\right) = \begin{cases} \{0\} & \text{if } l \neq \dim \mathfrak{a}_o^{\mathcal{G}} \text{ or } \mathcal{Q} \neq \mathcal{G} \\ \mathbb{C} & \text{if } l = \dim \mathfrak{a}_o^{\mathcal{G}} \text{ and } \mathcal{Q} = \mathcal{G}. \end{cases}$$

Combining these two facts, we get

$$H^k\left(C^*(\tilde{\boldsymbol{M}}_\bullet)\right) = H^{k+\dim \mathfrak{a}_o^{\mathcal{G}}-1}\left(C^*(\boldsymbol{N}_\bullet^\bullet)\right) = \begin{cases} \{0\} & \text{if } k \neq 1 \\ \mathbb{C} & \text{if } k = 1. \end{cases}$$

As was mentioned earlier, this implies the theorem. □

We complete this section with a rather elementary lemma. For a parabolic subgroup $\mathcal{R}$ of $\mathcal{G}$, let

$$\boldsymbol{E}(\mathcal{R})^{\mathcal{P}*} = \begin{cases} \Lambda^*(\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{P}}) & \text{if } \mathcal{P} \supseteq \mathcal{R} \\ \{0\} & \text{if } \mathcal{P} \not\supseteq \mathcal{R}. \end{cases} \tag{4.4}$$

The transition homomorphism $\boldsymbol{E}(\mathcal{R})^{\tilde{\mathcal{P}}\subseteq\mathcal{P}*}$ is given by the projection $\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{P}} \to \check{\mathfrak{a}}_{\mathcal{R}}^{\tilde{\mathcal{P}}}$. $\boldsymbol{E}(\mathcal{R})^{\bullet *}$ is a functor from $\mathfrak{P}$ into the category of graded vector spaces.

Lemma 4.6. *The projection*

$$\boldsymbol{E}(\mathcal{R})^{\mathcal{G}*} = \Lambda^*(\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}) \to \det \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}[-\dim \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}]$$

defines an isomorphism on cohomology

$$H^*\left(C^*\left(\boldsymbol{E}(\mathcal{R})^{\bullet *}\right)\right) \cong \det \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}[-\dim \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{G}}].$$

By the determinant of a finite dimensional vector space, we understand its highest exterior power.

Proof. Let $\mathcal{R}_1, \ldots, \mathcal{R}_k$ be the parabolic subgroups containing $\mathcal{R}$ with the property that $\dim \mathfrak{a}_{\mathcal{R}}^{\mathcal{R}_i} = 1$. Then

$$C^* \left(\boldsymbol{E}(\mathcal{R})^{\bullet *} \right) \cong \bigotimes_{i=1}^{k} \Big((\mathbb{C} \oplus \check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{R}_i}) \to \mathbb{C} \Big),$$

proving the lemma. □

5 The space of automorphic forms

It is known that $H^*(\mathcal{G}, \mathbb{C})$ can be evaluated by using the cohomology of the de Rham complex, which is isomorphic to the standard complex for evaluating the $(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}_\infty^o)$-cohomology $H^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}_\infty)}(C^\infty(\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})))$. Let

$$C^\infty_{\text{umg}}(\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})) \subset C^\infty(\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})) \qquad (5.1)$$

be the subspace of functions of uniformly moderate growth. Let $\mathcal{J} = \mathfrak{U}(\mathfrak{g})\mathfrak{g} \cap \mathfrak{Z}(\mathfrak{g})$ be the annihilator of the constant representation in $\mathfrak{Z}(\mathfrak{g})$. Let

$$A_{\mathcal{J}} \colon = \left\{ f \in C^\infty_{\text{umg}}(\mathcal{A}_{\mathcal{G}}(\mathbb{R})^+\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})) \,\middle|\, \mathcal{J}^n f = \{0\} \text{ for } n \gg 0 \right\}. \qquad (5.2)$$

Borel has verified that the inclusion (5.1) defines an isomorphism on cohomology and conjectured that the inclusion $A_{\mathcal{J}} \subset C^\infty_{\text{umg}}$ also defines an isomorphism on cohomology with constant coefficients. After partial results by Casselman, Harder, and Speh, this has been verified in [9], where we denoted C^∞_{umg} by S_∞ and $A_{\mathcal{J}}$ by $\mathfrak{F}in_{\mathcal{J}} S_\infty$ since we worked in a more general situation.

Let S be a set of finite primes which contains all but finitely many primes. It is a consequence of well-known finiteness properties of the space of automorphic forms (cf. [11, Proposition 2.3]) that the space of $\boldsymbol{K}_S$-spherical vectors $A_{\mathcal{J}}^{\boldsymbol{K}_S} \subset A_{\mathcal{J}}$ has a decomposition into associated Hecke eigenspaces

$$A_{\mathcal{J}}^{\boldsymbol{K}_S} = \coprod_{\tilde{\mathcal{I}}} A_{\mathcal{J}, \tilde{\mathcal{I}}}^{\boldsymbol{K}_S},$$

where the sum is over maximal ideals $\tilde{\mathcal{I}} \subset \mathfrak{H}_S$ and

$$A_{\mathcal{J}, \tilde{\mathcal{I}}}^{\boldsymbol{K}_S} = \left\{ f \in A_{\mathcal{J}} \,\middle|\, \tilde{\mathcal{I}}^n f = \{0\} \text{ for } n \gg 0 \right\}.$$

It is clear from the proven Borel conjecture that the cohomology of $A_{\mathcal{J}, \tilde{\mathcal{I}}}^{\boldsymbol{K}_S}$ is isomorphic to the space of $\boldsymbol{K}_S$-spherical vectors in $H^*(\mathcal{G}, \mathbb{C})$ which are annihilated by a power of $\tilde{\mathcal{I}}$. Recall the maximal ideal $\mathcal{I}_S \subset \mathfrak{H}_S$, which is the annihilator of the constant representation. We put

$$A_{\mathcal{J}, \mathcal{I}} \colon = \operatorname*{colim}_{S} A_{\mathcal{J}, \mathcal{I}_S}^{\boldsymbol{K}_S}. \qquad (5.3)$$

The aim of this section is to study $A_{\mathcal{J}, \mathcal{I}}$.

There are two methods available for studying the space of automorphic forms. One method is to define a filtration on the space of automorphic forms, and to show that its quotients are spanned by principal values of cuspidal and residual Eisenstein series. This method was used in [9]. It is particularly useful in a general situation, where one has only the facts proved in Langlands' book [15] available. The second method, which was proposed by Harder in [14] before [9] was written, is to generate the space of automorphic forms by the coefficients of the Laurent expansions of cuspidal Eisenstein series at a certain point. In [11], we derived from the result of [9] that this procedure really gives the space of all automorphic forms. This method gives a complete description of the space of automorphic forms (and not just the quotients of a filtration), but is useful only if the precise structure of the singularities of the cuspidal Eisenstein series near the point where they have to be evaluated is known. For the Eisenstein series which contribute to $A_{\mathcal{J},\mathcal{I}}$, we are in the fortunate situation to have such information available. We will therefore generate the space $A_{\mathcal{J},\mathcal{I}}$ by cuspidal Eisenstein series. At the beginning, the procedure will be quite similar to the methods used by Speh in [22]. However, Speh studied only a certain subspace of $A_{\mathcal{J},\mathcal{I}}$, which was sufficient for her examples of the noninjectivity of the Borel map, and for which only Eisenstein series depending on one parameter were needed.

Let $\mathcal{P}$ be a standard parabolic subgroup. Recall the standard height function $H_{\mathcal{P}}\colon \mathcal{G}(\mathbb{A}) \to \mathfrak{a}_{\mathcal{P}}$, which is defined by

$$\langle H_{\mathcal{P}}(g), \chi\rangle = \sum_{v} \log |\chi(p_v)|_v \,, \tag{5.4}$$

where $g = pk$ with $p \in \mathcal{P}(\mathbb{A})$ and $k \in \boldsymbol{K}$. The scalar product $\langle .,. \rangle$ on the left side is the pairing between $\mathfrak{a}_{\mathcal{P}}$ and $\check{\mathfrak{a}}_{\mathcal{P}}$, and $\chi \in X^*(\mathcal{P}) \subset \check{\mathfrak{a}}_{\mathcal{P}}$. It is clear that (5.4) characterizes $H_{\mathcal{P}}(g)$ uniquely, and that $H_{\mathcal{P}}(g)$ does not depend on the choice of the Iwasawa decomposition $g = pk$. If $\mathcal{Q} \supseteq \mathcal{P}$, then $H_{\mathcal{Q}}(g)$ is the projection of $H_{\mathcal{P}}(g)$ to $\mathfrak{a}_{\mathcal{Q}}$.

We have to recall a few facts about the Eisenstein series starting from the constant representation of a Levi component. Proofs can be found in [10, Lemma 2.7], although the results about the Eisenstein series were almost certainly known previously. If $\phi \in C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$, the Eisenstein series starting from ϕ is defined by

$$E^{\mathcal{G}}_{\mathcal{P}}(\phi, \lambda) = \sum_{\gamma\in\mathcal{P}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q})} \phi(\gamma g)\, \mathrm{e}^{\langle\lambda+\rho_{\mathcal{P}}, H_{\mathcal{P}}(\gamma g)\rangle} \,. \tag{5.5}$$

This series converges for sufficiently regular $\Re\lambda$ in the positive Weyl chamber, and has an analytic continuation to $\lambda \in (\check{\mathfrak{a}}_{\mathcal{P}})_{\mathbb{C}}$. The singular hyperplanes of this function which cross through $\rho_{\mathcal{P}}$ are precisely the hyperplanes $\langle \lambda - \rho_{\mathcal{P}}, \check{\alpha}\rangle = 0$, where $\alpha \in \Delta_{\mathcal{P}}$ and $\check{\alpha}$ is the corresponding coroot. The residues may be described as follows. Let for $\lambda \in (\check{\mathfrak{a}}_{\mathcal{P}})_{\mathbb{C}}$

$$q_{\mathcal{P}}^{\mathcal{Q}}(\lambda) = \prod_{\alpha \in \Delta_{\mathcal{P}}^{\mathcal{Q}}} \langle \check{\alpha}, \lambda - \rho_{\mathcal{P}} \rangle.$$

Then the function $q_{\mathcal{P}}^{\mathcal{Q}}(\lambda) E_{\mathcal{P}}^{\mathcal{G}}(\phi, \lambda)$ is regular on an open dense subset of $\rho_{\mathcal{P}}^{\mathcal{Q}} + (\check{\mathfrak{a}}_{\mathcal{Q}})_{\mathbb{C}}$. Its restriction to $\rho_{\mathcal{P}}^{\mathcal{Q}} + (\check{\mathfrak{a}}_{\mathcal{Q}})_{\mathbb{C}}$ can be described as follows. If $\phi \in C^\infty(\mathcal{P}(\mathbb{A}) \backslash \mathcal{G}(\mathbb{A}))$, then

$$\mathrm{e}^{\langle H_{\mathcal{P}}(\cdot), 2\rho_{\mathcal{P}} \rangle} \phi(\cdot) \in \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}_{2\rho_{\mathcal{P}}},$$

and let $\mathbb{C}_{2\rho_{\mathcal{P}}}$ be the one-dimensional vector space on which $p \in \mathcal{P}(\mathbb{A})$ acts by multiplication by $\mathrm{e}^{\langle H_{\mathcal{P}}(p), 2\rho_{\mathcal{P}} \rangle}$. There exists a unique nonvanishing homomorphism

$$\tau_{\mathcal{P}}^{\mathcal{Q}} \colon \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}_{2\rho_{\mathcal{P}}} \to \operatorname{Ind}_{\mathcal{Q}}^{\mathcal{G}} \mathbb{C}_{2\rho_{\mathcal{Q}}}$$

with the following property. For generic $\vartheta \in \check{\mathfrak{a}}_{\mathcal{Q}}$ we have

$$\left(q_{\mathcal{P}}^{\mathcal{Q}}(\cdot) E_{\mathcal{P}}^{\mathcal{G}}(\cdot) \right)(\phi, \vartheta + \rho_{\mathcal{P}}^{\mathcal{Q}}) = E_{\mathcal{Q}}^{\mathcal{G}} \left(\mathrm{e}^{-\langle 2\rho_{\mathcal{Q}}, H_{\mathcal{Q}}(\cdot) \rangle} \tau_{\mathcal{P}}^{\mathcal{Q}} \left(\mathrm{e}^{\langle 2\rho_{\mathcal{P}}, H_{\mathcal{P}}(\cdot) \rangle} \phi \right), \vartheta \right). \quad (5.6)$$

It is easy to verify

$$\tau_{\mathcal{Q}}^{\mathcal{R}} \tau_{\mathcal{P}}^{\mathcal{Q}} = \tau_{\mathcal{P}}^{\mathcal{R}}$$

and to see that $\tau_{\mathcal{P}}^{\mathcal{Q}}$ is independent of $\boldsymbol{K}_f$.

Let $S(\check{\mathfrak{a}}_o^{\mathcal{G}})$ be the symmetric algebra of $\check{\mathfrak{a}}_o^{\mathcal{G}}$. It can be identified with the algebra of differential operators with constant coefficients on $\check{\mathfrak{a}}_o^{\mathcal{G}}$. After we choose a basis for $\check{\mathfrak{a}}_o^{\mathcal{G}}$, we have elements $\frac{\partial^\alpha}{\partial \lambda^\alpha} \in S(\check{\mathfrak{a}}_o^{\mathcal{G}})$ for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_{\dim \check{\mathfrak{a}}_o^{\mathcal{G}}})$. Elements of $S(\check{\mathfrak{a}}_o^{\mathcal{G}})$ can also be viewed as polynomials on $\mathfrak{a}_{\mathcal{P}}^{\mathcal{G}} \subset \mathfrak{a}_{\mathcal{P}}$. Let H^α be the polynomial in $H \in \mathfrak{a}_{\mathcal{P}}$ belonging to $\frac{\partial^\alpha}{\partial \lambda^\alpha}$. We define a $\mathcal{G}(\mathbb{A}_f)$-action on

$$S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes C^\infty(\mathcal{P}(\mathbb{A}) \backslash \mathcal{G}(\mathbb{A})) \quad (5.7)$$

by

$$\left(h\left(\frac{\partial^\alpha}{\partial \lambda^\alpha} \otimes \phi\right) \right)(g) = \sum_{\alpha = \beta + \gamma} \left(\prod_{i=1}^{\dim \mathfrak{a}_{\mathcal{P}}^{\mathcal{G}}} \frac{\alpha_i!}{\beta_i! \gamma_i!} \right) \frac{\partial^\beta}{\partial \lambda^\beta} \otimes \left(\left(H_{\mathcal{P}}(gh) - H_{\mathcal{P}}(g) \right)^\gamma \mathrm{e}^{2\langle \rho_{\mathcal{P}}, H_{\mathcal{P}}(gh) - H_{\mathcal{P}}(g) \rangle} \phi(gh) \right) \quad (5.8)$$

for $h \in \mathcal{G}(\mathbb{A}_f)$. In a similar way, one obtains a $(\mathfrak{g}, \boldsymbol{K}_\infty)$-module structure on (5.7) by taking the differential of the $\mathcal{G}(\mathbb{R})$-action which would be given by (5.8) if there was no condition of K_∞-finiteness for elements of $C^\infty(\mathcal{P}(\mathbb{A}) \backslash \mathcal{G}(\mathbb{A}))$. Let a $\mathcal{P}(\mathbb{A})$-action on $S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})$ be defined by

$$p \colon D \to \mathrm{e}^{-\langle H_{\mathcal{P}}(p), \cdot \rangle} D \, \mathrm{e}^{\langle H_{\mathcal{P}}(p), \cdot \rangle}.$$

At the infinite place, the $\mathcal{P}(\mathbb{R})$-action gives rise to the structure of a $(\mathfrak{p}, \boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R}))$-module. There is a homomorphism of $(\mathfrak{p}, \boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{A}_f))$-modules

$$S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) \to S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}$$
$$D \otimes \phi(g) \to D \otimes \phi(1)$$

which defines an isomorphism

$$S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) \to \operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}. \tag{5.9}$$

Using this isomorphism and the regularity of $q^{\mathcal{G}}_{\mathcal{P}}(\cdot)E^{\mathcal{G}}_{\mathcal{P}}(\phi, \cdot)$ at $\rho_{\mathcal{P}}$, we get a homomorphism of $(\mathfrak{g}, \boldsymbol{K}_\infty, \mathcal{G}(\mathbb{A}_f))$-modules

$$\varXi^{\mathcal{G}}_{\mathcal{P}} \colon S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) \cong \operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} \to A_{\mathcal{J},\mathcal{I}} \tag{5.10}$$

which maps $D \otimes \phi$ to $\left(Dq^{\mathcal{G}}_{\mathcal{P}}(\cdot)E^{\mathcal{G}}_{\mathcal{P}}(\phi, \cdot)\right)(\rho_{\mathcal{P}})$.

To see that the functions in the image of $\varXi^{\mathcal{G}}_{\mathcal{P}}$ are annihilated by sufficiently high powers of $\mathcal{I}_S$ and $\mathcal{J}$, it suffices to note that $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}$ is the union of an ascending sequence of subrepresentations with quotients isomorphic to $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} \mathbb{C}_{2\rho_{\mathcal{P}}}$, and that $\mathcal{I}_S$ and $\mathcal{J}$ trivially act on $\operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} \mathbb{C}_{2\rho_{\mathcal{P}}}$.

We first prove the surjectivity of $\varXi^{\mathcal{G}}_{\mathcal{P}_o}$.

Theorem 5.1. *$\varXi^{\mathcal{G}}_{\mathcal{P}_o}$ is surjective. It is independent of the choice of $\boldsymbol{K}_f$.*

Proof. The fact that $\varXi$ is independent of $\boldsymbol{K}_f$ is established by an easy computation, using the fact that both $E^{\mathcal{G}}_{\mathcal{P}_o}$ and the identification

$$S(\check{\mathfrak{a}}^{\mathcal{G}}_{o}) \otimes C^\infty(\mathcal{P}_o(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) \cong \operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}_o} S(\check{\mathfrak{a}}^{\mathcal{G}}_{o}) \otimes \mathbb{C}_{2\rho_o}$$

depend on $\boldsymbol{K}_f$, and these dependencies cancel out.

We will derive the surjectivity of $\varXi^{\mathcal{G}}_{\mathcal{P}_o}$ from the description of the space of automorphic forms in [11, §1]. Recall from [11, Theorem 1.4] that the space $A_{\mathcal{J}}$ as a composition

$$A_{\mathcal{J}} = \bigoplus_{\{P\}} \coprod_{\varphi \in \varPhi_{\mathbb{C},\{P\}}} \mathfrak{A}_{\mathbb{C},\{P\},\varphi}, \tag{5.11}$$

where the first sum is over classes $\{P\}$ of associate parabolic subgroups and the second sum is over $\varPhi_{\mathbb{C},\{P\}}$, a set of equivalence classes of cuspidal automorphic representations π of the Levi components of the elements of $\{P\}$. Here two cuspidal automorphic representations belong to the same equivalence class if they can be identified by a Weyl group substitution. An equivalence class belongs to $\varPhi_{\mathbb{C},\{P\}}$ if and only if it is in a certain way compatible with the infinitesimal character $\mathcal{J}$ of the constant representation. For a precise definition, we refer to [11, §1.2]. Note that our notation is slightly different from the notation in [11], where the space of automorphic forms was denoted $\mathfrak{A}_{\mathcal{E}}$ with a finite-dimensional representation $\mathcal{E}$, which in our case is $\mathbb{C}$. Therefore,

$A_{\mathcal{J}}$ in our notations is $\mathfrak{A}_{\mathbb{C}}$ in [11]. The notations on the left side of (5.11) are, however, the same as in [11].

By [11, Theorem 1.4], the space $\mathfrak{A}_{\mathbb{C},\{P\},\varphi}$ can be spanned by the coefficients of the Laurent expansion of cuspidal Eisenstein series starting from elements of φ. In particular, [11, Theorem 1.4] says that, for the special case $\{P\} = \{\mathcal{P}_o\}$ and $\varphi = \{\mathbb{C}_{w\cdot\rho_o}\}_{w\in W(\mathcal{A}_o:\mathcal{G}(\mathbb{Q}))}$, we have

$$\text{image of } \Xi^{\mathcal{G}}_{\mathcal{P}_o} = \mathfrak{A}_{\mathbb{C},\{\mathcal{P}_o\},\{\mathbb{C}_{w\cdot\rho_o}\}_{w\in W(\mathcal{A}_o:\mathcal{G}(\mathbb{Q}))}}. \tag{5.12}$$

Let us fix $\{P\}$ and $\varphi \in \Phi_{\mathbb{C},\{P\}}$. Let $\mathcal{P} \in \{P\}$ and let π be an irreducible cuspidal automorphic representation of $\mathcal{L}_{\mathcal{P}}$ which belongs to $\varphi_{\mathcal{P}}$. Let $\chi_\pi\colon \mathcal{A}_{\mathcal{P}}(\mathbb{A})/\mathcal{A}_{\mathcal{P}}(\mathbb{Q}) \to \mathbb{C}^\times$ be the central character of π, and let $\lambda_\pi \in \check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}$ be the differential of the restriction of χ_π to $\mathcal{A}_{\mathcal{P}}(\mathbb{R})$. By applying a Weyl group substitution to $\mathcal{P}$ and π, we may assume $\lambda_\pi \in \overline{\check{\mathfrak{a}}^{\mathcal{G}+}_{\mathcal{P}}}$. Let S be a set of non-archimedean primes of $\mathbb{Q}$ which has a finite complement. We assume that π is unramified at the places of S. Let $v \in S$. By [11, Theorem 2.3], we have an ideal $\mathcal{I}_{\varphi,v} \subset \mathfrak{H}_v$ associated to φ such that all $\boldsymbol{K}_v$-spherical vectors in $\mathfrak{A}_{\mathbb{C},\{P\},\varphi}$ are annihilated by some power of $\mathcal{I}_{\varphi,v}$. Recall the annihilator $\mathcal{I}_v \subset \mathfrak{H}_v$ of the constant representation. If $\mathfrak{A}_{\mathbb{C},\{P\},\varphi} \cap A_{\mathcal{J},\mathcal{I}} \neq \{0\}$, then we must have $\mathcal{I}_{\varphi,v} = \mathcal{I}_v$ for all but finitely many places. We will verify that this implies $\{P\} = \{\mathcal{P}_o\}$ and $\varphi = \{w \cdot \mathbb{C}_{2\rho_o}\}_{w\in W(\mathcal{A}_o:\mathcal{G}(\mathbb{Q}))}$. By (5.12), this will complete the proof of the theorem.

