Elementary Euclidean Geometry
An Introduction

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Points and Lines

Lines play a fundamental role in geometry. It is not just that they occur widely in the analysis of physical problems – the geometry of more complex curves can sometimes be better understood by the way in which they intersect lines. For some readers the material of this chapter will be familiar from linear algebra, in which case it might be best just to scan the contents and proceed to Chapter 2. Even so, you are advised to look carefully at the basic definitions. It is worth understanding the difference between a linear function and its zero set: it may seem unduly pedantic, but blurring the distinction introduces a potential source of confusion. Much of this text depends on the mechanics of handling lines efficiently and for that reason Section 1.4 is devoted to practical procedures. In Section 1.5 we consider lines from the parametric viewpoint, which will be of use later when we look at the properties of conics in more detail. Finally, we go one step further by considering pencils of lines, which will play a key role in introducing axes in Chapter 7.

1.1 The Vector Structure

Throughout this text $\mathbb{R}$ will denote the set of real numbers. For linguistic variety we will refer to real numbers as *constants* (or *scalars*).[1] We will work in the familiar real plane $\mathbb{R}^2$ of elementary geometry, whose elements $Z = (x, y)$ are called *points* (or *vectors*). Recall that we can add vectors, and multiply them by constants λ, according to the familiar rules

$$(x, y) + (x', y') = (x + x', y + y'), \qquad \lambda(x, y) = (\lambda x, \lambda y).$$

Two vectors $Z = (x, y)$, $Z' = (x', y')$ are *linearly dependent* when there exist constants λ, λ' (not both zero) for which $\lambda Z + \lambda' Z' = 0$: otherwise they are

[1] In this text the first occurrence of an expression is always italicized, the context defining its meaning. Now and again we also italicize expressions for emphasis.

linearly independent. Thus non-zero vectors Z, Z' are linearly dependent when each is a constant multiple of the other. By linear algebra Z, Z' are linearly independent if and only if $xy' - x'y \neq 0$: and in that case linear algebra tells us that any vector can be written uniquely in the form $\lambda Z + \lambda' Z'$ for some scalars λ, λ'.

Example 1.1 The relation of linear dependence on *non-zero* vectors is an equivalence relation on the plane (with the origin deleted) and the resulting equivalence classes are *ratios*. The ratio associated to the point (x, y) is denoted $x : y$. Provided $y \neq 0$ the ratio $x : y$ can be identified with the constant x/y, whilst the ratio $(1 : 0)$ is usually denoted ∞.

1.2 Lines and Zero Sets

Our starting point is to give a careful definition of what we mean by a line. A *linear function* in x, y is an expression $ax + by + c$, where the *coefficients* a, b, c are constants, and at least one of a, b is non-zero. Suppose we have two linear functions

$$L(x, y) = ax + by + c, \qquad L'(x, y) = a'x + b'y + c'.$$

We say that L, L' are *scalar multiples* of each other when there exists a real number $\lambda \neq 0$ with $a' = \lambda a$, $b' = \lambda b$, $c' = \lambda c$. For instance, any two of the following linear functions are scalar multiples of each other

$$x - y + 1, \qquad 2x - 2y + 2, \qquad -x + y - 1.$$

This relation on linear functions is an equivalence relation, and an equivalence class is called a *line*. Our convention is that the line associated to a linear function L is denoted by the same symbol. Associated to any linear function L is its *zero set*

$$\{(x, y) \in \mathbb{R}^2 : \; L(x, y) = 0\}.$$

Note that any scalar multiple of L has the same zero set, so the concept makes perfect sense for lines. Instead of saying that $P = (x, y)$ is a point in the zero set, we shall (for linguistic variety) say that P *lies on* L, or that L *passes through* P.

At this point you should pause, long enough to be sure you have absorbed the preceding definitions. A line is a linear function, up to scalar multiples: it is a quite distinct object from its zero set, a set of points in the plane. The zero set of a line is completely determined by that line. In the next section we will show that conversely, a line is completely determined by its zero set, so it may seem pedantic to separate the concepts. However the ‘conics’ we will meet in the Chapter 4 are not necessarily determined by their zero sets, so it is wise to get into the habit of maintaining the distinction.

1.3 Uniqueness of Equations

Though elementary, the following result is conceptually important. It is the first of a sequence of results linking two disparate notions.