Let $v \in S$ such that $\mathcal{I}_{\varphi,v} = \mathcal{I}_v$. We recall Satake's description of $\mathfrak{H}_v$. Let $\mathcal{P}_v \subset \mathcal{P}_o$ be a minimal $\mathbb{Q}_v$-rational parabolic subgroup with Levi component $\mathcal{L}_v$. Let

$$\check{\mathfrak{a}}_v = X^*(\mathcal{P}_v)_{\mathcal{Q}_v} \otimes_{\mathbb{Z}} \mathbb{R},$$

where $X^*_{\mathbb{Q}_v}$ are the characters defined over $\mathbb{Q}_v$, and let $\mathfrak{a}_v$ be the dual of $\check{\mathfrak{a}}_v$. Let $\boldsymbol{T}_v$ be the group of unramified characters of $\mathcal{L}_v(\mathbb{Q}_v)$, i.e., of continuous characters $\chi\colon \mathcal{L}_v(\mathbb{Q}_v) \to \mathbb{C}^\times$ which are trivial on the projection of $\mathcal{P}_v(\mathbb{Q}_v)\cap\boldsymbol{K}_v$ to $\mathcal{L}_v(\mathbb{Q}_v)$. The map

$$\begin{aligned}(\check{\mathfrak{a}}_v)_{\mathbb{C}} &\to \boldsymbol{T}_v \\ \lambda &\to \chi_\lambda(l) = \mathrm{e}^{\langle H_{\mathcal{P}_v}(l),\chi\rangle}\end{aligned} \tag{5.13}$$

is surjective, and $\boldsymbol{T}_v$ has the structure of a complex torus which is isomorphic to $(\check{\mathfrak{a}}_v)_{\mathbb{C}}/\Gamma_v$, where Γ_v is a lattice in $i\check{\mathfrak{a}}_v$. Let $\mathcal{O}(T_v)$ be the ring of algebraic functions on the complex torus $\boldsymbol{T}_v$. The Weyl group $W(\mathcal{A}_v : \mathcal{G}(\mathbb{Q}_v))$ of $\mathcal{A}_v$ in $\mathcal{G}(\mathbb{Q}_v)$ acts on $\mathfrak{T}_v$, and we have the Satake isomorphisms

$$\boldsymbol{S}_{\mathcal{G}(\mathbb{Q}_v)}\colon \mathfrak{H}_v \to \mathcal{O}(\boldsymbol{T}_v)^{W\left(\mathcal{A}_v:\mathcal{G}(\mathbb{Q}_v)\right)}$$

$$\boldsymbol{S}_{\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)}\colon \mathfrak{H}_v(\mathcal{L}_{\mathcal{P}}) \to \mathcal{O}(\boldsymbol{T}_v)^{W\left(\mathcal{A}_v:\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)\right)}$$

(cf. [8, Theorem 4.1]) for $\mathcal{G}$ and for the Levi components of standard parabolic subgroups. Here $\mathfrak{H}_v(\mathcal{L}_{\mathcal{P}})$ is the Hecke algebra for $\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)$, defined by the projection of $\boldsymbol{K}_v \cap \mathcal{P}(\mathbb{Q}_v)$ to $\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)$.

Let $\mathfrak{H}_v(\mathcal{L}_{\mathcal{P}})$ act on the $\boldsymbol{K}_v$-spherical vector of π by multiplication by the character $(\boldsymbol{S}_{\mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v)}h)(t_v)$ for $t_v \in \boldsymbol{T}_v$. The $W(\mathcal{A}_v : \mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v))$-orbit of t_v is uniquely determined by π. Let $\tilde{t}_v \in \check{\mathfrak{a}}_v$ be a lifting of t_v. It is well-known that the ideal $\mathcal{I}_v$ corresponds to the image of ρ_v in $\boldsymbol{T}_v$ by (5.13), where ρ_v is one half the sum of the positive roots of $\mathcal{A}_v$. If $\mathcal{I}_v = \mathcal{I}_{\varphi,v}$, then the $W(\mathcal{A}_v : \mathcal{G}(\mathbb{Q}_v))$-orbit of that image must contain t_v. By changing $\tilde{t}_v$ in its Γ_v-orbit, we may assume

$$t_v = w\rho_v \tag{5.14}$$

for some $w \in W(\mathcal{A}_v : \mathcal{G}(\mathbb{Q}_v))$. Let $\check{\mathfrak{a}}_v^{\mathcal{P}}$ be defined in a similar way as $\check{\mathfrak{a}}_o^{\mathcal{P}}$, and let $t_v = t_v^{\mathcal{P}} + t_{v\mathcal{P}}$ be the decomposition of t_v according to $\check{\mathfrak{a}}_v = \check{\mathfrak{a}}_v^{\mathcal{P}} \oplus \check{\mathfrak{a}}_{\mathcal{P},v}$, where $\check{\mathfrak{a}}_{\mathcal{P},v} \supseteq \check{\mathfrak{a}}_{\mathcal{P}}$ is the $\mathbb{Q}_v$-character group of $\mathcal{P}$ made into a real vector space. By changing t_v in its $W(\mathcal{A}_v : \mathcal{L}_{\mathcal{P}}(\mathbb{Q}_v))$-orbit, we may assume that $t_v^{\mathcal{P}}$ belongs to the closure of the positive Weyl chamber $\overline{\check{\mathfrak{a}}_v^{\mathcal{P}+}}$.

Let Δ_v and $\Delta_v^{\mathcal{P}}$ be the same as in the proof of Proposition 4.3. For a root α of $\mathcal{A}_v$, let n_α be its multiplicity. If α is positive and reduced, then we have the inequality

$$\langle \check{\alpha}, \rho_v \rangle \geq n_\alpha + 2n_{2\alpha}, \tag{5.15}$$

for which equality occurs if and only if α is simple. This is easily verified by comparing the expressions

$$s_\alpha \rho_v = \rho_v - \alpha \langle \check{\alpha}, \rho_v \rangle = \rho_v - \sum_{\substack{\beta > 0 \\ s_\alpha \beta < 0}} n_\beta \beta.$$

From (5.14) and (5.15), we get for $\alpha \in \Delta_v^{\mathcal{P}}$

$$\begin{aligned} \left| \langle \check{\alpha}, t_v^{\mathcal{P}} \rangle \right| &= |\langle \check{\alpha}, t_v \rangle| \\ &\geq n_\alpha + n_{2\alpha} \\ &= \langle \check{\alpha}, \rho_v^{\mathcal{P}} \rangle. \end{aligned}$$

This implies $t_v^{\mathcal{P}} \in \rho_v^{\mathcal{P}} + \overline{\check{\mathfrak{a}}_v^{\mathcal{P}+}}$. By the boundedness of the matrix coefficients of the unitary representation π, this may happen only if $t_v^{\pi} = \rho_v^{\mathcal{P}}$. But then the local factor π_v of π at v is multiplication by an unramified character of $\mathcal{L}(\mathbb{Q}_v)$. Since this has to be the case at all but finitely many primes, weak approximation proves that π must be one-dimensional. Since π is cuspidal, this implies $\mathcal{P} = \mathcal{P}_o$.

To show that $\pi = \mathbb{C}_{\rho_o}$, it remains to verify that $t_{v\mathcal{P}_o} = \rho_o$. Fix a Weyl group invariant scalar product on $\check{\mathfrak{a}}_v$ and consider the following inequality:

$$\begin{aligned} \left| t_{v\mathcal{P}_o} \right|^2 &= \langle t_{v\mathcal{P}_o}, t_v \rangle \\ &\leq \langle t_{v\mathcal{P}_o}, \rho_v \rangle \\ &= \langle t_{v\mathcal{P}_o}, \rho_o \rangle \\ &\leq \left| t_{v\mathcal{P}_o} \right| \left| \rho_o \right|. \end{aligned} \tag{5.16}$$

The equalities are easy orthogonality relations. The inequality on the second line follows from $t_v = w\rho_v \in \rho_v - \overline{{}^+\check{\mathfrak{a}}_v}$, where $\overline{{}^+\check{\mathfrak{a}}_v}$ is the closed positive cone spanned by the positive roots, plus the fact that by our assumption on π we have $t_{v\mathcal{P}_o} = \lambda_\pi \in \overline{\check{\mathfrak{a}}_o^{\mathcal{G}+}}$ for the central character $\lambda + \pi$ of π. The inequality on the last line of (5.16) is the Cauchy-Schwarz inequality. We also have the equality

$$\left|t_{v\mathcal{P}_o}\right|^2 = |t_v|^2 - \left|t_v^{\mathcal{P}_o}\right|^2 = |\rho_v|^2 - \left|\rho_v^{\mathcal{P}_o}\right|^2 = |\rho_o|^2 .$$

Comparing this with (5.16), we see that equality must occur on the last line of (5.16). By Cauchy–Schwarz, this implies $t_{v\mathcal{P}_o} = \rho_o$, and we have finally verified that $\mathcal{P} = \mathcal{P}_o$ and $\pi = \mathbb{C}_{\rho_o}$. As was mentioned earlier, this completes the proof. □

Our next task is to determine the kernel of $\varXi_{\mathcal{P}_o}^{\mathcal{G}}$. We start with a few facts about the kernel of the operators $\tau_{\mathcal{P}_o}^{\mathcal{P}}$. The operator $\tau_{\mathcal{P}_o}^{\mathcal{G}}$ is a $\mathcal{G}(\mathbb{A})$-invariant linear functional on $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \mathbb{C}_{2\rho_o}$ and induces a duality

$$\begin{aligned} C^\infty(\mathcal{P}_o(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}) \otimes \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} \mathbb{C}_{2\rho_o} &\to \mathbb{C} \\ \phi \otimes \tilde{\phi} &\to \tau_{\mathcal{P}_o}^{\mathcal{G}}(\phi\tilde{\phi}). \end{aligned}$$

With respect to this pairing, for any standard parabolic subgroup $\mathcal{P}$ with $\dim \mathfrak{a}_o^{\mathcal{P}} = 1$, the orthogonal complement of $C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$ is $\operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})}$. For arbitrary $\mathcal{P} \neq \mathcal{P}_o$, the orthogonal complement of $C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$ is the kernel of $\tau_{\mathcal{P}}$. By Theorem 4.2 applied to $\boldsymbol{C}(\mathcal{L}_{\mathcal{P}}, \mathcal{P}_o/\mathcal{N}_{\mathcal{P}}, \mathbb{A})^\bullet$, we have

$$C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})) = \bigcap_{\substack{\mathcal{Q}\subset\mathcal{P} \\ \dim \mathfrak{a}_o^{\mathcal{Q}}=1}} C^\infty(\mathcal{Q}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})),$$

and the orthogonal complement of the intersection is the sum of the orthogonal complements since any $\mathbb{K}$-type occurs with finite multiplicity. We get

$$\ker \tau_{\mathcal{P}_o}^{\mathcal{P}} = \sum_{\substack{\mathcal{Q}\subset\mathcal{P} \\ \dim \mathfrak{a}_o^{\mathcal{Q}}=1}} \operatorname{Ind}_{\mathcal{Q}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{Q}}(\mathbb{A})}. \tag{5.17}$$

We will now give the description of the kernel of $\varXi_{\mathcal{P}_o}^{\mathcal{G}}$.

Theorem 5.2. *We have*

$$\ker \varXi_{\mathcal{P}_o}^{\mathcal{G}} = \sum_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}_o^{\mathcal{P}}=1}} \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}. \tag{5.18}$$

Proof. It is clear from (5.6) that the right-hand side of (5.18) is really contained in the kernel of $\varXi_{\mathcal{P}_o}^{\mathcal{G}}$. Conversely, let $f \in \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} S(\check{\mathfrak{a}}_o^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_o}$ belong to the kernel of $\varXi_{\mathcal{P}_o}^{\mathcal{G}}$. Define $\delta_{\mathcal{P}} \in S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})$ by

$$\delta_{\mathcal{P}} = \prod_{\alpha \in \Delta_{\mathcal{P}}} \omega_\alpha,$$

where ω_α is defined by

$$\langle \omega_\alpha, \check{\beta} \rangle = \begin{cases} \{0\} & \text{if } \beta \in \Delta_{\mathcal{P}} \setminus \{\alpha\} \\ 1 & \text{if } \alpha = \beta. \end{cases}$$

There is a unique decomposition

$$f = \sum_{\mathcal{P} \in \mathfrak{P}} f^{(\mathcal{P})} \delta_{\mathcal{P}}$$

with

$$f_{(\mathcal{P})} \in \operatorname{Ind}_{\mathcal{P}_o}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_o}.$$

Of course, the map $f \to f^{(\mathcal{P})}$ is only a map of vector spaces. From the fact that $f \in \ker \Xi_{\mathcal{P}_o}^{\mathcal{G}}$ we derive

$$(\mathrm{Id} \otimes \tau_{\mathcal{P}_o}^{\mathcal{P}}) f^{(\mathcal{P})} = 0 \in \operatorname{Ind}_{\mathcal{P}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{P}}}. \tag{5.19}$$

By (5.17) this implies

$$f^{(\mathcal{P})} \delta_{\mathcal{P}} \in \sum_{\substack{\mathcal{Q} \subseteq \mathcal{P} \\ \dim \mathfrak{a}_o^{\mathcal{Q}} = 1}} \operatorname{Ind}_{\mathcal{Q}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{Q}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}$$

and proves (5.18).

Let T be a bijective map from the set of vectors $\rho_{\mathcal{P}}$ for $\mathcal{P} \in \mathfrak{P}$ to the set $\{0; 1; \ldots; 2^{\dim \mathfrak{a}_o^{\mathcal{G}}} - 1\}$ with the following property: If $\rho_{\mathcal{Q}} \in \rho_{\mathcal{P}} - \overline{{}^{+}\check{\mathfrak{a}}_o^{\mathcal{G}}}$, then $T(\rho_{\mathcal{P}}) \le T(\rho_{\mathcal{Q}})$. Here $\overline{{}^{+}\check{\mathfrak{a}}_o^{\mathcal{G}}}$ is the closed cone spanned by Δ_o. It easy to verify the existence of such a function T. Let $\mathcal{P}^{(i)}$ be the unique parabolic subgroup with $T(\rho_{\mathcal{P}^{(i)}}) = i$. Then $\mathcal{P}^{(0)} = \mathcal{P}_o$.

It is a consequence of (5.6) that

$$\Xi_{\mathcal{P}_o}^{\mathcal{G}} f = \sum_{\mathcal{P}} \Xi_{\mathcal{P}}^{\mathcal{G}} \left((\mathrm{Id} \otimes \tau_{\mathcal{P}_o}^{\mathcal{P}}) f^{(\mathcal{P})} \delta_{\mathcal{P}} \right). \tag{5.20}$$

We will prove (5.19) for $\mathcal{P} = \mathcal{P}^{(i)}$ by induction on i by an investigation of the constant term of the Eisenstein series occurring in (5.20). Recall that for a continuous function ψ on $\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A})$, the constant term with respect to $\mathcal{P}$ is defined by

$$\psi_{\mathcal{P}}(g) = \int_{\mathcal{N}_{\mathcal{P}}(\mathbb{Q}) \backslash \mathcal{N}_{\mathcal{G}}(\mathbb{A})} \psi(ng)\, dn,$$

where the Haar measure dn is normalised by $1_{\mathcal{P}} = 1$. The necessary facts about the constant term of Eisenstein series are summarised in the following lemma, which will be proved after the proof of Theorem 5.2 is complete.

Lemma 5.3. *There exists a finite set $\mathfrak{W}_i$ of affine maps $\check{\mathfrak{a}}_{\mathcal{P}^{(i)}} \to \check{\mathfrak{a}}_o$ such that*

$$\left(E^{\mathcal{G}}_{\mathcal{P}^{(i)}}(\phi,\lambda)\right)_{\mathcal{P}_o}(g) = \sum_{w\in\mathfrak{W}_i} \left(N_i(w,\lambda)\phi\right)(g)\,\mathrm{e}^{\langle w\lambda+\rho_o, H_{\mathcal{P}_o}(g)\rangle}, \tag{5.21}$$

where $N_i(w,\lambda)$ is a meromorphic function from $\check{\mathfrak{a}}_{\mathcal{P}}$ to the space of $\mathbb{K}$-invariant homomorphisms from $C^\infty(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$ to $C^\infty(\mathcal{P}_o(\mathbb{A})\backslash\mathcal{G}(\mathbb{A}))$. If w_i is defined by

$$\begin{aligned} w_i\colon \check{\mathfrak{a}}_{\mathcal{P}^{(i)}} &\to \check{\mathfrak{a}}_o \\ w_i\lambda &= \lambda - \rho_o^{\mathcal{P}^{(i)}}, \end{aligned}$$

then $w_i \in \mathfrak{W}_i$ and $N_i(w_i,\lambda)\phi = \phi$. Furthermore, if $w\in\mathfrak{W}_j$ and if $w\rho_{\mathcal{P}^{(j)}} = \rho_{\mathcal{P}^{(i)}} - \rho_o^{\mathcal{P}^{(i)}}$, then $j\le i$.

Let us assume that (5.19) has been proved for $\mathcal{P}=\mathcal{P}^{(j)}$ with $j<i$. If $i=0$, this assumption is void. In any case, the induction assumption implies that the only summands in (5.20) which are possibly different from zero belong to the parabolic subgroups $\mathcal{P}^{(j)}$ with $j\ge i$. As a consequence of (5.21), the constant term of $\Xi^{\mathcal{G}}_{\mathcal{P}_o} f$ may be written as

$$\left(\Xi^{\mathcal{G}}_{\mathcal{P}_o} f\right)_{\mathcal{P}_o}(g) = \sum_{\lambda\in\Lambda} f_\lambda(g)\,\mathrm{e}^{\langle\lambda+\rho_o, H_{\mathcal{P}_o}(g)\rangle},$$

where Λ is a finite subset of $\check{\mathfrak{a}}_o$ and where f_λ is a continuous function on $\mathcal{G}(\mathbb{A})$ with the property that for any $g\in\mathcal{G}(\mathbb{A})$, the function $f_\lambda(pg)$ of $p\in\mathcal{P}_o(\mathbb{A})$ is a polynomial in $H_{\mathcal{P}_o}(p)$. Since f is in the kernel of $\Xi^{\mathcal{G}}_{\mathcal{P}_o}$, we have $f_\lambda = 0$ for any λ.

Let $N=\dim\mathfrak{a}^{\mathcal{G}}_{\mathcal{P}^{(i)}}$, let $\alpha_1,\ldots,\alpha_N$ be the elements of $\Delta_{\mathcal{P}^{(i)}}$, and let $\omega_i = \omega_{\alpha_i}$. We have a unique representation

$$(\mathrm{Id}\otimes\tau^{\mathcal{P}}_{\mathcal{P}_o})f^{(\mathcal{P})} = \sum_{a_1,\ldots,a_N=0}^{\infty} \Big(\prod_{k=1}^{N}\omega_i^{a_i}\Big)\otimes f_{a_1,\ldots,a_N}$$

with $f_{a_1,\ldots,a_N}\in\mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}^{(i)}}\mathbb{C}_{2\rho_{\mathcal{P}^{(i)}}}$. By the induction assumption, (5.20), the definition of $\Xi^{\mathcal{G}}_{\mathcal{P}}$ and Lemma 5.3, we have

$$f_{\rho_{\mathcal{P}^{(i)}}-\rho_o^{\mathcal{P}^{(i)}}}(g) = \sum_{a_1,\ldots,a_N=0}^{\infty}\left(\prod_{k=1}^{N}(a_i+1)\langle\omega_i, H_{\mathcal{P}_o}(g)\rangle^{a_i}\right) f_{a_1,\ldots,a_N}(g).$$

This function vanishes identically if and only if $f_{a_1,\ldots,a_N}=0$ for all choices of the a_i. This establishes (5.19) and completes the proof of the theorem. □

Proof (of Lemma 5.3). The formula (5.21) is a general fact from the theory of Eisenstein systems (cf. [15, §7] or the modern exposition [18, §IV]). In general, the theory of Eisenstein systems provides for the possibility of additional polynomial factors of higher degree in the expression for the constant term. Since this may happen only in the case of singular infinitesimal character, in our case the expression for the constant term simplifies to (5.21).

To arrive at the assertion about $N_i(w_i,\lambda)$, we consider the partial Eisenstein series $E^{\mathcal{R}}_{\mathcal{P}}(\phi,\lambda)$, which is defined as in (5.5), but with the summation restricted to $\mathcal{P}(\mathbb{Q})\backslash\mathcal{R}(\mathbb{Q})$. As a general fact about Eisenstein systems, the constant term of $E^{\mathcal{R}}_{\mathcal{P}}(\phi,\lambda)$ is given as in (5.21), but with the summation restricted to those $w\in\mathfrak{W}_i$ whose linear part is the identity on $\check{\mathfrak{a}}_{\mathcal{R}}$. In the special case $\mathcal{R}=\mathcal{P}$, where

$$E^{\mathcal{P}}_{\mathcal{P}}(\phi,\lambda)=\mathrm{e}^{\langle\lambda+\rho_{\mathcal{P}},H_{\mathcal{P}}(\cdot)\rangle}\,\phi(\cdot),$$

this expression for the constant term boils down to the assertion about $N_i(w_i,\lambda)$.

Finally, the only Eisenstein series $E^{\mathcal{G}}_{\mathcal{P}}(\phi,\rho_{\mathcal{P}})$ which have an exponential term of the form $\mathrm{e}^{\langle 2\rho_{\mathcal{P}^{(i)}},H_{\mathcal{P}^{(i)}}\rangle}$ in their constant terms are the Eisenstein series starting from $\mathcal{P}=\mathcal{P}^{(j)}$ with $j\le i$. This fact is a consequence of our condition on T and the proof of the main theorem in [9, §6]. The proof of Lemma 5.3 is complete. □

The description of the kernel of $\Xi^{\mathcal{G}}_{\mathcal{P}_o}$ is a little too complicated to use directly. Therefore, we will use it to get a resolution of the space of automorphic forms by induced representations whose cohomology can be described easily. This is achieved in two steps. In the first step, we consider the functor

$$\boldsymbol{F}^{\mathcal{P}}=\left\{\begin{array}{ll}\mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}}\,S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}})\otimes\check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})}\otimes\mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P}\neq\mathcal{P}_o\\ \mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}_o}\,S(\check{\mathfrak{a}}^{\mathcal{G}}_{o})\otimes\mathbb{C}_{2\rho_o} & \text{if } \mathcal{P}=\mathcal{P}_o.\end{array}\right\}\subseteq\mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}_o}\,S(\check{\mathfrak{a}}^{\mathcal{G}}_{o})\otimes\mathbb{C}_{2\rho_o}.$$

The map $\boldsymbol{F}(\mathcal{G})^{\tilde{\mathcal{P}}\supseteq\mathcal{P}}$ is given by the inclusion $S(\check{\mathfrak{a}}^{\mathcal{G}}_{\tilde{\mathcal{P}}})\subset S(\check{\mathfrak{a}}_{\mathcal{P}})$, followed by the inclusion

$$\check{\mathfrak{S}}t_{\mathcal{L}_{\tilde{\mathcal{P}}}(\mathbb{A})}\subseteq\mathrm{Ind}^{\tilde{\mathcal{P}}}_{\mathcal{P}}\,\check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})}\otimes\mathbb{C}_{2\rho^{\tilde{\mathcal{P}}}_{\mathcal{P}}}$$

which holds because of the description of $\check{\mathfrak{S}}t_{\mathcal{G}(\mathbb{A})}$ as the orthogonal complement of

$$\sum_{\mathcal{P}\supset\mathcal{P}_o}C^{\infty}(\mathcal{P}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})).$$

Proposition 5.4. *The map $\Xi^{\mathcal{G}}_{\mathcal{P}_o}$ defines an isomorphism*

$$H^{\dim\mathfrak{a}^{\mathcal{G}}_o}\left(C^*\left(\boldsymbol{F}(\mathcal{G})^{\bullet}\right)\right)\cong A_{\mathcal{J},\mathcal{I}}.$$

This is the only nonvanishing cohomology group of $C^*\left(\boldsymbol{F}(\mathcal{G})^{\bullet}\right)$.

Proof. Let the functor $\tilde{\boldsymbol{F}}^{\bullet}$ be defined by $\tilde{\boldsymbol{F}}^{\mathcal{P}}=\boldsymbol{F}(\mathcal{G})^{\mathcal{P}}$ if $\mathcal{P}\supset\mathcal{P}_o$ and

$$\tilde{\boldsymbol{F}}^{\mathcal{P}_o}=\sum_{\substack{\mathcal{P}\in\mathfrak{P}\\ \dim\mathfrak{a}^{\mathcal{P}}_o=1}}\mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}}\,\check{\mathfrak{S}}t_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})}\otimes S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}})\mathbb{C}_{2\rho_{\mathcal{P}}}.$$

This is our expression for the kernel of $\Xi^{\mathcal{G}}_{\mathcal{P}_o}$. It is therefore sufficient to prove the acyclicity of the chain complex of $\tilde{\boldsymbol{F}}^\bullet$.

We have a filtration of functors

$$\operatorname{Fil}_k \tilde{\boldsymbol{F}}^{\mathcal{P}} = \begin{cases} \sum\limits_{\substack{\mathcal{Q}\supseteq\mathcal{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}}=k}} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}}) \otimes \check{\mathfrak{St}}_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} \supset \mathcal{P}_o \\ \sum\limits_{\substack{\mathcal{Q}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}}=k}} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{Q}}) \otimes \check{\mathfrak{St}}_{\mathcal{L}_{\mathcal{P}}(\mathbb{A})} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}} & \text{if } \mathcal{P} = \mathcal{P}_o \end{cases}$$

with quotients

$$(\operatorname{Fil}_k / \operatorname{Fil}_{k-1}) \, \tilde{\boldsymbol{F}}^\bullet = \sum_{\substack{\mathcal{R}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{R}}=k}} \boldsymbol{M}(\mathcal{R})^\bullet,$$

where

$$\boldsymbol{M}(\mathcal{R})^{\mathcal{P}} = \begin{cases} 0 & \text{if } \mathcal{Q} \not\subseteq \mathcal{R} \\ S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{R}}) \otimes \operatorname{Ind}^{\mathcal{G}}_{\mathcal{R}} \boldsymbol{D}(\mathcal{L}_{\mathcal{R}})^{\mathcal{P}/\mathcal{N}_{\mathcal{R}}} & \text{if } \mathcal{Q} \subseteq \mathcal{R}. \end{cases}$$

The acyclicity of the functors $\boldsymbol{D}(\mathcal{L}_{\mathcal{R}})^\bullet$ is the assertion of Theorem 4.5. This implies the acyclicity of the quotients of the filtration of $\tilde{\boldsymbol{F}}^\bullet$, and hence of $\tilde{\boldsymbol{F}}^\bullet$ itself. □

If $\mathcal{P} \supset \mathcal{P}_o$, then the cohomology of the representation $\boldsymbol{F}(\mathcal{G})^{\mathcal{P}}$ is still rather mysterious. We construct a second resolution for $A_{\mathcal{J},\mathcal{I}}$ by the bifunctor

$$\boldsymbol{G}(\mathcal{G})^{\mathcal{P}}_{\mathcal{Q}} = \begin{cases} \operatorname{Ind}^{\mathcal{G}}_{\mathcal{Q}} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}} & \text{if } \mathcal{Q} \subseteq \mathcal{P} \\ \{0\} & \text{if } \mathcal{Q} \not\subseteq \mathcal{P}. \end{cases}$$

The map $\boldsymbol{G}(\mathcal{G})^{\mathcal{P}}_{\tilde{\mathcal{Q}}\subseteq\mathcal{Q}}$ is given by $\tau^{\mathcal{Q}}_{\tilde{\mathcal{Q}}}$, and the map $\boldsymbol{G}(\mathcal{G})^{\mathcal{P}\subseteq\tilde{\mathcal{P}}}_{\mathcal{Q}}$ is given by the inclusion $S(\check{\mathfrak{a}}^{\mathcal{G}}_{\tilde{\mathcal{P}}}) \subseteq S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}})$.

Proposition 5.5. *The map*

$$\operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}_o} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}_o}) \otimes \mathbb{C}_{2\rho_o} = \boldsymbol{G}(\mathcal{G})^{\mathcal{P}_o}_{\mathcal{P}_o} \subset \mathcal{Z}^{\dim \mathfrak{a}^{\mathcal{G}}_o} (\boldsymbol{G}^\bullet_\bullet(\mathcal{G}))$$

induces a surjection

$$\operatorname{Ind}^{\mathcal{G}}_{\mathcal{P}_o} S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}_o}) \otimes \mathbb{C}_{2\rho_o} \to H^{\dim \mathfrak{a}^{\mathcal{G}}_o} (C^* (\boldsymbol{G}(\mathcal{G})^\bullet_\bullet))$$

whose kernel is equal to the kernel of $\Xi^{\mathcal{G}}_{\mathcal{P}_o}$. This gives us an isomorphism

$$H^{\dim \mathfrak{a}^{\mathcal{G}}_o} (C^* (\boldsymbol{G}(\mathcal{G})^\bullet_\bullet)) \cong A_{\mathcal{J},\mathcal{I}}.$$

The other cohomology groups of $C^* (\boldsymbol{G}(\mathcal{G})^\bullet_\bullet)$ vanish.

Proof. It suffices to construct an isomorphism

$$H^l\left(C^*\left(\boldsymbol{G}(\mathcal{G})^{\mathcal{P}}_{\bullet}\right)\right) = \begin{cases} \boldsymbol{F}(\mathcal{G})^{\mathcal{P}} & \text{if } l = 0 \\ \{0\} & \text{if } l > 0 \end{cases} \tag{5.22}$$

which is functorial in $\mathcal{P}$. Let us fix $\mathcal{P}$. Then

$$\boldsymbol{G}(\mathcal{G})^{\mathcal{P}}_{\bullet} = \mathrm{Ind}^{\mathcal{G}}_{\mathcal{P}}\, S(\check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{P}}) \otimes \boldsymbol{M}_{\bullet} \otimes \mathbb{C}_{2\rho_{\mathcal{P}}},$$

where

$$\boldsymbol{M}_{\mathcal{Q}} = \begin{cases} \mathrm{Ind}^{\mathcal{P}}_{\mathcal{Q}}\, \mathbb{C}_{2\rho^{\mathcal{P}}_{\mathcal{Q}}} & \text{if } \mathcal{Q} \subseteq \mathcal{P} \\ \{0\} & \text{if } \mathcal{Q} \not\subseteq \mathcal{P}. \end{cases}$$

If $\mathcal{Q} \subseteq \mathcal{P}$, then $\boldsymbol{M}_{\mathcal{Q}}$ is in duality with $\boldsymbol{C}\left(\mathcal{L}_{\mathcal{P}}, (\mathcal{P}_o/\mathcal{N}_{\mathcal{P}}), \mathbb{A}\right)$. An isomorphism (5.22) is therefore given by Theorem 4.2. It is easy to see that this isomorphism is functorial in $\mathcal{P}$. □

6 Construction of the isomorphism of equation (3.4)

Our final goal is to compute the $(\mathfrak{g}, \boldsymbol{K})$-hypercohomology of the chain complex $C^*(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\bullet})$ and to relate it to the topological model explained in Section 3. We first compute $H^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_{\infty})}\left(C^*\left(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\mathcal{Q}}\right)\right)$ for a given parabolic subgroup $\mathcal{Q}$.

We have the projection

$$\begin{aligned} C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_{\infty})}\left(C^*\left(\boldsymbol{G}(\mathcal{G})^{\bullet}_{\mathcal{Q}}\right)\right) &\to C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_{\infty})}\left(\boldsymbol{G}(\mathcal{G})^{\mathcal{G}}_{\mathcal{Q}}\right) \\ &= C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_{\infty})}\left(\mathrm{Ind}^{\mathcal{G}}_{\mathcal{Q}}\, \mathbb{C}_{2\rho_{\mathcal{Q}}}\right). \end{aligned} \tag{6.1}$$

By Frobenius reciprocity we have

$$\begin{aligned} &C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_{\infty})}\left(\mathrm{Ind}^{\mathcal{G}}_{\mathcal{Q}}\, \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \cong \\ &\quad \left(\mathrm{Ind}^{\mathcal{G}(\mathbb{A}_f)}_{\mathcal{Q}(\mathbb{A}_f)}\, \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \otimes \left(\mathrm{Hom}_{\boldsymbol{K}^o_{\infty} \cap \mathcal{Q}(\mathbb{R})}\left(\Lambda^*(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}}/\mathfrak{q} \cap \mathfrak{k}), \mathbb{C}\right)\right), \end{aligned} \tag{6.2}$$

where the $\mathcal{G}(\mathbb{A}_f)$-action on the second factor is trivial. The second factor carries the differential of the standard complex for computing $(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}^o_{\infty} \cap \mathcal{Q}(\mathbb{R}))$-cohomology. The embedding

$$\det(\mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} \oplus \mathfrak{n}_{\mathcal{Q}}) \otimes \Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}} \cap \mathfrak{k}) \subset \Lambda^*(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}}/\mathfrak{q} \cap \mathfrak{k})[\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} + \dim \mathfrak{n}_{\mathcal{Q}}]$$

defines a projection

$$\begin{aligned} &p_{\mathcal{Q}}\colon\ \mathrm{Hom}_{\boldsymbol{K}^o_{\infty} \cap \mathcal{Q}(\mathbb{R})}\left(\Lambda^*(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}}/\mathfrak{q} \cap \mathfrak{k}), \mathbb{C}\right) \\ &\to \mathrm{Hom}_{\boldsymbol{K}^o_{\infty} \cap \mathcal{Q}(\mathbb{R})}\left(\Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}} \cap \mathfrak{k}) \otimes \det(\mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} \oplus \mathfrak{n}_{\mathcal{Q}}), \mathbb{C}\right)[-\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} - \dim \mathfrak{n}_{\mathcal{Q}}]. \end{aligned} \tag{6.3}$$

This is a homomorphism of chain complexes, and the differential of its target vanishes. Let $\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^*$ be the graded vector space

$$\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^{*} = \left(\operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \tag{6.4}$$
$$\otimes \operatorname{Hom}_{\boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R})} \left(\Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}} \cap \mathfrak{k}) \otimes \det(\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} \oplus \mathfrak{n}_{\mathcal{Q}}), \mathbb{C}\right) [-\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} - \dim \mathfrak{n}_{\mathcal{Q}}],$$

which can also be viewed as a chain complex with zero differential. The composition of (6.1), (6.2), and (6.3) defines a projection

$$C^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}_\infty^\circ)} \left(C^* \left(\boldsymbol{G}(\mathcal{G})^\bullet_{\mathcal{Q}}\right)\right) \to \boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^{*}. \tag{6.5}$$

Proposition 6.1. *The projection (6.5) defines an isomorphism on cohomology.*

Proof. By Frobenius reciprocity and by Kostant's theorem on $\mathfrak{n}$-homology ([27, Theorem 9.6.2] or [26, Theorem 3.2.3]), there is an isomorphism

$$\begin{aligned}
&H^*_{(\mathfrak{m}_{\mathcal{G}}, \boldsymbol{K}_\infty^\circ)} \left(\operatorname{Ind}_{\mathcal{Q}}^{\mathcal{G}} S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}}) \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) [\dim \mathfrak{n}_{\mathcal{Q}}] \\
&\cong \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes H^*_{\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}} \left(S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})\right) \otimes \det \mathfrak{n}_{\mathcal{Q}}^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \\
&\cong \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes \Lambda^*(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{P}}) \otimes \det \mathfrak{n}_{\mathcal{Q}}^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \qquad (6.6) \\
&\cong \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes \boldsymbol{E}(\mathcal{Q})^{\mathcal{P}^*} \otimes \det \mathfrak{n}_{\mathcal{Q}}^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right),
\end{aligned}$$

where the factors in curved braces have trivial $\mathcal{Q}(\mathbb{A}_f)$-action. We have used the following isomorphism, which is easily constructed:

$$H^*_{\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}} \left(S(\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{G}})\right) \cong \Lambda^*(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{P}}) \cong \boldsymbol{E}(\mathcal{Q})^{\mathcal{P}^*},$$

where $\boldsymbol{E}(\mathcal{Q})^{\mathcal{P}} = \Lambda^*(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{P}})$ was considered at the end of Section 4. This isomorphism, and hence also (6.6), is functorial with respect to $\mathcal{P}$. (Recall that $\boldsymbol{E}(\mathcal{Q})^{\tilde{\mathcal{P}} \subseteq \mathcal{P}^*}$ is defined by the projection $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{Q}} \to \check{\mathfrak{a}}_{\tilde{\mathcal{P}}}^{\mathcal{Q}}$.)

If $\mathcal{P} = \mathcal{G}$, then the composition of the isomorphism (6.6) with the projection

$$\boldsymbol{E}(\mathcal{Q})^{\mathcal{G}} = \Lambda^*(\check{\mathfrak{a}}_{\mathcal{Q}}^{\mathcal{G}}) \to \det(\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}})^{-1}[-\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}] \tag{6.7}$$

is precisely the map defined by (6.2) and (6.3) on cohomology. By Lemma 4.6, the projection (6.7) defines an isomorphism

$$\begin{aligned}
&H^* \left(C^* \left(\operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes \boldsymbol{E}(\mathcal{Q})^{\mathcal{P}^*} \otimes \det \mathfrak{n}_{\mathcal{Q}}^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right)\right)\right) \\
&\cong \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \left(\left\{H^*_{(\mathfrak{m}_{\mathcal{Q}}, \boldsymbol{K}_\infty^\circ \cap \mathcal{Q}(\mathbb{R}))}(\mathbb{C}) \otimes \boldsymbol{E}(\mathcal{Q})^{\mathcal{P}^*} \otimes \det(\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} \oplus \mathfrak{n}_{\mathcal{Q}})^{-1}\right\} \otimes \mathbb{C}_{2\rho_{\mathcal{Q}}}\right) \\
&\qquad [-\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}],
\end{aligned}$$

which proves our claim. □

We now have to determine the structure of a covariant functor on $\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^*$ such that (6.5) becomes functorial in $\mathcal{Q}$. We have to introduce some new notation. For any $\mathcal{Q}$, let the Haar measure on $\boldsymbol{K}_\infty \cap \mathcal{Q}(\mathbb{R})$ be normalized by $\int_{\boldsymbol{K}_\infty \cap \mathcal{Q}(\mathbb{R})} dk = 1$. Then there is a unique homomorphism

$$\tau_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{Q}(\mathbb{A}_f)} \colon \operatorname{Ind}_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \mathbb{C}_{2\rho_{\mathcal{P}}} \to \operatorname{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)} \mathbb{C}_{2\rho_{\mathcal{P}}}$$

such that we have, for the standard model of the induced representation in the space of functions on the adelic group,

$$(\tau_{\mathcal{P}}^{\mathcal{Q}} f)(g_f g_\infty) = \tau_{\mathcal{P}(\mathbb{A}_f)}^{\mathcal{Q}(\mathbb{A}_f)} \mathrm{e}^{\langle H_{\mathcal{Q}}(g_\infty), 2\rho_{\mathcal{Q}}\rangle} \int_{\boldsymbol{K}_\infty \cap \mathcal{Q}(\mathbb{R})} f(g_f k k_\infty)\, dk, \tag{6.8}$$

where $g_f \in \mathcal{G}(\mathbb{A}_f)$ and $g_\infty = p_\infty k_\infty \in \mathcal{G}(\mathbb{R})$ with $p_\infty \in \mathcal{P}(\mathbb{R})$ and $k_\infty \in \boldsymbol{K}_\infty^o$. It is easy to see that the right-hand side of (6.8) is independent of the choice of the Iwasawa decomposition $g_\infty = p_\infty k_\infty$.