Theorem 1.1 *Through any two distinct points $P = (p_1, p_2)$, $Q = (q_1, q_2)$ there is a unique line $ax + by + c$.*

Proof To establish this fact we seek constants a, b, c (not all zero) for which

$$ap_1 + bp_2 + c = 0, \qquad aq_1 + bq_2 + c = 0. \tag{1.1}$$

That is a linear system of two equations in the three unknowns a, b, c with matrix

$$\begin{pmatrix} p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \end{pmatrix}.$$

Since P, Q are distinct, at least one of the 2×2 minors of this matrix is non-zero. (You really ought to check this.) By linear algebra, the matrix has rank 2, hence kernel rank 1. That means that there is a *non-trivial* solution (a, b, c), and that any other solution (a', b', c') is a non-zero scalar multiple. Non-triviality means that at least one of a, b, c is non-zero: in fact, at least one of a, b is non-zero, for if $a = b = 0$ then $c \neq 0$, and our linear system of equations fails to have a solution. Thus there is a line through P, Q and any other line with that property coincides with it. □

Thus *a line is determined by its zero set*, meaning that if the linear functions L, L' have the same zero sets they are scalar multiples of each other: we have only to pick two distinct points in the common zero set, and apply the above result. That justifies the time-honoured practice of referring to the *equation* $L = 0$ of a line L. Strictly, that is an abbreviation for the zero set of L, but since the zero set determines L it is not too misleading. Nevertheless, you are strongly advised to maintain a crystal-clear mental distinction between lines and their zero sets.

Example 1.2 The *slope* of a line $ax + by + c = 0$ is the ratio $-a : b$. Lines of infinite slope are *vertical* and can be written in the form $x = x_0$, whilst lines of zero slope are *horizontal* and can be written in the form $y = y_0$. For non-vertical lines the slope is identified with the constant $-a/b$. Any non-vertical line can be written $y = px + q$ for some constants p, q and has slope p: likewise, any non-horizontal line can be written $x = ry + s$ for some constants r, s and has slope $1/r$. Observe that any line can be expressed in one (or both) of these forms. It will also be convenient to refer to the ratio $-b : a$ as the *direction* of the line, and any representative of this ratio as a *direction vector*: in particular, $(-b, a)$ is a direction vector for the line.

Exercise

1.3.1 Two linear functions $a_1x + b_1y + c_1$, $a_2x + b_2y + c_2$ are such that $c_1 = a_1^2 + b_1^2$, $c_2 = a_2^2 + b_2^2$. Show that if the resulting lines coincide then $a_1 = a_2$, $b_1 = b_2$.

1.4 Practical Techniques

Much of the material in this book revolves around the sheer mechanics of handling lines. In this section we introduce a small number of practical techniques, which are well worth mastering.

Example 1.3 There is an easily remembered formula for the line through p, q of the previous example. Linear algebra (or direct substitution) tells us that a solution (a, b, c) of the equations (1.1) is given by $a = p_2 - q_2$, $b = q_1 - p_1$, $c = p_1q_2 - p_2q_1$. Substituting for a, b, c in $ax + by + c = 0$ we see that the equation of the line is

$$\begin{vmatrix} x & y & 1 \\ p_1 & p_2 & 1 \\ q_1 & q_2 & 1 \end{vmatrix} = 0. \tag{1.2}$$

Here is a useful application. A set of points is *collinear* when there exists one line on which all the points of the set lie. Assuming there are at least two distinct points in the set, it will be collinear if and only if every other point lies on the line joining these two. Thus to check that a given set of points is collinear we need a criterion for three points to be collinear.

Example 1.4 The condition for three distinct points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $P_3 = (x_3, y_3)$ to be collinear is that the following relation holds. Indeed they are collinear if and only if P_1 lies on the line joining P_2, P_3 so

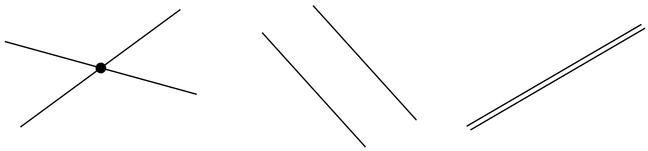


Fig. 1.1. Three ways in which lines can intersect

satifies the equation of the previous example

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The *intersection* of two lines is the set of points common to both zero sets. The point of the next example is that there are just three possibilities: the intersection is either a single point, or empty, or coincides with both zero sets.