It is clear that (6.1) is functorial with respect to $\mathcal{Q}$. Let $\tilde{\mathcal{Q}} \supseteq \mathcal{Q}$. Since $\tilde{\mathfrak{q}} \cap \mathfrak{m}_{\mathcal{G}} / \tilde{\mathfrak{q}} \cap \mathfrak{k} = \mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}} / \mathfrak{q} \cap \mathfrak{k}$, the formula

$$\left(i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}} \phi\right)(\lambda) = \int_{\boldsymbol{K}_\infty \cap \tilde{\mathcal{Q}}(\mathbb{R})} \phi(k\lambda)\, dk \tag{6.9}$$

for $\lambda \in \Lambda^*(\tilde{\mathfrak{q}} \cap \mathfrak{m}_{\mathcal{G}} / \tilde{\mathfrak{q}} \cap \mathfrak{k})$ and

$$\phi \in \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \mathcal{Q}(\mathbb{R})}(\tilde{\mathfrak{q}} \cap \mathfrak{m}_{\mathcal{G}} / \tilde{\mathfrak{q}} \cap \mathfrak{k}, \mathbb{C}) = \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \mathcal{Q}(\mathbb{R})}(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}} / \mathfrak{q} \cap \mathfrak{k}, \mathbb{C})$$

defines a map

$$i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}} \colon \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \mathcal{Q}(\mathbb{R})}(\mathfrak{q} \cap \mathfrak{m}_{\mathcal{G}} / \mathfrak{q} \cap \mathfrak{k}, \mathbb{C}) \to \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \tilde{\mathcal{Q}}(\mathbb{R})}(\tilde{\mathfrak{q}} \cap \mathfrak{m}_{\mathcal{G}} / \tilde{\mathfrak{q}} \cap \mathfrak{k}, \mathbb{C}).$$

It follows from (6.8) that the isomorphism (6.2) is functorial in $\mathcal{Q}$ if the transition homomorphism for its target is defined by $\tau_{\mathcal{Q}(\mathbb{A}_f)}^{\tilde{\mathcal{Q}}(\mathbb{A}_f)} \otimes i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}}$. It is clear that

$$\begin{aligned} i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}} \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \mathcal{Q}(\mathbb{R})}(\mathfrak{m}_{\mathcal{Q}} / \mathfrak{q} \cap \mathfrak{k} \otimes \det(\mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}} \oplus \mathfrak{n}_{\mathcal{Q}}), \mathbb{C}) \\ \subseteq \operatorname{Hom}_{\boldsymbol{K}_\infty^o \cap \tilde{\mathcal{Q}}(\mathbb{R})}(\mathfrak{m}_{\tilde{\mathcal{Q}}} / \tilde{\mathfrak{q}} \cap \mathfrak{k} \otimes \det(\mathfrak{a}_{\tilde{\mathcal{Q}}}^{\mathcal{G}} \oplus \mathfrak{n}_{\tilde{\mathcal{Q}}}), \mathbb{C}). \end{aligned}$$

Therefore, we may define $\boldsymbol{H}(\mathcal{G})_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}}$ by $\tau_{\mathcal{Q}(\mathbb{A}_f)}^{\tilde{\mathcal{Q}}(\mathbb{A}_f)} \otimes i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}}$. To verify that (6.3) is functorial in $\mathcal{Q}$, we have to verify that $p_{\tilde{\mathcal{Q}}} i_{\tilde{\mathcal{Q}} \supseteq \mathcal{Q}}$ vanishes on the kernel of $p_{\mathcal{Q}}$. This follows from the following lemma.

Lemma 6.2. *Let $\mathcal{H}$ be a semisimple algebraic group over $\mathbb{R}$, $\boldsymbol{K} \subset \mathcal{H}(\mathbb{R})$ a maximal compact subgroup, and let $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$ be a $\mathbb{R}$-parabolic subgroup of $\mathcal{H}$. Let $\mathfrak{h}$, $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ be the Lie algebras of $\mathcal{H}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$, $\mathcal{M}(\mathbb{R})$, $\mathcal{A}(\mathbb{R})$, $\mathcal{N}(\mathbb{R})$. If $\lambda \in \Lambda^i \mathfrak{a} \otimes \det \mathfrak{n} \otimes \Lambda^*(\mathfrak{m}/\mathfrak{k} \cap \mathfrak{m}) \subset \Lambda^*(\mathfrak{h}/\mathfrak{h} \cap \mathfrak{k})$ for $i < \dim \mathfrak{a}$, then*

$$\int_{\boldsymbol{K}^o} k\lambda\, dk = 0$$

in $\Lambda^*(\mathfrak{h}/\mathfrak{h} \cap \mathfrak{k})$.

Since (6.1), (6.2) and (6.3) are natural in $\mathcal{Q}$, the same is true for their composition (6.5). Therefore, Proposition 6.1 together with Proposition 5.5 and the proven Borel conjecture imply the following theorem.

Theorem 6.3. *Let $\boldsymbol{H}(\mathcal{G})_{\mathcal{Q}}{}^{*}$ be defined by (6.4), and let $\boldsymbol{H}(\mathcal{G})_{\tilde{\mathcal{Q}}\supseteq\mathcal{Q}} = \tau_{\mathcal{Q}(\mathbb{A}_f)}^{\tilde{\mathcal{Q}}(\mathbb{A}_f)} \otimes i_{\tilde{\mathcal{Q}}\supseteq\mathcal{Q}}$. Then we have an isomorphism of $\mathcal{G}(\mathbb{A}_f)$-modules*

$$H^k(\mathcal{G},\mathbb{C})_{\mathcal{I}} \cong H^{k-\dim \mathfrak{a}_o^{\mathcal{G}}}\left(C^*\left(\boldsymbol{H}(\mathcal{G})_{\bullet}{}^{*}\right)\right)$$

which respects the canonical real structures on its source and its target.

It remains to prove Lemma 6.2.

Proof (of Lemma 6.2). By Poincare duality, it suffices to verify that

$$\phi\left(\Lambda^j(\mathfrak{a}) \otimes \Lambda^*(\mathfrak{m}/\mathfrak{m}\cap\mathfrak{k})\right) = 0 \tag{6.10}$$

for $j > 0$ and any $\phi \in \mathrm{Hom}_{\boldsymbol{K}^o}\left(\Lambda^*(\mathfrak{h}/\mathfrak{k}),\mathbb{C}\right)$. Recall the definition of the compact homogeneous space $\boldsymbol{X}_{\mathcal{H}}^{(c)}$ and of the compact duals $\mathcal{H}^{(c)}$, $\mathcal{M}^{(c)}$, and $\mathcal{A}^{(c)}$ from the introduction. Then (6.10) admits a topological reformulation

$$\mathrm{im}\left(H^*(\boldsymbol{X}_{\mathcal{H}}^{(c)},\mathbb{C}) \to H^*(\boldsymbol{X}_{\mathcal{M}}^{(c)} \times \mathcal{A}^{(c)}(\mathbb{R}),\mathbb{C})\right) \subseteq H^*(\boldsymbol{X}_{\mathcal{M}}^{(c)},\mathbb{C}) \tag{6.11}$$

in terms of the pull-back of cohomology classes from $\boldsymbol{X}_{\mathcal{H}}^{(c)}$ to $\boldsymbol{X}_{\mathcal{M}}^{(c)} \times \mathcal{A}^{(c)}(\mathcal{R})$. Let J be an integer, and let

$$\begin{aligned} f_J \colon \mathcal{A}^{(c)}(\mathbb{R}) \times \boldsymbol{X}_{\mathcal{M}}^{(c)} &\to \boldsymbol{X}_{\mathcal{H}}^{(c)} \\ f_J(a,x) &= a^J x \end{aligned}$$

be defined by the action of $\mathcal{A}^{(c)}(\mathbb{R})$ on $\boldsymbol{X}_{\mathcal{H}}^{(c)}$ and the embedding $\boldsymbol{X}_{\mathcal{M}}^{(c)} \subset \boldsymbol{X}_{\mathcal{H}}^{(c)}$. To verify (6.11), it suffices to take some $J \neq 0$ and to verify

$$\mathrm{im}(f_J^*) = \mathrm{im}(f_0^*) \tag{6.12}$$

for the pull-back on cohomology with complex coefficients. For the right-hand side of (6.12) is always contained in the right-hand side of (6.11), and for $J \neq 0$ the left-hand sides of (6.11) and (6.12) agree.

As $\mathcal{H}$ was supposed to be semisimple, the fundamental group of $\mathcal{H}^{(c)}(\mathbb{R})$ is finite. Since $\mathcal{A}^{(c)}(\mathbb{R})$ is a product of circles, if J is divisible by a certain positive integer, the map

$$\begin{aligned} \mathcal{A}^{(c)}(\mathbb{R}) &\to \mathcal{H}^{(c)}(\mathbb{R}) \\ a &\to a^J \end{aligned}$$

will be homotopic to the identity. But then f_0 and f_J are homotopic, and this implies (6.12). The proof of Lemma 6.2 is complete. □

For those who are only interested in an algebraic formula for $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$, Theorem 6.3 would be the final result of this paper. It remains to derive the isomorphism (3.4) from this theorem.

Let

$$\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{Q}^*} = C^\infty(\mathcal{Q}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)) \otimes \mathrm{Hom}_{\boldsymbol{K}^o_\infty\cap\mathcal{Q}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}}\cap\mathfrak{k}),\mathbb{C}),$$

where the transition maps $\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{Q}\supseteq\tilde{\mathcal{Q}}^*}$ are given by the embedding

$$C^\infty(\mathcal{Q}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)) \subseteq C^\infty(\tilde{\mathcal{Q}}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f))$$

and the restriction to $\mathfrak{m}_{\tilde{\mathcal{Q}}}$

$$\mathrm{Hom}_{\boldsymbol{K}^o_\infty\cap\mathcal{Q}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_{\mathcal{Q}}/\mathfrak{m}_{\mathcal{Q}}\cap\mathfrak{k}),\mathbb{C}) \to \mathrm{Hom}_{\boldsymbol{K}^o_\infty\cap\tilde{\mathcal{Q}}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_{\tilde{\mathcal{Q}}}/\boldsymbol{k}\cap\mathfrak{m}_{\tilde{\mathcal{Q}}}),\mathbb{C}).$$

If $\mathcal{P}$ is a standard parabolic subgroup, then $\boldsymbol{K}^o_\infty\cap\mathcal{P}_o(\mathbb{R})$ meets every connected component of $\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})$ by Proposition 2.1. Consequently, there is a canonical isomorphism between

$$H^*\left(C^*\left(\tilde{\boldsymbol{H}}(\mathcal{G})^{\bullet^*}\right)\right)^{\pi_o(\boldsymbol{K}_\infty\cap\mathcal{P}_o(\mathbb{R}))}$$

and the invariants in the hypercohomology of the complex associated to the functor $\boldsymbol{A}(\mathcal{G},\mathbb{C})^{\mathcal{P}}$

$$H^*\left(C^*\left(\tilde{\boldsymbol{H}}(\mathcal{G})^{\bullet^*}\right)\right) \cong H^*_c(\mathcal{G},\mathbb{C})_{\mathcal{I}} \cong H^*(\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}, C^*(\boldsymbol{A}(\mathcal{G},\mathcal{C}))).$$

This isomorphism identifies the canonical real subspace of its source with

$$i^p H^p(\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}, C^*(\boldsymbol{A}(\mathcal{G},\mathcal{C}))).$$

To construct (3.4), we construct a duality between $\tilde{\boldsymbol{H}}(\mathcal{G})^{\bullet^*}$ and $\boldsymbol{H}(\mathcal{G})_\bullet{}^*$. Let o be an orientation of the real vector space $\mathfrak{m}_{\mathcal{G}}/\mathfrak{k}$. Multiplication by a square root i of -1 defines an isomorphism between $\mathfrak{m}_{\mathcal{G}}/\mathfrak{k}$ and the tangent space of $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}$ at the origin. Therefore, o and i define an orientation o_i of the differentiable manifold $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}$. There exists $\delta_o \in i^{\dim(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})}\det(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})$ such that

$$\int^{o_i}_{\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}} \delta_o = i^{\dim(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})}$$

if δ_o is viewed as a real $\dim(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})$-form on $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{G}}}$. We have

$$o_{-i} = (-1)^{\dim(\mathfrak{m}_{\mathcal{G}}/\mathfrak{k})} o_i,$$

hence δ_o is independent of the choice of i. Then δ_o defines a duality

$$\begin{aligned}\mathrm{Hom}_{\boldsymbol{K}_\infty^\circ\cap\mathcal{Q}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_\mathcal{Q}/\mathfrak{m}_\mathcal{Q}\cap\mathfrak{k})\otimes\det(\mathfrak{a}_\mathcal{Q}^\mathcal{G}\oplus\mathfrak{n}_\mathcal{Q}),\mathbb{C})\\ \otimes\mathrm{Hom}_{\boldsymbol{K}_\infty^\circ\cap\mathcal{Q}(\mathbb{R})}(\Lambda^*(\mathfrak{m}_\mathcal{Q}/\mathfrak{m}_\mathcal{Q}\cap\mathfrak{k}),\mathbb{C})\\ \to\mathbb{C}[-\dim(\mathfrak{m}_\mathcal{G}/\mathfrak{k})],\end{aligned}$$

and $\tau_q f Q^{\mathcal{G}(\mathbb{A}_f)}$ defines a duality between $\mathrm{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)}\mathbb{C}$ and $\mathrm{Ind}_{\mathcal{Q}(\mathbb{A}_f)}^{\mathcal{G}(\mathbb{A}_f)}\mathbb{C}_t rQ$. We get a duality

$$\tilde{\boldsymbol{H}}(\mathcal{G})^{\bullet^*}\boldsymbol{H}(\mathcal{G})_\bullet{}^*\to\mathbb{C}[-\dim(\mathfrak{m}_\mathcal{G}/\mathfrak{k})] \tag{6.13}$$

which defines an isomorphism (3.4) independent of o; (6.13) changes its sign if o is changed. Furthermore, (6.13) maps the real subspaces of $\boldsymbol{H}$ and $\check{\boldsymbol{H}}$ to $i^{\dim(\mathfrak{m}_\mathcal{G}/\mathfrak{k})}\mathbb{R}$, whence the assertion about real subspaces in Theorem 3.1 applies.

7 Some examples

7.1 Ghost classes in the image of the Borel map

It is rather easy to use the topological model to explicitly compute the kernel of the Borel map

$$I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^\circ}\to H^*(\mathcal{G},\mathbb{C}).$$

This allows us to give new examples of ghost classes. Recall that a cohomology class of $\mathcal{G}$ is called a ghost class if it trivially restricts to each boundary component of the Borel–Serre compactification and if its restriction to the full Borel–Serre boundary is not zero. This notion was coined by Borel. The first example of a ghost class was constructed by Harder in the cohomology of GL_3 over totally imaginary fields, using Eisenstein series starting from an algebraic Hecke character whose L-function vanishes at the center of the functional equation. Our computation of the kernel of the Borel map will make it clear that ghost classes abound in the image of the Borel map, at least for most groups of sufficiently high rank.

Recall that $H_c^*(\mathcal{G},\mathbb{C})_\mathcal{I}$ can be computed as the cohomology of the complex of graded vector spaces $C^*(\check{\boldsymbol{H}}^{\bullet *})$. The map

$$C^*(\check{\boldsymbol{H}}^{\bullet *})\to\check{\boldsymbol{H}}^{\mathcal{G}*}\to I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^\circ}$$

defines a homomorphism

$$H_o^*(\mathcal{G},\mathbb{C})_\mathcal{I}\to I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty^\circ} \tag{7.1}$$

which is easily identified with the Poincare dual of the Borel map. It can also be viewed as the restriction to the subspaces which are annihilated by the Hecke ideal $\mathcal{I}$ of the map from cohomology with compact support to L_2-cohomology. By the definition of the differential of the complex $C^*(\check{\boldsymbol{H}}^{\bullet *})$, the image of (7.1) is the space

$$(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty})_{\text{Image}} = \ker\left(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty} \to \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}=1}} I^*_{\mathcal{M}_{\mathcal{P}}(\mathbb{R}),\boldsymbol{K}^o_\infty\cap\mathcal{P}(\mathbb{R})} \right). \tag{7.2}$$

In other words, a cohomology class of the constant representation of $\mathcal{G}$ is in the image of the cohomology with compact support if and only if its restriction to the cohomology of the constant representation of any maximal Levi component vanishes. By Poincare duality, the kernel $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty})_{\text{Kernel}}$ of the Borel map is equal to the orthogonal complement of $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty})_{\text{Image}}$. Let $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty})_{\text{Ghost}}$ be the space of all invariant forms $i \in I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}$ such that, for any parabolic subgroup $\mathcal{P}$ with $\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}} = 1$, the image of j in

$$I^*_{\mathcal{M}_{\mathcal{P}}(\mathbb{R}),\boldsymbol{K}^o_\infty\cap\mathcal{P}(\mathbb{R})}$$

belongs to

$$(I^*_{\mathcal{M}_{\mathcal{P}}(\mathbb{R}),\boldsymbol{K}^o_\infty\cap\mathcal{P}(\mathbb{R})})_{\text{Kernel}}.$$

Then the space of ghost classes in the image of the Borel map is isomorphic to $(I_{\text{Ghost}}/I_{\text{Image}} + I_{\text{Kernel}})^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}$. This follows from: the fact ([20, 1.10]) that after identifying $(\mathfrak{g}, \boldsymbol{K})$- and de Rham-cohomology, the homomorphism defined on $(\mathfrak{g}, \boldsymbol{K})$-cohomology, by taking the constant term along $\mathcal{P}$, corresponds to restriction to the Borel–Serre boundary component belonging to $\mathcal{P}$; and (7.3) as presented below in Proposition 7.2.

Let us explain this a little more for the case of groups over totally imaginary fields. That is, let $\mathcal{G}$ be obtained by Weil restriction from a totally imaginary field. Then $\boldsymbol{X}^{(c)}_{\mathcal{G}}$ has a group structure. Therefore, its cohomology $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}$ is a Hopf algebra. By the Hopf structure theorem, it is an exterior algebra over a graded space $E^*(\mathcal{G})$ of primitive elements, which are of odd order. The same is true for all Levi components of parabolic subgroups of $\mathcal{G}$. Let

$$E^*_{\text{Top}}(\mathcal{G}) = \ker\left(E^*(\mathcal{G}) \to \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}=1}} E^*(\mathcal{M}_{\mathcal{G}}) \right)$$

and

$$E^*_{\text{Ghost}}(\mathcal{G}) = \ker\left(E^*(\mathcal{G}) \to \bigoplus_{\substack{\mathcal{P}\in\mathfrak{P} \\ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}=2}} E^*(\mathcal{M}_{\mathcal{G}}) \right).$$

Then

$$\begin{aligned}
\big(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}\big)_{\text{Image}} &= E^*_{\text{Top}}(\mathcal{G}) \wedge \Lambda^*\big(E^*(\mathcal{G})\big) \\
\big(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}\big)_{\text{Kernel}} &= \det\big(E^*_{\text{Top}}(\mathcal{G})\big) \wedge \Lambda^*\big(E^*(\mathcal{G})\big) \\
\big(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}\big)_{\text{Ghost}} &= \det\big(E^*_{\text{Ghost}}(\mathcal{G})/E^*_{\text{Top}}(\mathcal{G})\big) \wedge \Lambda^*\big(E^*(\mathcal{G})/E^*_{\text{Top}}(\mathcal{G})\big).
\end{aligned}$$

For instance, for SL_n over a totally imaginary field $\mathbb{K}$, we have primitive generators $\lambda_2^{(v)},\ldots,\lambda_n^{(v)}$ for each v in the set $\mathfrak{V}_{\mathbb{K}}^{\infty}$ of archimedean primes of $\mathbb{K}$, with the relation $\sum_{\mathfrak{V}_{\mathbb{K}}^{\infty}} \lambda_1^{(v)} = 0$. The degree of $\lambda_j^{(v)}$ is $2j-1$. The following fact is an obvious consequence of this discussion.

Theorem 7.1. *Then an invariant form is in the image of cohomology with compact support if and only if it is a sum of monomials which contain one of the classes $\lambda_n^{(v)}$. It is in the kernel of the Borel map if and only if it is divisible by $\bigwedge_{\mathfrak{V}_{\mathbb{K}}^{\infty}} \lambda_n^{(v)}$. It defines a ghost class if and only if it is a sum of monomials which contain all of the classes $\lambda_{n-1}^{(v)}$ but none of the classes $\lambda_n^{(v)}$.*

The space $E^*(\mathcal{G})$ is known for groups over totally imaginary fields by the known calculation of the cohomology of compact Lie groups. (See [1, §11] for a statement of the result and for references, and [12, §VI.7] for the case of the classical groups.) Therefore, the spaces $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\text{Image, Kernel, Ghost}}$ are at least in principle known for groups over totally imaginary fields.

Let us also formulate the result about the kernel of the Borel map and about ghost classes for SL_n over a field $\mathbb{K}$ which has real places. We first have to formulate the necessary facts about the cohomology of $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$. They can be obtained from the consideration of the Leray spectral sequence for the projection $\mathrm{SU}(n,\mathbb{R}) \to \mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$, either by hand or by the general theory (cf. [12, XI.4.4.]).

Proposition 7.2. *If n is odd, then the cohomology with complex coefficients of $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is an exterior algebra with generators $\tilde\lambda_3, \tilde\lambda_5,\ldots,\tilde\lambda_n$, where $\deg\tilde\lambda_i = 2i-1$. Furthermore, $\tilde\lambda_i$ can be obtained from the primitive element λ_i in the cohomology of $\mathrm{SU}(n,\mathbb{R})$ by pull-back via the map*

$$\begin{aligned} \mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) &\to \mathrm{SU}(n,\mathbb{R}) \\ \dot g &\to g\cdot g^{\mathrm{T}}. \end{aligned} \tag{7.3}$$

If n is even, then the cohomology of $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$ is an exterior algebra generated by elements $\tilde\lambda_3,\ldots,\tilde\lambda_{n-1}$ obtained in the same way as above, and by a class ε in degree n, which is the Euler class of the canonical n-dimensional orientable real bundle on $\mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R})$.

If $\sum_{i=1}^k n_i \le n$, then the restriction of $\tilde\lambda_l$ to

$$\prod_{i=1}^k \mathrm{SU}(n_i,\mathbb{R})/\mathrm{SO}(n_i,\mathbb{R}) \subset \mathrm{SU}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \tag{7.4}$$

is

$$\sum_{\substack{1\le i\le k\\ n_i\le l}} \tilde\lambda_l^{(i)},$$

where $\tilde\lambda_l^{(i)}$ is the copy of $\tilde\lambda_l$ for the i-th factor in (7.4). If n is even, then the restriction of the Euler class ε to (7.4) can be described as follows.

If $n = \sum_{i=1}^k n_i$ and if all the n_i are even, then the restriction of ε is given by

$$\varepsilon^{(1)} \wedge \dots \varepsilon^{(k)},$$

where ε^i is the copy of ε for the i-th factor in (7.4). If $n < \sum_{i=1}^k n_i$ or if some of the n_i are odd, then the restriction of the Euler class is zero.

Now let $\mathbb{K}$ be a field which has at least one real place. Let $\mathcal{G}$ be SL_n over $\mathbb{K}$. If n is odd, then the space of invariant forms is an exterior algebra with generators $\tilde{\lambda}_3^{(u)}$, $\tilde{\lambda}_5^{(u)}$,..., $\tilde{\lambda}_n^{(u)}$ for the real places u and $\lambda_2^{(v)}$,...,$\lambda_n^{(v)}$ for the complex places v (if there are any complex places). A monomial in these generators belongs to $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\mathrm{Image}}$ if and only if it contains one of the generators $\tilde{\lambda}_n^{(u)}$ for a real place u or one of the generators $\lambda_n^{(v)}$ for a complex place v. It belongs to the kernel of the Borel map if and only if it is divisible by

$$\bigwedge_{u \text{ real}} \tilde{\lambda}_n^{(u)} \wedge \bigwedge_{v \text{ imaginary}} \lambda_n^{(v)}.$$

If n is even, then $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty}$ is an exterior algebra with generators $\tilde{\lambda}_3^{(u)}$, $\tilde{\lambda}_5^{(u)}$,..., $\tilde{\lambda}_{n-1}^{(u)}$ and $\varepsilon^{(u)}$ for each real place u and $\lambda_2^{(v)}$,..., $\lambda_n^{(v)}$ for the complex places v. A monomial in these generators belongs to $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\mathrm{Image}}$ if and only if it contains one of the following factors:

- $\lambda_n^{(v)}$ for a imaginary place v;
- $\varepsilon^{(u)} \wedge \tilde{\lambda}_{n-1}^{(w)}$ for real places u and w;
- or $\varepsilon^{(u)} \wedge \lambda_{n-1}^{(v)}$ for a real place u and an imaginary place v.

A monomial belongs to the kernel of the Borel map if and only if it contains at least one of the following two factors:

$$\bigwedge_{u \text{ real}} \tilde{\lambda}_{n-1}^{(u)} \wedge \bigwedge_{v \text{ imaginary}} (\tilde{\lambda}_n^{(u)} \wedge \tilde{\lambda}_{n-1}^{(u)})$$

$$\text{or} \bigwedge_{u \text{ real}} \varepsilon^{(u)} \wedge \bigwedge_{v \text{ imaginary}} \lambda_n^{(v)},$$

where a product over the set of imaginary places is supposed to be one if the field is totally real. In particular, if $n > 2$ is even and if $\mathbb{K}$ is totally real, then $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\mathrm{Image}}$ does not contain $(I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty})_{\mathrm{Kernel}}$ completely.

We can use this to describe all ghost classes in the image of the Borel map. If n is odd, then a monomial in the generators of $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^\circ_\infty}$ is a ghost class if and only if it contains all the generators $\tilde{\lambda}_{n-2}^{(u)}$ for all the real places u and all the generators $\lambda_{n-1}^{(v)}$ and $\lambda_{n-2}^{(v)}$ for all the imaginary places v, but none of the generators $\tilde{\lambda}_n^{(u)}$ or $\lambda_n^{(v)}$. If $n = 3$, this means that there are no ghost classes in the image of the Borel map. (Recall our assumption that $\mathbb{K}$ is not purely imaginary.)

If n is even, then a monomial μ in the generators of $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}^o_\infty}$ defines a ghost class if and only if at least one of the following four conditions is satisfied:

- $\mathbb{K}$ is not totally real, and μ contains all the generators $\lambda^{(v)}_{n-1}$ for v complex and $\tilde{\lambda}^{(u)}_{n-1}$ for u imaginary, but none of the classes $\lambda^{(v)}_n$ nor any Euler class $\varepsilon^{(u)}$;
- $\mathbb{K}$ is not totally real, and μ contains all the generators $\varepsilon^{(u)}$ for u real and $\lambda^{(v)}_{n-2}$ for v imaginary, but none of the generators $\lambda^{(v)}_n$;
- $n \geq 6$ and $\mathbb{K}$ is not totally real, and μ contains at least one of the generators $\varepsilon^{(u)}$ and all of the generators $\tilde{\lambda}^{(u)}_{n-3}$, $\lambda^{(v)}_{n-3}$ and $\lambda^{(v)}_{n-2}$, but none of $\lambda^{(v)}_n$, $\lambda^{(v)}_{n-1}$ or $\tilde{\lambda}^{(u)}_{n-1}$;
- or $n \geq 6$ and $\mathbb{K} \neq \mathbb{Q}$ is totally real, and μ contains at least one but not all of the generators $\varepsilon^{(u)}$ and all of the generators $\tilde{\lambda}^{(u)}_{n-3}$, but none of $\tilde{\lambda}^{(u)}_{n-1}$.

If $n = 4$ and $\mathbb{K}$ is totally real or if $n > 4$ is even and $\mathbb{K} = \mathbb{Q}$, this means that there are no ghost classes in the image of the Borel map.

In our description of ghost classes, we have used the following fact[1].

Proposition 7.3. *Let $\mathcal{P}$ be a standard parabolic subgroup. Then the image of the restriction map*

$$H^*_{\mathfrak{m}_\mathcal{G},\boldsymbol{K}_\infty}(\mathbb{C}) \to H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C})$$

is contained in $H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n}_\mathcal{P},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C}) \subset H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C})$.

Proof. By an easy induction argument, it suffices to prove this assertion for maximal proper $\mathbb{Q}$-parabolic subgroups. In this case, it follows from Lemma 7.4 below that

$$H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C}) \cong H^*_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\mathbb{C}) \oplus H^{*-\dim\mathfrak{n}_\mathcal{P}}_{\mathfrak{m}_\mathcal{P}+\mathfrak{n},\boldsymbol{K}_\infty\cap\mathcal{P}(\mathbb{R})}(\det\mathfrak{n}_\mathfrak{P}),$$

and it follows from (6.2) that the restriction of an element of $H^*_{\mathfrak{m}_\mathcal{G},\boldsymbol{K}_\infty}(\mathbb{C})$ never has a nonvanishing projection to the second summand. □

Lemma 7.4. *Let $\mathcal{P}$ be a maximal proper $\mathbb{Q}$-parabolic subgroup of $\mathcal{G}$. Then*

$$H^*(\mathfrak{n}_\mathcal{P},\mathbb{C})^{\mathcal{M}_\mathcal{P}} = \mathbb{C} \oplus \det\mathfrak{n}_\mathcal{P}[-\dim\mathfrak{n}_\mathcal{P}]. \tag{7.5}$$

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{a}_o$ and is contained in $\mathfrak{a}_o \oplus \mathfrak{m}_\mathcal{P}$. Let $\mathcal{B} \subseteq \mathcal{P}_o$ be a Borel subgroup defined over $\mathbb{C}$ with $\mathfrak{h} \subseteq \mathfrak{b}$, and let Δ_η be the set of simple positive roots of $\mathfrak{h}$ determined by B. This set decomposes according to the restrictions to $\mathfrak{a}_o$:

$$\Delta_\mathfrak{h} = \bigcup_{\mathfrak{a}\in\Delta_o\cap\{0\}} \Delta_{\mathfrak{h},\alpha}.$$

[1] I am indebted to A. Kewenig and T. Rieband for pointing out that this is not self-evident.

By a theorem of Kostant ([27, Theorem 9.6.2] or [26, Theorem 3.2.3])

$$H^*(\mathfrak{n}_{\mathcal{P}}, \mathbb{C}) \cong \sum_{\substack{w \in \Omega(\mathfrak{h},\mathfrak{g}) \\ w^{-1}\Delta_{\mathfrak{h}}^{\mathfrak{m}_{\mathcal{P}}} > 0}} F_{w\rho_h - \rho_h}[-\ell(w)]. \tag{7.6}$$

In fact, it follows from the proof given in the above references that (7.6) even holds in the derived category of $(\mathfrak{m}_{\mathcal{P}}, \boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R}))$-modules. This implies the splitting of the Leray spectral sequence, which will be used below. Since both summands on the right-hand side of (7.5) are accounted for by this formula, it suffices to show that there are at most two w for which the corresponding summand in (7.6) contributes to (7.5).

Indeed, if the summand belonging to w in (7.6) contributes to (7.5), then

$$\langle \check{\alpha}, w\rho_{\mathfrak{h}} \rangle = \langle \check{\alpha}, \rho_{\mathfrak{h}} \rangle = 1 \tag{7.7}$$

for all $\alpha \in \Delta_{\mathfrak{h}}^{\mathfrak{m}_{\mathcal{P}}}$ and

$$\langle \check{\alpha}, w\rho_{\mathfrak{h}} \rangle = \langle \check{\beta}, w\rho_{\mathfrak{h}} \rangle \tag{7.8}$$

for $\alpha, \beta \in \Delta_{\mathfrak{h},\gamma}$, where $\Delta_o = \Delta_o^{\mathcal{P}} \cup \{\gamma\}$. The first of these conditions implies that $w^{-1}\alpha$ is not only positive but also a simple positive root. It follows from the second condition, (7.8), either $w^{-1}\Delta_{\mathfrak{h},\gamma} > 0$ or $w^{-1}\Delta_{\mathfrak{h},\gamma} < 0$. In the first case, w^{-1} maps every positive root to a positive root, and w is the identity. In the second case, let γ be a root of $\mathfrak{h}$ in $\mathfrak{n}_{\mathcal{P}}$. Then $\gamma = \gamma' + \gamma''$, where γ' is a linear combination of elements of $\Delta_{\mathfrak{h},\gamma}$ with nonnegative coefficients, and γ'' is a linear combination of elements of $\Delta_{\mathfrak{h}}^{\mathcal{P}}$. By our assumption, $w^{-1}\gamma'$ is a linear combination of simple roots with nonpositive coefficients. Since γ' does not vanish on $\mathfrak{a}_{\mathcal{P}}$, it is not a linear combination of elements of $\Delta_{\mathfrak{h}}^{\mathcal{P}}$. Therefore, there is an element $\alpha \in \Delta_{\mathfrak{h}} - w^{-1}\Delta_{\mathfrak{h}}^{\mathcal{P}}$ which occurs with a negative coefficient in the representation of $w^{-1}\gamma'$ as a linear combination of the elements of $\Delta_{\mathfrak{h}}$. Since $w^{-1}\gamma''$ is a linear combination of the elements of $w^{-1}\Delta_{\mathfrak{h}}^{\mathcal{P}} \subset \Delta_{\mathfrak{h}}$, this means that α occurs with a negative coefficient in the representation of $w^{-1}\gamma$ as a linear combination of positive roots. But this means that w^{-1} maps all positive roots of $\mathfrak{h}$ which do not occur in $\mathfrak{l}_{\mathcal{P}}$ to negative roots. Therefore, the length of w is the largest possible, and the contribution of w to (7.6) is in the highest possible degree, which is one-dimensional and coincides with the second summand in (7.5) □

7.2 SL_n over imaginary quadratic fields

Let $\mathbb{K}$ be an imaginary quadratic field, and let $\mathcal{G} = \mathrm{res}_{\mathbb{Q}}^{\mathbb{K}} \mathrm{SL}_n$. We want to explicitly compute $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$. We will directly use the complex $C^*\left(\check{\boldsymbol{H}}(\mathcal{G})^{\bullet *}\right)$. Let us first describe this complex explicitly.

Recall that the cohomology of the constant representation of SL_n over an imaginary quadratic field is the exterior algebra with generators $\lambda_2, \ldots, \lambda_n$.

The degree of λ_n is $2n-1$, and, using the coalgebra structure of the cohomology of $\boldsymbol{X}_{\mathcal{G}}^{(c)} = \mathrm{SU}(n,\mathbb{C})$ coming from the group law, λ_n is characterised up to multiplication by a nonvanishing number as the primitive element in degree $2n-1$. We will assume that for $k \leq l < n$, the restriction of λ_k on SL_n is λ_k on SL_l.

Let the minimal parabolic subgroup be the stabilizer of the full flag $V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{K}^n$. Then any standard parabolic subgroup $\mathcal{P}$ is the stabilizer of a flag $V_{i_1} \subset V_{i_2} \subset \cdots \subset V_{i_K}$ for some sequence $0 < i_1 < i_2 < \cdots < i_K = n$. Then

$$\mathcal{M}_{\mathcal{P}} = \prod_{l=1}^{K} \mathrm{Res}_{\mathbb{Q}}^{\mathbb{K}} \mathrm{SL}_{i_l - i_{l-1}}, \qquad i_0 \colon = 0$$

hence the cohomology of $\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}$ is an exterior algebra with generators

$$\lambda_2^{(1)}, \ldots, \lambda_{i_1}^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_{i_2-i_1}^{(2)}, \ldots, \lambda_2^{(K)}, \ldots, \lambda_{i_K - i_{K-1}}^{(K)},$$

where the supercript $^{(l)}$ stands for the l-th simple factor of the Levi component. If $i_l - i_{l-1} = 1$, there is primitive element of $H^*(\mathcal{M}_{\mathcal{G}})$ belonging to the l-th factor. Furthermore, the restriction from $\boldsymbol{X}_{\mathcal{G}}^{(c)}$ to $\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}$ of the primitve generator λ_k is given by

$$\mathrm{res}_{\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}}^{\boldsymbol{X}_{\mathcal{G}}^{(c)}} \lambda_k = \sum_{i_l - i_{l-1} \geq k} \lambda_k^{(l)}. \tag{7.9}$$

Finally,

$$\boldsymbol{K}_\infty \cap \mathcal{P}(\mathbb{R}) = \boldsymbol{S}(\boldsymbol{U}(i_1) \times \boldsymbol{U}(i_2 - i_1) \times \cdots \times \boldsymbol{U}(i_K - i_{K-1}))$$

is connected, hence its group of connected components does not interfere with the computation of the functor $\check{\boldsymbol{H}}(\mathcal{G})^{\bullet *}$. Therefore, we get an explicit description of the functor $\check{\boldsymbol{H}}(\mathcal{G})^{\bullet *}$ which we now want to describe.

Let $\mathfrak{L}_n(\mathbb{K})$ be the set of functions

$$\mathfrak{l} \colon \{2, \ldots, n\} \to \{0, 1, \ldots\}$$

such that

$$\sum_{j=1}^{\infty} \max\{k \,|\, \mathfrak{l}(k) \geq j\} \leq n. \tag{7.10}$$

If the parabolic subgroup $\mathcal{P}$ corresponds to $0 < i_1 < \cdots < i_K = n$, let $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ be the set of functions

$$\mathfrak{y} \colon \{(k,l) \,|\, 2 \leq k \leq n, 1 \leq l \leq \mathfrak{l}(k)\} \to \{1, \ldots, k\}$$

with the property that

$$i_{\mathfrak{y}(k,l)} - i_{\mathfrak{y}(k,l)-1} \geq k \tag{7.11}$$

and

$$0 < \mathfrak{y}(k,1) < \mathfrak{y}(k,2) < \cdots < \mathfrak{y}(k,\mathfrak{l}(k)). \tag{7.12}$$

Note that $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ is not functorial with respect to $\mathcal{P}$. Let $\tilde{\mathcal{P}} \supseteq \mathcal{P}$, $\mathfrak{y} \in \mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$, $\tilde{\mathfrak{y}} \in \mathfrak{Y}_{\mathfrak{l},\tilde{\mathcal{P}}}$. Let $\mathcal{P}$ belong to the sequence $0 < i_1 < \cdots < i_K = n$ and let $\tilde{\mathcal{P}}$ correspond to $0 < \tilde{i}_1 < \cdots < \tilde{i}_{\tilde{K}} = n$. Then $\{\tilde{i}_1, \ldots, \tilde{i}_{\tilde{K}}\} \subseteq \{i_1, \ldots, i_K\}$. We will write $\tilde{\mathfrak{y}} \trianglerighteq \mathfrak{y}$ if

$$\tilde{i}_{\tilde{\mathfrak{y}}(k,l)-1} < i_{\mathfrak{y}(k,l)} \leq \tilde{i}_{\mathfrak{y}(k,l)}. \tag{7.13}$$

It is clear that for given $\mathcal{P}$, $\tilde{\mathcal{P}}$, and $\mathfrak{y}$ there is at most one $\tilde{\mathfrak{y}}$ with $\tilde{\mathfrak{y}} \trianglerighteq \mathfrak{y}$. Let $\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}$ be the vector space with base $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$. Then $\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}$ is a contravariant functor from $\mathfrak{P}$ to the category of vector spaces if we put for $\tilde{\mathfrak{y}} \in \mathfrak{Y}_{\mathfrak{l},\tilde{\mathcal{P}}} \subset (\boldsymbol{I}_{\mathcal{P}})_{\mathfrak{l}}$

$$\boldsymbol{I}_{\mathfrak{l}}^{\tilde{\mathcal{P}} \supseteq \mathcal{P}}(\tilde{\mathfrak{y}}) = \sum_{\substack{\mathfrak{y} \in \mathfrak{Y}_{\mathfrak{l},\mathcal{P}} \\ \tilde{\mathfrak{y}} \trianglerighteq \mathfrak{y}}} \mathfrak{y}. \tag{7.14}$$

By (7.9), the map

$$\begin{aligned} \boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-\deg \mathfrak{l}] &\to \check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*} \\ \mathfrak{y} &\to \bigwedge_{k=2}^{n} \bigwedge_{l=1}^{\mathfrak{l}(k)} \lambda_k^{\mathfrak{y}(k,l)}, \end{aligned}$$

where

$$\deg \mathfrak{l} = \sum_{k=2}^{n} (2k-1)\mathfrak{l}(k) \tag{7.15}$$

is a functormorphism. We get a direct sum decomposition

$$\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*} \cong \bigoplus_{\mathfrak{l} \in \mathfrak{L}_n(\mathbb{K})} \boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-\deg \mathfrak{l}] \otimes C^\infty(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)). \tag{7.16}$$

If $\boldsymbol{K}_f$ is a good maximal compact subgroup of $\mathcal{G}(\mathbb{A}_f)$, we also get a direct sum decomposition for spherical vectors

$$(\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*})^{\boldsymbol{K}_f} \cong \bigoplus_{\mathfrak{l} \in \mathfrak{L}_n(\mathbb{K})} \boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-\deg \mathfrak{l}]. \tag{7.17}$$

Let us first formulate our result for spherical vectors in the cohomology.