Example 1.5 The intersections of two lines L, L' are the common solutions of two linear equations

$$ax + by + c = 0, \qquad a'x + b'y + c' = 0.$$

Provided (a, b), (a', b') are linearly independent there is a unique solution, given by Cramer's Rule

$$x = \frac{bc' - b'c}{ab' - a'b}, \qquad y = \frac{a'c - ac'}{ab' - a'b}.$$

Otherwise, there are two possibilities. The first is that L, L' have no intersection, and are said to be *parallel*: and the second is that L, L' have identical zero sets, so coincide. Thus the lines parallel to $ax + by + c = 0$ are those of the form $ax + by + d = 0$ with $d \neq c$. More generally, a set of lines is *parallel* when no two of them have a common point.

A set of lines is *concurrent* when there exists a point through which every line in the set passes. Assuming there are at least two distinct lines in the set, it will be concurrent if and only if every other line passes through their intersection. It would therefore be helpful to have a criterion for three general lines to be concurrent.

Lemma 1.2 *A necessary and sufficient condition for three distinct non-parallel lines* $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_3x + b_3y + c_3 = 0$ *to be concurrent is that the relation (1.3) below holds*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{1.3}$$

Proof By linear algebra (1.3) is a necessary and sufficient condition for the following homogeneous sytem of linear equations to have a non-trivial solution (x, y, z)

$$a_1x + b_1y + c_1z = 0, \qquad a_2x + b_2y + c_2z = 0, \qquad a_3x + b_3y + c_3z = 0. \tag{1.4}$$

If the lines are concurrent, there is a point (p, q) lying on all three, and hence a non-trivial solution $x = p$, $y = q$, $z = 1$ of the system with $z \neq 0$. And, conversely, if there is a solution with $z \neq 0$ then the point (p, q) with $p = x/z$, $q = y/z$ lies on all three lines, so they are concurrent. It remains to consider the possibility when (1.4) has a non-trivial solution (x, y, z) with $z = 0$, so there is a non-trivial solution (x, y) for the homogeneous system

$$a_1x + b_1y = 0, \qquad a_2x + b_2y = 0, \qquad a_3x + b_3y = 0.$$

However, in that case linear algebra tells us that the vectors (a_1, b_1), (a_2, b_2), (a_3, b_3) are linearly dependent, so the lines are parallel, contrary to assumption. □

Exercises

1.4.1 In each of the following cases find the equation of the line L through the given points P, Q:

(i) $P = (1, -1)$, $Q = (2, -3)$,
(ii) $P = (1, 7)$, $Q = (3, -4)$,
(iii) $P = (3, -2)$, $Q = (5, -1)$.

1.4.2 In each of the following cases find the points of intersection of the given lines:

(i) $2x - 5y + 1 = 0$, $x + y + 4 = 0$,
(ii) $7x - 4y + 1 = 0$, $x - y + 1 = 0$,
(iii) $ax + by - 1 = 0$, $bx + ay - 1 = 0$.

1.4.3 In each of the following cases determine whether P, Q, R are collinear, and if so find the line through them:

(i)	$P = (1, -3)$,	$Q = (-1, -5)$,	$R = (2, -2)$,
(ii)	$P = (3, 1)$,	$Q = (-1, 2)$,	$R = (19, -3)$,
(iii)	$P = (4, 3)$,	$Q = (-2, 1)$,	$R = (1, 2)$.

1.4.4 Find the value of λ for which $P = (3, 1)$, $Q = (5, 2)$, $R = (\lambda, -3)$ are collinear.

1.4.5 Show that for any choice of a, b the points $(a, 2b)$, $(3a, 0)$, $(2a, b)$, $(0, 3b)$ are collinear.

1.4.6 In each of the following cases show that the given lines are concurrent:

(i)	$3x - y - 2 = 0$,	$5x - 2y - 3 = 0$,	$2x + y - 3 = 0$,
(ii)	$2x - 5y + 1 = 0$,	$x + y + 4 = 0$,	$x - 3y = 0$,
(iii)	$7x - 4y + 1 = 0$,	$x - y + 1 = 0$,	$2x - y = 0$.

1.4.7 Find the unique value of λ for which the lines $x - 3y + 3 = 0$, $x + 5y - 7 = 0$, $2x - 2y - \lambda = 0$ are concurrent.