Theorem 7.5. *For $\mathfrak{l} \in \mathfrak{L}_n(\mathbb{K})$, $\mathfrak{e} \in \{0,1\}$, and $N \leq 0$, let $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ be the set of ordered $(N+1)$-tuples $\mathfrak{x} = (X_0, \ldots, X_N)$ of subsets of $\{2, \ldots, n\}$ with the following properties:*

- *Each number k with $2 \leq k \leq n$ belongs to precisely $\mathfrak{l}(k)$ of the sets X_i;*
- *We have*

$$\sum_{i=0}^{N} \max \#\{X_i\} = n - \mathfrak{e}.$$

If $\mathfrak{l} = 0$*, we put* $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}} = \emptyset$*. Then for each* $\mathfrak{x} \in \mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$*,* $H^*(C^*(\boldsymbol{I}_{\mathfrak{l}}^{\bullet}))$ *has a generator* $\{\mathfrak{x}\}$ *in degree* $N + \mathfrak{e}$*, and we have*

$$H^i(C^*(\boldsymbol{I}_{\mathfrak{l}}^{\bullet})) = \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in\mathfrak{X}_{i-\mathfrak{e},\mathfrak{e},\mathfrak{l}}} \mathbb{C} \cdot \{\mathfrak{x}\}. \tag{7.18}$$

Consequently,

$$H_c^j(\mathcal{G}, \mathbb{C})_{\mathcal{I}} \cong \bigoplus_{\mathfrak{l}\in\mathfrak{L}_n(\mathbb{K})} \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in\mathfrak{X}_{j-\mathfrak{e}-\deg \mathfrak{l}}} \mathbb{C} \cdot \{\mathfrak{x}\}.$$

Moreover, let the ordering $\prec$ *on the roots which was used to define the complex* $C^*(\boldsymbol{F}^{\bullet})$ *be*

$$x_1 - x_2 \prec x_2 - x_3 \prec \cdots \prec x_{n-1} - x_n.$$

Then for $\mathfrak{x} = (X_0, \ldots, X_N) \in \mathfrak{X}_{N,0,\mathfrak{l}}$ *a representative of the cohomology class* $\{\mathfrak{x}\}$ *is given by the element*

$$\bigwedge_{i=0}^{N} \bigwedge_{\substack{j=2 \\ j\in X_i}}^{\#(X_i)} \lambda_j^{(i)} \tag{7.19}$$

in the cohomology of $\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}$*, where* $\mathcal{P} \in \mathfrak{P}$ *is the stabilizer of the standard flag of vector spaces with dimensions*

$$0 < \#(X_0) < \#(X_0) + \#(X_1) < \cdots < \sum_{i=0}^{N-2} \#(X_i) < \sum_{i=0}^{N-1} \#(X_i) = n.$$

If $\mathfrak{x} = (X_0, \ldots, X_N) \in \mathfrak{X}_{N,1,\mathfrak{l}}$ *and if* $0 \leq k \leq N + 1$*, then a representative of the cohomology class* $\{\mathfrak{x}\}$ *is given by the element*

$$(-1)^k \bigwedge_{i=0}^{k-1} \bigwedge_{\substack{j=2 \\ j\in X_i}}^{\#(X_i)} \lambda_j^{(i)} \wedge \bigwedge_{i=k}^{N} \bigwedge_{\substack{j=2 \\ j\in X_i}}^{\#(X_i)} \lambda_j^{(i+1)} \tag{7.20}$$

in the cohomology of $\boldsymbol{X}_{\mathcal{M}_{\mathcal{P}}}^{(c)}$*, where* $\mathcal{P} \in \mathfrak{P}$ *is the stabilizer of the standard flag of vector spaces with dimensions*

$$\begin{aligned} 0 < \#(X_0) < \#(X_0) + \#(X_1) < \cdots < \sum_{i=0}^{k-1} \#(X_i) \\ < 1 + \sum_{i=0}^{k-1} \#(X_i) < \cdots 1 + \sum_{i=1}^{N-2} \#(X_i) < 1 + \sum_{i=1}^{n-1} \#(X_i) = n. \end{aligned}$$

For instance, if $n = 2$, then the only spherical vector in $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$ is the volume form in degree 2. This is in good keeping with the results of Harder for SL_2 and also with the computation of R. Staffeldt [25, Theorem IV.1.3.] which implies that $H^*(\mathrm{SL}_2(\mathbb{Z}[i]), \mathbb{C})$ vanishes in positive dimension. In particular, there are no harmonic cusp forms for $\mathrm{SL}_2(\mathbb{Z}[i])$. If $n = 3$, then $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$ contains the following three spherical vectors:

- In degree 4, the cohomology class belonging to

$$\mathfrak{l}(k) = \begin{cases} 1 \text{ if } k = 2 \\ 0 \text{ otherwise} \end{cases}$$

  and $\mathfrak{x} = \{\{2\}\} \in \mathfrak{X}_{0,1,\mathfrak{l}}$;
- In degree 5, the cohomology class belonging to

$$\mathfrak{l}(k) = \begin{cases} 1 \text{ if } k = 3 \\ 0 \text{ otherwise} \end{cases}$$

  and $\mathfrak{x} = \{\{3\}\} \in \mathfrak{X}_{0,0,\mathfrak{l}}$ (This class maps to λ_3 in $I^*_{\mathcal{G}(\mathbb{R}),\boldsymbol{K}_\infty}$.);
- And in degree 8, the volume form belonging to

$$\mathfrak{l}(k) = \begin{cases} 1 \text{ if } k = 2 \text{ or } k = 3 \\ 0 \text{ otherwise} \end{cases}$$

  and $\mathfrak{x} = \{\{2, 3\}\} \in \mathfrak{X}_{0,0,\mathfrak{l}}$.

In the case $\mathbb{K} = \mathbb{Q}(i)$, this can be compared with the computation by R. Staffeldt ([25, Theorem IV.1.4.] combined with the Borel–Serre duality theorem [3, Theorem 11.4.1.]). It turns out that in this case all cohomology classes of $\mathrm{SL}_3(\mathbb{Z}[i])$ can be generated by Eisenstein series starting from the constant representation or by the constant representation itself. In particular, there are no harmonic cusp forms modulo $\mathrm{SL}_3(\mathbb{Z})$.

7.3 Homotopy type of a poset of partitions

As the main combinatorial tool in our computation of $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$ for GL_n over imaginary quadratic fields, we use the description of the homotopy type of a partially ordered set of partitions.

In the following, we shall write 'poset' for 'partially ordered set'. Let $\boldsymbol{B}X$ be the classifying space of the poset X. Notions from homotopy theory applied to posets or morphisms of posets will have the meaning of these notions, applied to the classifying space of the poset or morphism of posets. We will freely use the basic techniques for investigating the homotopy of the classifying space of a category (cf. [19] or the textbook [24]).

By an ordered partition of an integer n, we mean a tuple $(M, x_0, \ldots, x_M)$, where M is the number of intervals in the partition and $0 = x_0 < x_1 <$

$\cdots < x_M = n$ are the vertices of these intervals. We will say that a partition $(M, x_0, \ldots, x_M)$ is finer than or equal to $(N, y_0, \ldots, y_N)$, and write

$$(M, x_0, \ldots, x_M) \trianglelefteq (N, y_0, \ldots, y_N),$$

if $\{y_0, \ldots, y_N\} \subseteq \{x_0, \ldots, x_M\}$.

Consider a finite set $\mathfrak{S}$ and a function $F\colon \mathfrak{S} \to \{1, 2, \ldots\}$. Let $P_{n,\mathfrak{S},F}$ be the set of pairs $(f, (M, x_1, \ldots, x_M))$, where $(M, x_0, \ldots, x_M)$ is a partition of n and $f\colon \mathfrak{S} \to \{1, \ldots, M\}$ such that $x_{f(s)} - x_{f(s)-1} \geq F(s)$ for $s \in \mathfrak{S}$. In other words, elements of $P_{n,\mathfrak{S},F}$ are ordered partitions of n in which for each element $s \in \mathfrak{S}$ an interval of length $\geq F(s)$ is marked. The intervals associated to different elements of $\mathfrak{S}$ are not supposed to be different.

There is a partial order $\trianglelefteq$ on $P_{n,\mathfrak{S},F}$ for which

$$(f, (M, x_1, \ldots, x_M)) \trianglelefteq (g, (N, y_1, \ldots, y_N))$$

if and only if $(M, x_0, \ldots, x_M) \trianglelefteq (N, y_0, \ldots, y_N)$ and $y_{f(s)-1} \leq x_{f(s)-1} < x_{f(s)} \leq y_{f(s)}$ for $s \in \mathfrak{S}$. In other words, the partition $(M, x_0, \ldots, x_M)$ has to be finer than $(N, y_0, \ldots, y_N)$ and the interval in $(M, x_0, \ldots, x_M)$ associated to s by f must be contained in the interval in $(N, y_0, \ldots, y_N)$ associated to s by g.

If $n < \max\limits_{s \in \mathfrak{S}} F(s)$, the poset $P_{n,\mathfrak{S},F}$ is empty. Otherwise, it is contractible since it has a final object $(\mathbf{1}, (1, 0, n))$, where $\mathbf{1}$ is the constant function $s \to 1$ on $\mathfrak{S}$. Let

$$\tilde{P}_{n,\mathfrak{S},F} = p_{n,\mathfrak{S},F} - \{(\mathbf{1}, (1, 0, n))\}.$$

We will investigate the homotopy type of $\tilde{P}_{n,\mathfrak{S},F}$. It will turn out that it is a wedge of spheres. Before formulating our result, we have to define the index sets over which the wedge is taken. For $1 \leq k \leq \#(\mathfrak{S})$ and $\mathfrak{e} \in \{0, 1\}$, let $M_{n,\mathfrak{S},F,\mathfrak{e},k}$ be the set of ordered k-tuples $(\mathfrak{S}_1, \ldots, \mathfrak{S}_k)$ of nonempty mutually disjoint subsets of $\mathfrak{S}$ such that $\mathfrak{S} = \bigcup_{l=1}^k \mathfrak{S}_k$ and $n = \mathfrak{e} + \sum_{l=1}^k \max\limits_{s \in \mathfrak{S}_l} F(s)$.

Proposition 7.6. *If $n = \max\limits_{s \in \mathfrak{S}} F(s)$, $\tilde{P}_{n,\mathfrak{S},F}$ is empty. Let $n > \max\limits_{s \in \mathfrak{S}} F(s)$. Fix the basepoint* $✠ = \left(\mathbf{1}, (2, 0, \max\limits_{s \in \mathfrak{S}} F(s), n)\right)$ *of the poset $\tilde{P}_{n,\mathfrak{S},F}$. We have a homotopy equivalence of pointed spaces*

$$\phi_{n,\mathfrak{S},F}\colon \left(\boldsymbol{B}\tilde{P}_{n,\mathfrak{S},F}\right) \cong \bigvee_{\mathfrak{e}=0}^{1} \bigvee_{k=1}^{\#\mathfrak{S}} \bigvee_{M_{n,\mathfrak{S},F,\mathfrak{e},k}} S^{k+\mathfrak{e}-2}, \tag{7.21}$$

where S^l is the pointed l-sphere (a set of two points if $l = 0$). It is assumed that the wedge over an empty index set is a contractible space.

Moreover, if $\mathfrak{s} = (\mathfrak{S}_1, \ldots, \mathfrak{S}_k) \in M_{n,\mathfrak{S},F,0,k}$, then the reduced cohomology class of its factor in (7.21) is given by the unrefinable chain of length $k-1$

$$x_1(\mathfrak{s}) \lhd x_2(\mathfrak{s}) \lhd \cdots \lhd x_{k-1}(\mathfrak{s}),$$

where

$$x_l(\mathfrak{s}) = \left(f_l^{\mathfrak{s}}, \left(k-l, \sum_{j=1}^{l} \#\mathfrak{S}_j, \sum_{j=1}^{l+1} \#\mathfrak{S}_j, \ldots, \sum_{j=1}^{k} \#\mathfrak{S}_j = n\right)\right)$$

and

$$f_l(s) = \begin{cases} 1 & \text{if } s \in \bigcup_{i=1}^{l} \mathfrak{S}_i \\ k+1 & \text{if } s \in \mathfrak{S}_{l+k} \text{ with } k > 0. \end{cases}$$

Similarly, if $(\mathfrak{S}_1, \ldots, \mathfrak{S}_k) \in M_{n,\mathfrak{S},F,1,k}$*, then any of the following unrefinable chains of length k is a representative for the reduced cohomology class defined by the corresponding factor in the wedge (7.21). Take* $1 \leq m \leq k+1$*, define*

$$f_l^{(m)}(s) = \begin{cases} 1 & \text{if } l < m \text{ and } s \in \bigcup_{i=1}^{l} \mathfrak{S}_i \\ 2 & \text{if } l \geq m \text{ and } s \in \bigcup_{i=1}^{l} \mathfrak{S}_i \\ k+1 & \text{if } s \in \mathfrak{S}_{k+l} \text{ with } k > 0 \text{ and } k+l < m \\ k+2 & \text{if } s \in \mathfrak{S}_{k+l} \text{ with } k > 0 \text{ and } k+l \geq m \end{cases}$$

for $1 \leq l \leq k$ *and consider the chain*

$$x_1^{(m)} \lhd x_2 \lhd \cdots \lhd x_k^{(m)}$$

with

$$x_l^{(m)} = \Bigg(f_l^{(m)}, \Big(k+2-l, \sum_{i=1}^{l} \#\mathfrak{S}_i, \ldots \sum_{i=1}^{m-1} \#\mathfrak{S}_i, 1+\sum_{i=1}^{m-1} \#\mathfrak{S}_i, 1+\sum_{i=1}^{m} \#\mathfrak{S}_i, \ldots, n\Big)\Bigg)$$

if $l < m$ *and*

$$x_l^{(m)} = \left(f_l^{(m)}, \left(k+2-l, 1+\sum_{i=1}^{m-1} \#\mathfrak{S}_i, 1+\sum_{i=1}^{m} \#\mathfrak{S}_i, \ldots, n\right)\right)$$

otherwise.

Proof. It is clear that $\tilde{P}_{n,\mathfrak{S},F}$ is empty if $n = \max_{s \in \mathfrak{S}} F(s)$. If $n = 1 + \max_{s \in \mathfrak{S}} F(s)$, it is easy to see that $\tilde{P}_{n,\mathfrak{S},F}$ consists of two points without relation, and the theorem follows.

Let $n > 1 + \max_{s \in \mathfrak{S}} F(s)$. Let A be the poset of all elements $(f, (M, x_1, \ldots, x_M)) \in \tilde{P}_{n,\mathfrak{S},F}$ which satisfy one of the following two conditions:

- $f^{-1}(1)$ is empty;
- or $x_1 > \max_{s \in f^{-1}(1)} F(s)$.

Since $n > 1 + \max_{s\in\mathfrak{S}} F(s)$, $P_{n-1,\mathfrak{S},F}$ is contractible. We have an embedding

$$i_1\colon P_{n-1,\mathfrak{S},F} \to A$$

defined by

$$\begin{aligned} &i_1\left((g,(N,y_1,\dots,y_N))\right) \\ &\quad = \left(g+\mathbf{1},(N+1,0,1=y_1+1,y_2+1,\dots,n=y_N+1)\right), \end{aligned}$$

where $g+\mathbf{1}$ is the function $s \to g(s)+1$ on $\mathfrak{S}$. We also have a retraction for i_1

$$r_1\colon A \to P_{n-1,\mathfrak{S},F}$$

which is defined by

$$\begin{aligned} &r_1\left((f,(M,x_1,\dots,x_M))\right) \\ &\quad = \begin{cases} (f-\mathbf{1},(M-1,0=x_1-1,x_2-1,\dots,n-1=x_M-1)) & \text{if } x_1 = 1 \\ (f,(M,0,x_1-1,\dots,x_M-1=n-1)) & \text{if } x_1 > 1. \end{cases} \end{aligned}$$

Since $i_1 r_1\left(f,(M,x_1,\dots,x_M)\right) \trianglelefteq \left(f,(M,x_1,\dots,x_M)\right)$, $P_{n-1,\mathfrak{S},F}$ is a deformation retract of A, hence A is contractible.

Let $B \subset \tilde{P}_{n,\mathfrak{S},F}$ be the poset of all $(f,(M,x_1,\dots,x_M))$ which satisfy at least one of the following two conditions:

- $(f,(M,x_1,\dots,x_M)) \in A$;
- or $M > 2$.

We have the obvious inclusion $i_2\colon A \to B$ and a retraction $r_2\colon B \to A$ which is defined as follows. If $(f,(M,x_1,\dots,x_M)) \in A$, we put

$$r_2((f,(M,x_1,\dots,x_M))) = (f,(M,x_1,\dots,x_M)).$$

If $(f,(M,x_1,\dots,x_M)) \in B - A$, we define a function

$$h\colon \mathfrak{S} \to 1,\dots,M-1$$

by

$$h(s) = \begin{cases} 1 & \text{if } f(s) = 1 \\ f(s)-1 & \text{if } f(s) > 1 \end{cases} \tag{7.22}$$

and put

$$r_2((f,(M,x_1,\dots,x_M))) = (h,(M-1,x_0,x_2,\dots,x_n)). \tag{7.23}$$

It is easy to see that r_2 is a morphism of posets, that $r_2 i_2 = \mathrm{Id}$, and that $i_2 r_2\left(f,(M,x_1,\dots,x_M)\right) \trianglerighteq \left(f,(M,x_1,\dots,x_M)\right)$. Therefore, A is a deformation retract of B, and B is contractible.

For $x \in \tilde{P}_{n,\mathfrak{S},F} - B$, let $B_-(x)$ be the poset of all $y \in B$ with $y \lhd x$. The set of all x for which $B_-(x)$ is empty can be identified with $M_{n,\mathfrak{S},F,0,2}$. Since no element of $\tilde{P}_{n,\mathfrak{S},F}$ can be coarser than x, the fact that B is contractible gives us a homotopy equivalence

$$\boldsymbol{B}\tilde{P}_{n,\mathfrak{S},F} \cong \bigvee_{\substack{x\in \tilde{P}_{n,\mathfrak{S},F}-B\\ B_-(x)\neq\emptyset}} \Sigma(\boldsymbol{B}B_-(x)) \vee \bigvee_{M_{n,\mathfrak{S},F,0,2}} S^0, \tag{7.24}$$

where Σ is the suspension functor.

Let us first assume that $\mathfrak{S}$ consists of a single element s. The assumption made at the beginning of the proof means that $n > F(s)+1$. Then all sets $M_{n,\mathfrak{S},F,\mathfrak{e},k}$ are empty, and we have to show that $\tilde{P}_{n,\mathfrak{S},F}$ is contractible. The only factor in (7.24) is $B_-(\maltese)$, which has an initial object

$$(\mathbf{1}, (n-F(s), 0, F(s), F(s)+1, \dots, n))\,.$$

This completes the proof of the proposition if $\mathfrak{S}$ has only one element.

Now we assume by induction that the proposition has been verified for all subsets of $\mathfrak{S}$. As above, $B_-(\maltese)$ has an initial object and is contractible. The other elements of $\tilde{P}_{n,\mathfrak{S},F} - B$ for which $B_-(x)$ is not empty are of the form

$$x = \left(\begin{pmatrix} 1 \text{ on } T \\ 2 \text{ on } \mathfrak{S}-T \end{pmatrix}, \left(2, 0, \max_{s\in T} F(s), n\right)\right),$$

where T is a nonempty subset of $\mathfrak{S}$ such that $\max\limits_{s\in T} F(s) + \max\limits_{s\notin T} F(s) \leq n$. For such x we have

$$B_-(x) = \tilde{P}_{n-\max\limits_{s\in T} F(s),\mathfrak{S}-T,F}\,.$$

Then a combination of (7.24) with the induction assumption gives us

$$\begin{aligned}
\boldsymbol{B}\tilde{P}_{n,\mathfrak{S},F} &\cong \bigvee_{\substack{T\subset\mathfrak{S}\\ T\neq\emptyset\\ \max_{s\in T}F(s)+\max_{s\notin T}F(s)<n}} \Sigma(\boldsymbol{B}\tilde{P}_{n-\max_{s\in T}F(s),\mathfrak{S}-T,F}) \vee \bigvee_{M_{n,\mathfrak{S},F,0,2}} S^0 \\
&\cong \left(\bigvee_{\substack{T\subset\mathfrak{S}\\ T\neq\emptyset\\ \max_{s\in T}F(s)+\max_{s\notin T}F(s)<n}} \bigvee_{\mathfrak{e}=0}^{1} \bigvee_{k=1}^{\#(\mathfrak{S}-T)} \bigvee_{M_{n-\max_{s\in T}F(s),\mathfrak{S}-T,F,\mathfrak{e},k}} S^{k+\mathfrak{e}-1}\right) \\
&\qquad \vee \bigvee_{M_{n,\mathfrak{S},F,0,2}} S^0.
\end{aligned}$$

Since the maps

$$\begin{aligned}
M_{n-\max_{s\in T}F(s),\mathfrak{S}-T,F,\mathfrak{e},k} &\to M_{n,\mathfrak{S},F,\mathfrak{e},k+1}\\
(\mathfrak{S}_1,\dots,\mathfrak{S}_{k-1}) &\to (T,\mathfrak{S}_1,\dots,\mathfrak{S}_{k-1})
\end{aligned}$$

define an isomorphism

$$\bigcup_{\substack{T\subset\mathfrak{S}\\ T\neq\emptyset\\ \max\limits_{s\in T}F(s)+\max\limits_{s\notin T}F(s)<n}} M_{n-\max\limits_{s\in T}F(s),\mathfrak{S}-T,F,\mathfrak{e},k}\cong M_{n,\mathfrak{S},F,\mathfrak{e},k+1},$$

this completes the induction argument. The explicit formula for the reduced cohomology classes defined by the individual factors in the wedge (7.21) can easily be verified by induction. □

We now want to explain how one can translate homology computations for certain posets into assertions about the homology of functors from $\mathfrak{P}$ to abelian groups. Let

$$p\colon (X,\lhd)\to(\mathfrak{P},\subset)$$

be a morphism of posets such that $p^{-1}(\mathcal{G})$ is empty and such that

a. for $\mathcal{G}\supset\mathcal{P}\supseteq\mathcal{Q}$ and $x\in p^{-1}(\mathcal{Q})$, there is a unique $y\in p^{-1}(\mathcal{P})$ with $y\unrhd x$.

We define a functor $\boldsymbol{J}^{\bullet}_{X,p}$ by

$$\boldsymbol{J}^{\mathcal{P}}_{X,p}=\begin{cases}\bigoplus_{x\in p^{-1}(\mathcal{P})}\mathbb{C}x \text{ if } \mathcal{P}\subset\mathcal{G}\\ \mathbb{C}\text{if } \mathcal{P}=\mathcal{G}\end{cases}\tag{7.25}$$

and

$$\boldsymbol{J}^{\mathcal{Q}\subseteq\mathcal{P}}_{X,p}(x)=\sum_{\substack{y\,\in\, p^{-1}(\mathcal{Q})\\ y\,\lhd\, x}} y$$

for $x\in p^{-1}(\mathcal{P})$ with $\mathcal{P}\subset\mathcal{G}$ and

$$\boldsymbol{J}^{\mathcal{Q}\subseteq\mathcal{G}}_{X,p}1=\sum_{y\in p^{-1}(\mathcal{Q})} y.$$

Proposition 7.7. *Assume condition a. above and assume moreover the condition*

b. If $x_1,\dots,x_k\in X$ such that $p(x_i)$ is a maximal parabolic subgroup for $1\le i\le k$, then there is at most one $y\in p^{-1}(p(x_1)\cap\dots\cap p(x_k))$ with $y\unlhd x_i$ for all $1\le i\le k$.

Under these circumstances, we have a canonical isomorphism

$$H^*(C^*(\boldsymbol{J}^{\bullet}_{X,p}))[1]\cong\tilde{H}^*(\boldsymbol{B}X).\tag{7.26}$$

Moreover, let us assume that the differential on $C^*(\boldsymbol{J}^{\bullet}_{X,p})$ was defined using the order $\prec$ on Δ_o. Let $\xi=(x_1\lhd x_2\lhd\dots x_k)$ be an unrefinable chain in X, defining a reduced cohomology class $[\xi]$ in degree $k-1$ on $\boldsymbol{B}X$. Then x_1 cannot be refined, hence it defines a cohomology class in degree k for $\boldsymbol{J}_{X,p}$. Let α_i be the unique element of $\Delta_o^{p(x_{i+1})}-\Delta_o^{p(x_i)}$ if $i<k$ and the unique element of $\Delta_o-\Delta_o^{p(x_k)}$ if $i=k$. Further, let $\varepsilon\in\{1;-1\}$ be the orientation with respect to $\prec$ of $\alpha_1,\dots,\alpha_k$. Then (7.26) maps $[\xi]$ to εx_k.

Proof. Let us define a simplicial complex (Y, Σ) as follows. The set of vertices Y is the set of $x \in X$ such that $p(x)$ is a standard maximal parabolic subalgebra. A k-tuple $(x_1, \ldots, x_k)$ of vertices belongs to the set Σ of simplices if there exists an $x \in X$ with $x \trianglelefteq x_k$ for all k. By conditions a. and b. above, the reduced chain complex for computing the cohomology of (Y, Σ) is $C^*(\boldsymbol{J}^\bullet_{X,p})[1]$. The proposition now follows from the well-known fact that the barycentric subdivision of a simplicial complex is the nerve of its poset of simplices, which in the case of (Y, Σ) is X. □

7.4 Proof of Theorem 7.5

We now prove the explicit formulas for the Eisenstein cohomology which we announced earlier. We are considering the group $\mathcal{G} = \mathrm{res}^{\mathbb{K}}_{\mathbb{Q}} \mathcal{GL}_n$ for an imaginary quadratic field $\mathbb{K}$.

To prove Theorem 7.5, consider $\mathfrak{l} \in \mathfrak{L}_n(\mathbb{K})$. If $\mathfrak{l}(n) = 1$, then $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ is empty unless $\mathcal{P} = \mathcal{G}$, in which case it has precisely one element. It follows that $C^*(\mathfrak{I}^\bullet_{\mathfrak{l}})$ has a one-dimensional cohomology group in dimension zero, and no other cohomology. Also, $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ is empty unless $N = 1$ and $\mathfrak{e} = 0$, in which case it consists of a single element. This proves the theorem for those $\mathfrak{l}$ with $\mathfrak{l}(n) = 1$. The case $\mathfrak{l}(n) > 1$ is excluded by the condition (7.10). Therefore we suppose for the remaining part of this proof that $\mathfrak{l}(n) = 0$.

We define the set $\mathfrak{S}_{\mathfrak{l}}$ by

$$\mathfrak{S}_{\mathfrak{l}} = \{(k, l) \,|2 \le k \le n, 1 \le l \le \mathfrak{l}(k)\}$$

and define the function $F \colon \mathfrak{S}_{\mathfrak{l}} \to \{1, 2, \ldots\}$ by $F((k, l)) = k$. Since $\mathfrak{l}(n) = 0$, the poset $\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F}$ defined in the last subsection is not empty. We have the map

$$p_{\mathfrak{l}} \colon \tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F} \to \mathfrak{P}$$

from $\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F}$ to the poset $\mathfrak{P}$ of standard parabolic subgroups which associates to the tuple $(f, (M, i_1, \ldots, i_M))$ the parabolic subgroup of type $0 < i_1 < \cdots < i_M = n$, i.e., the stabilizer of the standard flag of subspaces of succesive dimension i_k. The formula (7.25) now defines us a functor $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$ from $\mathfrak{P}$ to vector spaces whose homology is known by Proposition 7.6 and Proposition 7.7. We will express $\boldsymbol{I}^{\mathcal{P}}_{\mathfrak{l}}$ as an "antisymmetrization" of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$.

The product of the symmetric groups $\prod_{k=2}^n S_{\mathfrak{l}(k)}$ acts on the set $\mathfrak{S}_{\mathfrak{l}}$ by permutation of the second entry of the pairs (k, l) which form $\mathfrak{S}_{\mathfrak{l}}$. This permutation leaves F invariant, therefore it extends to an action of the group $\prod_{k=2}^n S_{\mathfrak{l}(k)}$ on the poset $\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F}$. This action leaves $p_{\mathfrak{l}}$ invariant, therefore it extends to an action of $\prod_{k=2}^n S_{\mathfrak{l}(k)}$ on the functor $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$. We want to consider the antisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$ with respect to this action.

For each parabolic subgroup $\mathcal{P}$, we have an injective map of sets

$$\mathfrak{i}\colon \mathfrak{Y}_{\mathfrak{l},\mathcal{P}} \to p_{\mathfrak{l}}^{-1}(\mathcal{P})$$

which maps the element $\mathfrak{y} \in \mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ to the element

$$(\mathfrak{y}, (K, i_1, \ldots, i_K)) \in \mathfrak{S}_{\mathfrak{l}}.$$

By condition (7.11), this element really belongs to $\mathfrak{S}_{\mathfrak{l}}$. Consider an element $(f, (K, i_1, \ldots, i_K))$ of $p^{-1}(\mathcal{P})$. If there exists a k and if $1 \le l_1 < l_2 \le \mathfrak{l}(k)$, then exchanging (k, l_1) and (k, l_2) is an odd element of $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$ which leaves $(f, (K, i_1, \ldots, i_K))$ fixed. Therefore, the image of $(f, (K, i_1, \ldots, i_K))$ in the anisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$ vanishes. Otherwise, the $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$-orbit of $(f, (K, i_1, \ldots, i_K))$ contains an element in the image of $\mathfrak{i}$, which is unique by (7.12). This identifies $\boldsymbol{I}_{\mathfrak{l}}^{\bullet}$ with the antisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$.

By Proposition 7.6 and Proposition 7.7, the cohomology of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$ is a graded vector space with a basis given by the sets $M_{n,\mathfrak{S}_{\mathfrak{l}},F,\mathfrak{e},k}$. A permutation $\mathfrak{p}$ in $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$ acts on these sets by

$$\pi\colon (\mathfrak{S}_1, \ldots, \mathfrak{S}_k) \to (\pi(\mathfrak{S}_1), \ldots, \pi(\mathfrak{S}_k)),$$

and this action commutes with the action on the cohomology of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$. We have the map

$$\mathfrak{j}\colon \mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}} \to M_{n,\mathfrak{S}_{\mathfrak{l}},F,\mathfrak{e},N+1}$$

which maps the collection $X_0, \ldots, X_N$ of subsets of $\{2, \ldots, n\}$ to the disjoint partition $\mathfrak{S}_{\mathfrak{l}} = \bigcup_{j=1}^{N+1} \mathfrak{S}_j$, where

$$\mathfrak{S}_j = \Big\{ (k, l) \in \mathfrak{S}_{\mathfrak{l}} \;\Big|\; k \in X_{j+1}, \text{ and there are precisely } l-1 \text{ elements } i \text{ with } 0 \le i < j \text{ and } k \in X_{i+1} \Big\}.$$

If $(S_1, \ldots, S_{N+1}) \in M_{n,\mathfrak{S}_{\mathfrak{l}},F,\mathfrak{e},N+1}$ and if there exists $2 \le k < n$ and $1 \le l_1 < l_2 \le \mathfrak{l}(k)$, then exchanging (k, l_1) and (k, l_2) is an odd element of $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$ which leaves $(S_1, \ldots, S_{N+1})$ fixed. Therefore, the image of the generator belonging to $(S_1, \ldots, S_{N+1})$ vanishes in the antisymmetrization of the cohomology of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$. Otherwise, the $\prod_{k=2}^{n} S_{\mathfrak{l}(k)}$-orbit of $(S_1, \ldots, S_{N+1})$ contains a unique element in the image of $\mathfrak{j}$.

We have identified $\boldsymbol{I}_{\mathfrak{l}}^{\bullet}$ with the antisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$ and the right-hand side of (7.18) with the antisymmetrization of the homology of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$. This proves (7.18). By the remarks made before the formulation of Theorem 7.5, this also completes the computation of the spherical subspace of $H_c^*(\mathcal{G}, \mathbb{C})_{\mathcal{I}}$.

To get a result about the nonspherical vectors in the cohomology, we have to investigate the cohomology of the functor

$$\tilde{\boldsymbol{J}}^{\mathcal{P}}_{n,\mathfrak{S},F} = \boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S},F},p_{\mathfrak{l}}} \otimes C^{\infty}(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)),$$

where $\mathfrak{S}$ is a finite set and F is a function from $\mathfrak{S}$ to integers. Since both the formulation and the proof of the result are straightforward but quite unpleasant,the result will be formulated precisely but the proof will only be sketched. Let $\mathfrak{Q}_k(\mathfrak{S})$ be the set of partitions

$$\mathfrak{s}\colon \mathfrak{S} = \bigcup_{l=1}^{k} \mathfrak{S}_l$$

into k disjoint pieces. For $\mathfrak{s} \in \mathfrak{Q}_k(\mathfrak{S})$, let $\mathfrak{A}_{\mathfrak{s},\mathfrak{S},F}$ be the set of pairs $(\mathcal{P}, \mathfrak{f})$ with the following properties:

- $\mathcal{P}$ is a standard parabolic subgroup, stabilizing the standard flag of subspaces of dimensions
$$0 = i_0^{\mathcal{P}} < i_1^{\mathcal{P}} < \cdots < i^{\mathcal{P}}_{\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}} = n.$$
- $\mathfrak{f}$ is a monotonous map from $\{1, \ldots, k\}$ to $\{1, \ldots, \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}\}$ such that
$$i_{\mathfrak{f}(j)} - i_{\mathfrak{f}(j)-1} = \max_{s\in\mathfrak{S}_j} F(s).$$
- If $j \in \{1, \ldots, \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}\} - \mathfrak{f}(\{1, \ldots, k\})$, then $i_j - i_{j-1} = 1$.

Note that the rank of $\mathcal{P}$ is uniquely determined; it is equal to

$$\mathfrak{d}(\mathfrak{s}) = \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}} = k + n - \sum_{j=1}^{k} \#(\mathfrak{S}_k).$$

For $(\mathcal{P}, \mathfrak{f}) \in \mathfrak{A}_{\mathfrak{s},\mathfrak{S},F}$, let $x_{\mathcal{P},\mathfrak{f}} \in P_{n,\mathfrak{S},F}$ be the element

$$\left(\mathfrak{f}^{\spadesuit}, (\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}, i_1, \ldots, i_{\dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{P}}})\right),$$

where $\mathfrak{f}^{\spadesuit}$ is equal to $\mathfrak{f}(j)$ on $\mathfrak{S}_j$. This is a minimal element of $P_{n,\mathfrak{S},F}$ which lies over $\mathcal{P}$. We get a homomorphism

$$\mathfrak{a}_{\mathfrak{s}}\colon \bigoplus_{(\mathcal{P},\mathfrak{f})\in\mathfrak{A}_{\mathfrak{s}}} C^{\infty}(\mathcal{P}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f)) \to H^{\mathfrak{d}(\mathfrak{s})}\left(C^*(\tilde{\boldsymbol{I}}^{\bullet}_{n,\mathfrak{S},F})\right)$$

$$(f_{\mathcal{P},\mathfrak{f}})_{(\mathcal{P},\mathfrak{f}\in\mathfrak{A}_{\mathfrak{s}})} \to \sum_{(\mathcal{P},\mathfrak{f})\in\mathfrak{A}_{\mathfrak{s}}} f_{\mathcal{P},\mathfrak{f}} \otimes x_{\mathcal{P},\mathfrak{f}}.$$

If $\mathcal{Q}$ is a parabolic subgroup of rank $> \mathfrak{d}(\mathfrak{s})$, let $\mathfrak{B}^{\dagger}_{\mathcal{Q},\mathfrak{s}}$ be the set of all pairs $\left((\mathcal{P}, \mathfrak{f}), (\tilde{\mathcal{P}}, \tilde{\mathfrak{f}})\right)$ with the following properties:

- We have $(\mathcal{P}, \mathfrak{f}), (\tilde{\mathcal{P}}, \tilde{\mathfrak{f}}) \in \mathfrak{A}_{\mathfrak{s}}$ and $\mathcal{Q} \supset \mathcal{P}$, $\mathcal{Q} \supset \tilde{\mathcal{P}}$.

- Let $0 = i_0^{\mathcal{Q}} < i_1^{\mathcal{Q}} < \cdots < i^{\mathcal{Q}}_{\dim \mathfrak{a}_{\mathcal{Q}}^{\mathcal{G}}} = n$ be the dimension of the spaces in the standard flag defining $\mathcal{Q}$. For each $j \in \{1, \ldots, k\}$, there exists an l with
$$i^{\mathcal{Q}}_{l-1} < i^{\mathcal{P}}_{\mathfrak{f}(j)} \leq i^{\mathcal{Q}}_l$$
and
$$i^{\mathcal{Q}}_{l-1} < i^{\tilde{\mathcal{P}}}_{\tilde{\mathfrak{f}}(j)} \leq i^{\mathcal{Q}}_l.$$
In other words, the intervals $[i^{\mathcal{P}}_{\mathfrak{f}(j)-1} + 1; i^{\mathcal{P}}_{\mathfrak{f}(j)}]$ and $[i^{\tilde{\mathcal{P}}}_{\mathfrak{f}(j)-1} + 1; i^{\tilde{\mathcal{P}}}_{\mathfrak{f}(j)}]$ are contained in the same interval of the partition $i^{\mathcal{Q}}_m$.
- We have
$$\sum_{\substack{1 \leq j \leq k \\ i^{\mathcal{Q}}_{l-1} < i^{\mathcal{P}}_{\mathfrak{f}(j)} \leq i^{\mathcal{Q}}_l}} \#\mathfrak{S}_j = i^{\mathcal{Q}}_l - i^{\mathcal{Q}}_{l-1} - 1.$$
An empty sum is supposed to be zero. Note that by the previous assumption, the sum on the left-hand side of the inequality is also equal to
$$\sum_{\substack{1 \leq j \leq k \\ i^{\mathcal{Q}}_{l-1} < i^{\tilde{\mathcal{P}}}_{\tilde{\mathfrak{f}}(j)} \leq i^{\mathcal{Q}}_l}} \#\mathfrak{S}_j.$$

We have the homomorphism
$$\mathfrak{b}^{\dagger}_{\mathfrak{s}} \colon \bigoplus_{\mathcal{Q} \in \ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} > \mathfrak{d}(\mathfrak{s})} \ \bigoplus_{\left((\mathcal{P},\mathfrak{f}),(\tilde{\mathcal{P}},\tilde{\mathfrak{f}})\right) \in \mathfrak{B}^{\dagger}_{\mathcal{Q},\mathfrak{s}}} \to \bigoplus_{(\mathcal{P},\mathfrak{f}) \in \mathfrak{A}_{\mathfrak{s}}} C^{\infty}(\mathcal{P}(\mathbb{A}_f) \backslash \mathcal{G}(\mathbb{A}_f))$$
which for $\left((\mathcal{P}, \mathfrak{f}), (\tilde{\mathcal{P}}, \tilde{\mathfrak{f}})\right)$ maps $f \in C^{\infty}(\mathcal{Q}(\mathbb{A}_f) \backslash \mathcal{G}(\mathbb{A}_f))$ to $f \otimes (\mathcal{P}, \mathfrak{f}) - f \otimes (\tilde{\mathcal{P}}, \tilde{\mathfrak{f}})$. Similarly, let $\mathfrak{C}^{\dagger}_{\mathcal{Q},\mathfrak{s}}$ be the set of all $(\mathcal{P}, \mathfrak{f}) \in \mathfrak{A}_{\mathfrak{s}}$ such that $\mathcal{P} \subset \mathcal{Q}$ and such that there exists an l with
$$i^{\mathcal{Q}}_l - i^{\mathcal{Q}}_{l-1} - 1 > \sum_{\substack{1 \leq j \leq k \\ i^{\mathcal{Q}}_{l-1} < i^{\mathcal{P}}_{\mathfrak{f}(j)} \leq i^{\mathcal{Q}}_l}} \#\mathfrak{S}_j.$$

Let $\mathfrak{c}^{\dagger}_{\mathfrak{s}}$ be the obvious map
$$\bigoplus_{\mathcal{Q} \in \ \dim \mathfrak{a}^{\mathcal{G}}_{\mathcal{Q}} > \mathfrak{d}(\mathfrak{s})} \ \bigoplus_{(\mathcal{P},\mathfrak{f}) \in \mathfrak{C}^{\dagger}_{\mathcal{Q},\mathfrak{s}}} C^{\infty}(\mathcal{Q}(\mathbb{A}_f) \backslash \mathcal{G}(\mathbb{A}_f)) \to \bigoplus_{(\mathcal{P},\mathfrak{f}) \in \mathfrak{A}_{\mathfrak{s}}} C^{\infty}(\mathcal{P}(\mathbb{A}_f) \backslash \mathcal{G}(\mathbb{A}_f)).$$

Theorem 7.8. *The kernel of $\tilde{a}_{\mathfrak{s}}$ is equal to the image of $\mathfrak{b}^{\dagger}_{\mathfrak{s}} \oplus \mathfrak{c}^{\dagger}_{\mathfrak{s}}$, and we have an isomorphism of $\mathcal{G}(\mathbb{A}_f)$-modules*
$$H^*\left(C^*(\tilde{\boldsymbol{J}}_{n,\mathfrak{S},F})\right) \cong \bigoplus_k \bigoplus_{\mathfrak{s} \in \mathfrak{Q}_k(\mathfrak{S})} \operatorname{coker}(\mathfrak{b}^{\dagger}_{\mathfrak{s}} \oplus \mathfrak{c}^{\dagger}_{\mathfrak{s}})[-\mathfrak{d}(\mathfrak{s})]. \tag{7.27}$$

To prove the theorem, one filters $\tilde{\mathfrak{I}}^{\mathcal{P}}_{n,\mathfrak{S},f}$ by the subspaces

$$\bigoplus_{\substack{\mathcal{R}\in\mathfrak{P}\\ \mathcal{R}\subseteq\mathcal{P}\\ \dim\mathfrak{a}^{\mathcal{G}}_{\mathcal{R}}\leq k}} C^\infty(\mathcal{R}(\mathbb{A}_f)\backslash\mathcal{G}(\mathbb{A}_f))\otimes \boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}},$$

to define a similar filtration on the sources of $\mathfrak{a}_{\mathfrak{s}}$, $\mathfrak{b}^\dagger_{\mathfrak{s}}$ and $\mathfrak{c}^\dagger_{\mathfrak{s}}$, and to derive the theorem for the grading from Proposition 7.6 and Proposition 7.7.

To compute $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$, recall that $\boldsymbol{I}^{\mathcal{P}}_{\mathfrak{l}}$ is the antisymmetrization of $\tilde{\boldsymbol{J}}_{n,\mathfrak{S}_{\mathfrak{l}},F}$ with respect to the group $\prod_{k=2}^n S_{\mathfrak{l}(k)}$ and note that (7.27) identifies the action of this group on the cohomology of $\tilde{\boldsymbol{J}}_{n,\mathfrak{S}_{\mathfrak{l}},F}$ with the action on the right-hand side of (7.27) derived by permutation of the elements of the set $\mathfrak{Q}_k(\mathfrak{S}_{\mathfrak{l}})$. If therefore $\mathfrak{Q}^{\mathrm{mon}}_k(\mathfrak{S}_{\mathfrak{l}})$ is the set of all $\mathfrak{s}=(\mathfrak{S}_1,\ldots,\mathfrak{S}_k)\in\mathfrak{Q}_k(\mathfrak{S}_{\mathfrak{l}})$ such that, if $1\leq l_1<l_2<\mathfrak{l}(m)$ and $(m,l_1)\in\mathfrak{S}_{i_1}$ and $(m,l_2)\in\mathfrak{S}_{i_2}$ then $i_1<i_2$, we then get

Theorem 7.9. *We have a canonical isomorphism*

$$H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}\cong\bigoplus_{\mathfrak{l}\in\mathfrak{L}_n(\mathbb{K})}\bigoplus_{k}\bigoplus_{\mathfrak{s}\in\mathfrak{Q}^{\mathrm{mon}}_k(\mathfrak{S}_{\mathfrak{l}})}\operatorname{coker}(\mathfrak{b}^\dagger_{\mathfrak{s}}\oplus\mathfrak{c}^\dagger_{\mathfrak{s}})[-\mathfrak{d}(\mathfrak{s})-\deg\mathfrak{l}]. \tag{7.28}$$

7.5 The case $\mathbf{SL}_{\boldsymbol{n}}(\mathbb{Z})$

Here we consider the case $\mathcal{G}=\mathrm{SL}_n$. We want to explicitly compute the space of spherical vectors in $H^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}$ and to compare the result with computations by C. Soulé and J. Schwermer for $n=3$ and by R. Lee and R. H. Szczarba for $n=4$.