1.5 Parametrized Lines

So far we have viewed lines as sets of points in the plane, defined by a single equation. The next step is to take a different viewpoint, and think of lines as 'parametrized' in a natural way. It is a small step, but it develops into a different viewpoint of the subject.

Lemma 1.3 *Let $P = (p_1, p_2)$, $Q = (q_1, q_2)$ be distinct points on a line $ax + by + c = 0$. For any constant t the point $Z(t) = (x(t), y(t))$, where $x(t)$, $y(t)$ are defined below, also lies on the line*

$$x(t) = (1 - t)p_1 + tq_1, \qquad y(t) = (1 - t)p_2 + tq_2. \tag{1.5}$$

Conversely, any point $Z = (x, y)$ on the line has this form for some constant t.

Proof The first claim follows from the following identity, as both expressions in braces are zero

$$ax(t) + by(t) + c = (1 - t)\{ap_1 + bp_2 + c\} + t\{aq_1 + bq_2 + c\}.$$

Conversely, for any point $Z = (x, y)$ on L the relation (1.2) is satisfied. Thus the rows of the matrix are linearly dependent, and there are constants s, t for

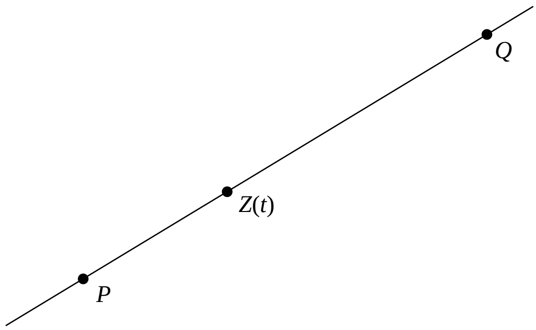


Fig. 1.2. Parametrization of a line

which the following relation holds. The relations (1.5) follow immediately

$$(x, y, 1) = s(p_1, p_2, 1) + t(q_1, q_2, 1).$$

□

The relations (1.5) are the *standard parametrization* of the line with respect to the points P, Q. The mental picture is that the line is traced by a moving particle having the position $Z(t)$ at time t: at time $t = 0$ the particle is at $P = Z(0)$, and at time $t = 1$ it is at $Q = Z(1)$.

Example 1.6 Consider the line $2x - 3y + 3 = 0$. By inspection the line passes through the points $P = (0, 1)$, $Q = (3, 3)$ giving the parametrization $x(t) = 3t$, $y(t) = 2t + 1$. A different choice gives rise to a different parametrization. For instance $P = (-3, -1)$, $Q = (6, 5)$ produces $x(t) = 3(3t - 1)$, $y(t) = 6t - 1$.

The proof of Lemma 1.3 shows that the zero set of any line is infinite, since different values of t correspond to different points $Z(t)$ on the line. The *midpoint* of the line is the point R with parameter $t = 1/2$, i.e. the point

$$R = \frac{P + Q}{2}.$$

Exercises

1.5.1 In each of the following cases find the standard parametrization of the line L relative to the points P, Q:

(i) $L = x - 2y - 5$, $P = (3, -1)$, $Q = (7, 1)$,
(ii) $L = 3x + y - 1$, $P = (3, -8)$, $Q = (-1, -2)$.

1.5.2 In each of the following cases find parametrizations (with integral coefficients if possible) for the given lines:

(i)	$x + 3y - 7 = 0,$	(iv)	$2x + 6y - 5 = 0,$
(ii)	$3x - 4y - 13 = 0,$	(v)	$2x - 3y + 1 = 0,$
(iii)	$7x - 3y - 8 = 0,$	(vi)	$5x - 3y + 1 = 0.$

1.5.3 Find equations for the following parametrized lines:

(i)	$x = 2 + 3t,$	$y = -1 + 4t,$
(ii)	$x = \frac{1}{2} + \frac{3}{4}t,$	$y = -3 + t,$
(iii)	$x = -3 - t,$	$y = 1 - 2t.$

1.5.4 Show that the parametrized lines $x = 2 + 3t$, $y = -1 + 4t$ and $x = -4 + 6t$, $y = -9 + 8t$ coincide.

1.5.5 Find the three intersections of the following parametrized lines:

(i)	$x = 2 + 3t,$	$y = 1 - t,$
(ii)	$x = 4 + 4t,$	$y = 1 - 2t,$
(iii)	$x = -3 - t,$	$y = 2 + 3t.$

1.5.6 Show that any non-vertical line has a parametrization of the form $x(t) = t$, $y(t) = \alpha + \beta t$, and that any non-horizontal line has a parametrization of the form $x(t) = \gamma + \delta t$, $y(t) = t$.