We start with an explicit description of the spaces $\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*\boldsymbol{K}_f}$. Recall that the minimal parabolic subgroup $\mathcal{P}_o$ is the stabilizer of a standard full flag $V_1\subset V_2\subset\cdots\subset V_n=\mathbb{K}^n$. Let $\mathcal{P}$ be the stabilizer of the subflag $V_{i_1}\subset V_{i_2}\subset\cdots\subset V_{i_K}$ for some sequence $0<i_1<i_2<\cdots<i_K=n$. Then

$$\mathcal{M}_{\mathcal{P}}=\prod_{l=1}^K\mathrm{SL}_{i_l-i_{l-1}}.$$

By Proposition 7.2, the cohomology of $\boldsymbol{X}^{(c)}_{\mathcal{M}_{\mathcal{P}}}$ is an exterior algebra which, for $1\leq l\leq K$, has the following generators:

$$\begin{array}{ll}\tilde{\lambda}^{(l)}_3,\tilde{\lambda}^{(l)}_5,\ldots,\tilde{\lambda}^{(l)}_n & \text{if } i_l-i_{l-1} \text{ is odd}\\ \tilde{\lambda}^{(l)}_3,\tilde{\lambda}^{(l)}_5,\ldots,\tilde{\lambda}^{(l)}_{n-1},\varepsilon^{(l)} & \text{if } i_l-i_{l-1} \text{ is even.}\end{array} \tag{7.29}$$

The group

$$\pi_0\big(\mathrm{SO}(n,\mathbb{R})\cap\mathcal{L}_{\mathcal{P}}(\mathbb{R})\big)\cong\left\{\sigma_1,\ldots,\sigma_K\in\{\pm1\}\;\middle|\;\prod_{l=1}^K\sigma_l=1\right\}$$

acts on this cohomology algebra, and only the invariants will contribute to $\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*\boldsymbol{K}_f}$. Using the fact that $\tilde{\lambda}_i^{(l)}$ is obtained by pull-back with respect to (7.3), one easily sees that the classes $\tilde{\lambda}_i^{(l)}$ are $\pi_0\big(\mathrm{SO}(n,\mathbb{R})\cap\mathcal{L}_{\mathcal{P}}(\mathbb{R})\big)$-invariant. However, conjugation by an element of $\boldsymbol{O}(n,\mathbb{R})-\mathrm{SO}(n,\mathbb{R})$ changes the orientation of the canonical n-dimensional real bundle on $\mathrm{SU}(n,\mathbb{R})/\,\mathrm{SO}(n,\mathbb{R})$, hence $\big(\sigma_i\big)_{i=1}^K\in\pi_0\big(\mathrm{SO}(n,\mathbb{R})\cap\mathcal{L}_{\mathcal{P}}(\mathbb{R})\big)$ maps $\varepsilon^{(l)}$ to $\sigma_l\varepsilon^{(l)}$. This means that a monomial μ in the generators (7.29) is $\pi_0\big(\mathrm{SO}(n,\mathbb{R})\cap\mathcal{L}_{\mathcal{P}}(\mathbb{R})\big)$-invariant if and only if one of the following cases occurs:

- μ contains no Euler class $\varepsilon^{(l)}$;
- or the numbers i_l-i_{l-1} are all even, and μ contains all Euler classes $\varepsilon^{(l)}$.

It follows that

$$\check{\boldsymbol{H}}(\mathcal{G})^{\mathcal{P}*\boldsymbol{K}_f}=\begin{cases}\boldsymbol{M}^{*\mathcal{P}}_{(n)} & n \text{ odd}\\ \boldsymbol{M}^{*\mathcal{P}}_{(n)}\oplus{}^{e}\boldsymbol{M}^{*\mathcal{P}}_{(n)} & n \text{ even,}\end{cases}\tag{7.30}$$

where

$$\boldsymbol{M}^{*\mathcal{P}}_{(n)}=\left\{\text{monomials in the }\tilde{\lambda}_i^{(l)}\right\}$$

and

$${}^{e}\boldsymbol{M}^{*\mathcal{P}}_{(n)}=\begin{cases}\prod_{l=1}^{K}\varepsilon^{(l)}\cdot\left\{\text{monomials in the }\tilde{\lambda}_i^{(l)}\right\} & \begin{array}{l}\text{if all the numbers } i_l,\\ 1\le l\le K,\text{ are even}\end{array}\\ \{0\} & \text{otherwise.}\end{cases}$$

The explicit formulas for the restriction of cohomology classes in Proposition 7.2 show that, for n even, the decomposition (7.30) is functorial in $\mathcal{P}$.

We first give an explicit formula for the first summand in (7.30). Let

$$\mathrm{Odd}_{\le n}:=\begin{cases}\{3,\ldots,n\} & \text{if } n \text{ is odd}\\ \{3,\ldots,n-1\} & \text{if } n \text{ is even.}\end{cases}$$

Let $\mathfrak{L}_n(\mathbb{Q})$ be the set of functions

$$\mathfrak{l}\colon \mathrm{Odd}_{\le n}\to\{0,1,\ldots\}$$

satisfying the condition

$$\sum_{j=1}^{\infty}\max\left\{k\in\mathrm{Odd}_{\le n}\,|\,\mathfrak{l}(k)\ge j\right\}\le n.\tag{7.31}$$

If the parabolic subgroup $\mathcal{P}$ corresponds to $0<i_1<\cdots<i_K=n$, let $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ be defined in the same way as in the case of imaginary quadratic fields, i.e., as the set of functions

$$\mathfrak{y}\colon\ \left\{(k,l)\,|\,k\in\mathrm{Odd}_{\le n},1\le l\le\mathfrak{l}(k)\right\}\to\{1,\ldots,k\}$$

with the properties (7.11) and (7.12). For $\tilde{\mathcal{P}} \supseteq \mathcal{P}$, $\mathfrak{y} \in \mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ and $\tilde{\mathfrak{y}} \in \mathfrak{Y}_{\mathfrak{l},\tilde{\mathcal{P}}}$, let the relation $\tilde{\mathfrak{y}} \trianglerighteq \mathfrak{y}$ be defined by (7.13). Then the vector space $\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}$ with base $\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ is functorial in $\mathcal{P}$ by formula (7.14), and there is a functor isomorphism

$$\boldsymbol{M}_{(n)}^{*\mathcal{P}} \cong \bigoplus_{\mathfrak{l}\in\mathfrak{L}_n(\mathbb{Q})} \boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-\deg \mathfrak{l}] \tag{7.32}$$

which maps $\mathfrak{y}$ to

$$\bigwedge_{k\in \mathrm{Odd}_{\leq n}} \bigwedge_{l=1}^{\mathfrak{l}(k)} \tilde{\lambda}_k^{\mathfrak{y}(k,l)}.$$

The degree $\deg \mathfrak{l}$ is defined in the same way as for imaginary quadratic fields, by (7.15).

Let

$$\mathfrak{S}_{\mathfrak{l}} = \{(k,l)\,|k \in \mathrm{Odd}_{\leq n}, 1 \leq l \leq \mathfrak{l}(k)\},$$

and let $F(k,l) = k$. As in the case of imaginary quadratic fields, $\boldsymbol{I}_{\mathfrak{l}}^{\bullet}$ can be identified with the antisymmetrization of $\boldsymbol{J}_{\tilde{P}_{n,\mathfrak{S}_{\mathfrak{l}},F,p_{\mathfrak{l}}}}$ with respect to the product of symmetric groups $\prod_{k\in\mathrm{Odd}_{\leq n}} S_{\mathfrak{l}(k)}$. As a result, we get a description for the first summand in (7.30) which is similar to (7.18).

Theorem 7.10. *For $\mathfrak{l} \in \mathfrak{L}_n(\mathbb{Q})$, $\mathfrak{e} \in \{0,1\}$ and $N \leq 0$, let $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ be the set of ordered $(N+1)$-tuples $\mathfrak{x} = (X_0,\dots,X_N)$ of subsets of $\{\mathrm{Odd}_{\leq n}\}$ with the following properties:*

- *Each number $k \in \mathrm{Odd}_{\leq n}$ belongs to precisely $\mathfrak{l}(k)$ of the sets X_i;*
- *We have*

$$\sum_{i=0}^{N} \max \#\{X_i\} = n - \mathfrak{e}.$$

If $\mathfrak{l} = 0$, we put $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}} = \emptyset$. Then for each $\mathfrak{x} \in \mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$, $H^*(C^*(\boldsymbol{I}_{\mathfrak{l}}^{\bullet}))$ has a generator $\{\mathfrak{x}\}$ in degree $N+\mathfrak{e}$, and we have

$$H^i\Big(C^*(\boldsymbol{I}_{\mathfrak{l}}^{\bullet})\Big) = \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in\mathfrak{X}_{i-\mathfrak{e},\mathfrak{e},\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

Consequently, the cohomology of the first summand in (7.30) is given by

$$H^j\Big(C^*(\boldsymbol{M}_{(n)}^{*\mathcal{P}})\Big) \cong \bigoplus_{\mathfrak{l}\in\mathfrak{L}_n(\mathbb{K})} \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in\mathfrak{X}_{j-\mathfrak{e}-\deg\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

These cohomology classes are given by formulas similar to (7.19) and (7.20), with $\lambda_j^{(i)}$ replaced by $\tilde{\lambda}_j^{(i)}$.

If n is odd, this is the only summand in (7.30), and the computation of $H_c^*(\mathcal{G},\mathbb{C})_{\mathcal{I}}^{\boldsymbol{K}_f}$ is complete in this case. For example, if $n=3$, the only possible $\mathfrak{l}$ is $\mathfrak{l}(3)=1$ or $\mathfrak{l}(3)=0$. In the second case, the $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ are empty by definition. In the first case, the only element of the sets $\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ is $\{\{2\}\}\in\mathfrak{X}_{0,0,\mathfrak{l}}$ which gives us the volume form in degree 5. This compares well to the result of Soulé [21, Theorem 4] which implies that $H^*(\mathrm{SL}_3(\mathbb{Z}),\mathbb{C})$ vanishes in positive dimension. In particular, there are no harmonic cusp forms for $\mathrm{SL}_3(\mathbb{Z})$.

If n is even, then we still have to compute the cohomology of the second summand ${}^e\boldsymbol{M}_{(n)}^{*\mathcal{P}}$ in (7.30). Let ${}^e\mathfrak{L}_n(\mathbb{Q})$ be the set of functions

$$\mathfrak{l}\colon \mathrm{Odd}_{\leq n-1}\to\{0,1,\ldots\}$$

satisfying the condition

$$\sum_{j=1}^{\infty}\left(1+\max\{k\in\mathrm{Odd}_{\leq n-1}\,|\,\mathfrak{l}(k)\geq j\}\right)\leq n. \tag{7.33}$$

If the parabolic subgroup $\mathcal{P}$ corresponds to $0<i_1<\cdots<i_K=n$, let ${}^e\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ be empty of one of the numbers i_l is odd, and be equal to the set of functions

$$\mathfrak{y}\colon\ \{(k,l)\,|k\in\mathrm{Odd}_{\leq n-1},1\leq l\leq\mathfrak{l}(k)\}\to\{1,\ldots,k\}$$

with the properties (7.11) and (7.12) if all numbers i_k are even. The vector space ${}^e\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}$ with base ${}^e\mathfrak{Y}_{\mathfrak{l},\mathcal{P}}$ is functorial in $\mathcal{P}$ by by formula (7.14), where the relation $\trianglerighteq$ is defined by (7.13), and there is a functor isomorphism

$${}^e\boldsymbol{M}_{(n)}^{*\mathcal{P}}\cong\bigoplus_{\mathfrak{l}\in{}^e\mathfrak{L}_n(\mathbb{Q})}{}^e\boldsymbol{I}_{\mathfrak{l}}^{\mathcal{P}}[-n-\deg\mathfrak{l}] \tag{7.34}$$

which maps $\mathfrak{y}$ to

$$\bigwedge_{j=1}^{K}\varepsilon^{(j)}\wedge\bigwedge_{l=1}^{\mathfrak{l}(k)}\tilde{\lambda}_k^{\mathfrak{y}(k,l)}.$$

Let

$${}^e\mathfrak{S}_{\mathfrak{l}}=\{(k,l)\,|k\in\mathrm{Odd}_{\leq n-1},1\leq l\leq\mathfrak{l}(k)\},$$

and let $F(k,l)=k$. Recall the poset $\tilde{P}_{n,{}^e\mathfrak{S}_{\mathfrak{l}},F}$ consisting of partitions of n which have for each $\mathfrak{s}\in{}^e\mathfrak{S}_{\mathfrak{l}}$ a piece of length $\geq F(\mathfrak{s})$ marked, and recall the projection

$$p_{\mathfrak{l}}\colon\tilde{P}_{n,{}^e\mathfrak{S}_{\mathfrak{l}},F}\to\mathfrak{P}$$

which sends a partition of n to the corresponding parabolic subgroup of GL_n. Let $\hat{P}_{n,\mathfrak{l}}\subset\tilde{P}_{n,{}^e\mathfrak{S}_{\mathfrak{l}},F}$ be the subposet of all partitions of n into even pieces, together with a map which for each $\mathfrak{s}\in{}^e\mathfrak{S}_{\mathfrak{l}}$ marks a piece of length $\geq F(\mathfrak{s})$, and let $\hat{p}_{\mathfrak{l}}$ be the restriction of $p_{\mathfrak{l}}$ to $\hat{P}_{n,\mathfrak{l}}$. Then ${}^e\boldsymbol{I}_{\mathfrak{l}}^{\bullet}$ can be identified with the antisymmetrization of $\boldsymbol{J}^{\bullet}_{\hat{P}_{n,\mathfrak{l}},\hat{p}_{\mathfrak{l}}}$ with respect to the product of symmetric groups $\prod_{k\in\mathrm{Odd}_{\leq n-1}} S_{\mathfrak{l}(k)}$. Proposition 7.7 can be applied to $\boldsymbol{J}^\bullet_{\tilde{P}_{n,{}^e\mathfrak{S}_{\mathfrak{l}},F},p_{\mathfrak{l}}}$ and gives us an isomorphism

$$H^*\left(C^*(\boldsymbol{J}^\bullet_{\hat{P}_{n,\mathfrak{l}},\hat{p}_{\mathfrak{l}}})\right) \cong \tilde{H}^*(\boldsymbol{B}\hat{P}_{n,\mathfrak{l}}).$$

On the other side, the homotopy type of the poset

$$\hat{P}_{n,\mathfrak{l}} \cong \tilde{P}_{\frac{n}{2},{}^e\mathfrak{S}_{\mathfrak{l}},\frac{1+F}{2}}$$

is given by Proposition 7.6. We arrive at the following explicit description of the second summand in (7.30).

Theorem 7.11. *If $n = 2$, we have*

$$H^*\left(C^*\left({}^e\boldsymbol{M}^{*\bullet}_{(n)}\right)\right) \cong \mathbb{C}[2].$$

For $n > 2$ and $\mathfrak{l} \in {}^e\mathfrak{L}_n(\mathbb{Q})$, $\mathfrak{e} \in \{0,1\}$, and $N \leq 0$, let ${}^e\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$ be the set of ordered $(N+1)$-tuples $\mathfrak{x} = (X_0,\ldots,X_N)$ of subsets of $\{\mathrm{Odd}_{\leq n-1}\}$ with the following properties:

- *Each number $k \in \mathrm{Odd}_{\leq n-1}$ belongs to precisely $\mathfrak{l}(k)$ of the sets X_i;*
- *We have*
$$\sum_{i=0}^{N}(1+\max\{X_i\}) = n - 2\mathfrak{e}.$$

If $\mathfrak{l} = 0$, we put ${}^e\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}} = \emptyset$. Then for each $\mathfrak{x} \in {}^e\mathfrak{X}_{N,\mathfrak{e},\mathfrak{l}}$, $H^*(C^*({}^e\boldsymbol{I}^\bullet_{\mathfrak{l}}))$ has a generator $\{\mathfrak{x}\}$ in degree $N + \mathfrak{e}$, and we have

$$H^i\left(C^*\left({}^e\boldsymbol{I}^\bullet_{\mathfrak{l}}\right)\right) = \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in{}^e\mathfrak{X}_{i-\mathfrak{e},\mathfrak{e},\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

Consequently, the cohomology of the second summand in (7.30) is given by

$$H^j\left(C^*\left({}^e\boldsymbol{M}^{*\bullet}_{(n)}\right)\right) \cong \bigoplus_{\mathfrak{l}\in{}^e\mathfrak{L}_n(\mathbb{K})} \bigoplus_{\mathfrak{e}=0}^{1} \bigoplus_{\mathfrak{x}\in{}^e\mathfrak{X}_{j-\mathfrak{e}-n-\deg\mathfrak{l}}} \mathbb{C}\cdot\{\mathfrak{x}\}.$$

Moreover, let the ordering $\prec$ on the roots which was used to define the complex $C^*(\boldsymbol{F}^\bullet)$ be

$$x_1 - x_2 \prec x_2 - x_3 \prec \cdots \prec x_{n-1} - x_n.$$

Then for $\mathfrak{x} = (X_0,\ldots,X_N) \in {}^e\mathfrak{X}_{N,0,\mathfrak{l}}$, a representative of the cohomology class $\{\mathfrak{x}\}$ is given by the element

$$\bigwedge_{i=0}^{N}\left(\varepsilon^{(i)} \wedge \bigwedge_{\substack{j=2\\ j\in X_i}}^{\#(X_i)} \tilde{\lambda}_j^{(i)}\right) \tag{7.35}$$

in the cohomology of $\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{P}}$, *where* $\mathcal{P} \in \mathfrak{P}$ *is the stabilizer of the standard flag of vector spaces with dimensions*

$$0 < 1 + \#(X_0) < 2 + \#(X_0) + \#(X_1) < \dots < N - 1 + \sum_{i=0}^{N-2} \#(X_i) < N + \sum_{i=0}^{N-1} \#(X_i) = n.$$

If $\mathfrak{x} = (X_0, \dots, X_N) \in {}^e\mathfrak{X}_{N,1,\mathfrak{l}}$ *and if* $0 \le k \le N+1$, *then a representative of the cohomology class* $\{\mathfrak{x}\}$ *is given by the element*

$$(-1)^k \bigwedge_{i=0}^{N-1} \varepsilon^{(i)} \bigwedge_{i=0}^{k-1} \bigwedge_{\substack{j=2 \\ j \in X_i}}^{\#(X_i)} \tilde{\lambda}_j^{(i)} \wedge \bigwedge_{i=k}^{N} \bigwedge_{\substack{j=2 \\ j \in X_i}}^{\#(X_i)} \tilde{\lambda}_j^{(i+1)} \tag{7.36}$$

in the cohomology of $\boldsymbol{X}^{(c)}_{\mathcal{M}_\mathcal{P}}$, *where* $\mathcal{P} \in \mathfrak{P}$ *is the stabilizer of the standard flag of vector spaces with dimensions*

$$0 < 1 + \#(X_0) < 2 + \#(X_0) + \#(X_1) < \dots < k + \sum_{i=0}^{k-1} \#(X_i) < k + 2 + \sum_{i=0}^{k-1} \#(X_i) < \cdots N + 2 + \sum_{i=1}^{N-2} \#(X_i) < 1 + \sum_{i=1}^{n-1} \#(X_i) = n.$$

In the case $n = 4$, we have the vector degree 6 in the first summand in (7.30) defined by $\mathfrak{l}(3) = 1$ and $\mathfrak{x} = \big\{\{3\}\big\} \in \mathfrak{X}_{0,1,\mathfrak{l}}$. In the second summand, we have the cohomology class defined by $\mathfrak{l}(3) = 1$ and $\mathfrak{x} = \big\{\{3\}\big\} \in {}^e\mathfrak{X}_{0,0,\mathfrak{l}}$. It is the volume form in degree 9. These are all spherical vectors in the cohomology with compact support, since $H^i\big(\mathrm{SL}_4(\mathbb{Z}), \mathbb{Z}\big)$ is of dimension one if $i \in \{0; 3\}$ and zero otherwise, by the computation of Lee and Szczarba [16, Theorem 2]. Once again there are no harmonic cusp forms modulo $\mathrm{SL}_4(\mathbb{Z})$. One may ask if this is true for all the groups $\mathrm{SL}_n(\mathbb{Z})$.

It is also possible to give a full computation of $H^*(\mathcal{G}, \mathbb{C})$ for $\mathcal{G} = \mathrm{SL}_n$. The result has a decomposition similar to (7.30) into a summand containing no Euler classes and, for n even, a summand containing the Euler classes. The first of these summands is given by (7.28). The second summand is similar to (7.28), however, the definition of the summands in (7.28) has to be modified to allow only parabolic subgroups corresponding to decompositions of n into even pieces. It is also possible to generalize this to SL_n over arbitrary number fields. The only difference to the cases treated here is that the cohomology with compact support of the Levi components has additional generators in dimension one, which complicate the formulation of the result even more.

Selective Index of Notation

This is a selective index of the mathematical notations which are most frequently used. They are listed according to the order in which they are introduced in the text.

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