1.6 Pencils of Lines

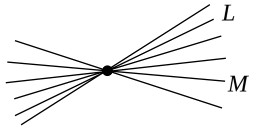
By the *pencil of lines* spanned by two distinct lines L, M we mean the set of all lines of the form $\lambda L + \mu M$, where λ, μ are constants, not both zero. The key *intersection property* of a pencil is that any two distinct lines L', M' in it have the same intersection as L, M. To this end, write

$$L' = \lambda L + \mu M, \qquad M' = \lambda' L + \mu' M.$$

Since L', M' are distinct, the vectors (λ, μ), (λ', μ') are linearly independent, and by linear algebra the relations $L' = 0$, $M' = 0$ are equivalent to $L = 0$, $M = 0$. That establishes the claim.

The first geometric possibility for the pencil of lines spanned by L, M is that L, M intersect at a point P. Then, by the intersection property any line in the pencil passes through P, and we refer to the pencil of lines *through* P. Any line $ax + by + c = 0$ through $P = (p, q)$ must satisfy $ap + bq + c = 0$, so can be written in the form

$$a(x - p) + b(y - q) = 0. \tag{1.6}$$



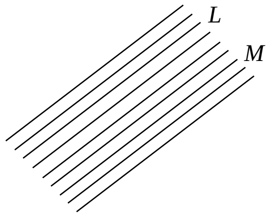


Fig. 1.3. Pencils of lines

Example 1.7 Let L, M be distinct lines in the pencil of all lines through a point P. We claim that *any* line N through P is in the pencil, so has the form $N = \lambda L + \mu M$ for some constants λ, μ. As above, we can write

$$\begin{cases} L(x, y) &= a(x - p) + b(y - q) \\ M(x, y) &= c(x - p) + d(y - q) \\ N(x, y) &= e(x - p) + f(y - q). \end{cases}$$

Since L, M have different directions, the vectors (a, b), (c, d) are linearly independent, and by linear algebra form a basis for the plane. Thus there exist unique constants λ, μ (not both zero) for which the displayed relation below holds. It follows that $N = \lambda L + \mu M$, as was required

$$(e, f) = \lambda(a, b) + \mu(c, d).$$

Example 1.8 In the pencil of lines through $P = (p, q)$ there is a unique vertical line $L(x, y) = x - p$, and a unique horizontal line $M(x, y) = y - q$. By the previous example, any line in the pencil is a linear combination of L, M, as is illustrated by equation (1.6).

The second geometric possibility for the pencil of lines spanned by L, M is that L, M are parallel, so by the intersection property *any* two distinct lines in the pencil are parallel. We call this a *parallel pencil* of lines, and think of it as a limiting case of a general pencil, where all the lines ‘pass through’ the same point at infinity. In such a pencil all the lines have the *same* direction $-b : a$, so it makes sense to refer to the parallel pencil in that direction.

Example 1.9 Any line in the direction $-b : a$ has an equation of the form $N = 0$, where $N = ax + by + c$ for some constant c. Conversely any line of this form must be in the pencil. Suppose indeed that $L = ax + by + l$,

$M = ax + by + m$ are two distinct lines in the pencil. Then we can write $N = \lambda L + \mu M$, where

$$\lambda = \frac{c - m}{l - m}, \qquad \mu = \frac{l - c}{l - m}.$$

Finally, it is worth noting one small difference between the two geometric possibilities described above. Consider the pencil of lines spanned by two distinct lines L, M. In the case when the lines intersect at a point P, *every* expression $\lambda L + \mu M$ is automatically a linear function, so defines a line. However, when the lines are parallel, there is a unique ratio $\lambda : \mu$ for which $\lambda L + \mu M$ fails to be a linear function. For instance, in the parallel pencil spanned by $L(x, y) = x$, $M(x, y) = 2x - 1$, the expression $2L - M = 1$ fails to be linear.

Exercise

1.6.1 Show that the pencil of lines spanned by the lines $2x + 3y - 8$, $4x - 7y + 10$ coincides with the pencil spanned by $3x + 4y - 11$, $2x - 5y + 8$.