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Preface

This book arose out of the *Workshop on structured ring spectra and their applications* held in Glasgow in January 2002. Although it is not intended to be a proceedings of this conference, nevertheless the articles reflect the subject matter of the conference and the papers of Elmendorf, Robinson and Schwede have their origins in series of overview talks which these authors gave in Glasgow. All the papers published here have been refereed.

We would like to thank the London Mathematical Society, Edinburgh Mathematical Society and Glasgow Mathematical Journal Trust Fund for their financial support for the Workshop.

Since the middle of the 1990's there has been a renewed interest in structured ring spectra and several new models for the homotopy category of spectra of Boardman or Adams have been constructed, for example the category of S-modules constructed by Elmendorf, Kriz, Mandell and May [8], the category of symmetric spectra of Hovey, Shipley and Smith [11], the category of Γ-spaces constructed by Lydakis [13], the category of orthogonal spectra defined in [15] and [16], and many more. All of these categories possess a smash product which is strictly associative, commutative and unital, and therefore it makes sense to talk about monoids and commutative monoids, i.e., associative and commutative ring spectra.

Before these constructions have been found, one merely had a smash product of spectra which fulfilled associativity, commutativity only up to homotopy and therefore multiplicative structures on spectra were always given up to homotopy as well. Nevertheless, the techniques of Boardman and Vogt [4] and May, Quinn, Ray and Tornehave [14] allowed for various stricter notions of stricter ring structures, namely ring spectra which are homotopy associative (resp. commutative) up to higher coherences, the so-called A_∞ (resp. E_∞) ring spectra. The coherences were encoded via *operads* in the May approach and via *categories of operators in standard form* in the work of Boardman and Vogt.

With the help of these methods it was possible to identify important spectra as E_∞ ring spectra. For instance, the complex cobordism spectrum MU and other bordism spectra are E_∞ spectra [14, IV §2], as are real and complex connective K-theory ko and ku [14, VIII §2]. But for instance, it is currently unknown whether the Brown-Peterson spectrum BP at a prime possesses an E_∞ structure.

The development of a strict smash product allows additional methods for proving that certain spectra have multiplicative structures and all examples of A_∞ and E_∞ spectra get a strict model by the comparison results of [8, II 4.5, II 4.6]. But many new examples can be gained by working in these strict monoidal categories. For instance, as Bousfield localizations preserve algebra structures on spectra [8, VIII 2.2], the commutative structures on ko

and ku can be used to prove that real and complex K-theory, KO and KU are commutative S-algebras [8, VIII 4.3].

In addition to the construction of algebra spectra as above there are other approaches to imposing associative or commutative algebra structures on a given spectrum: There are several obstruction theories where certain obstruction groups are identified whose vanishing implies the existence of an associative or commutative multiplication on a spectrum. The earliest work in this direction is contained in the paper by Alan Robinson [18] from 1989, which developed an obstruction theory for the detection of A_∞ spectra and applied it to prove that the Morava K-theories $K(n)$ at odd primes have uncountable many A_∞-structures. Later in [1] this method was used to show that completed versions of the Johnson-Wilson spectra possess A_∞ structures. Other obstruction theories for associative algebra structures are contained in the work of Hopkins and Miller [17], Goerss [9] and Lazarev [12].

Later obstruction theories for commutative structures were developed by Basterra [2], Goerss and Hopkins [10], and Robinson [19]. A common feature of all of these obstruction theories is the fact that the obstruction groups live in some generalized versions of André-Quillen cohomology. There will be a short overview about the different types of these cohomology theories for commutative algebras and comparison results between them in [3].

For example the Hopkins-Miller approach led to an A_∞ structure on each Lubin Tate spectrum E_n; the work of Goerss-Hopkins (compare the paper "Moduli Spaces of Commutative Ring Spectra" [10] in this volume) shows that this structure can be refined to an E_∞ multiplication.

The papers in this book deal with two different general topics. On the one hand the papers by Elmendorf, Schwede, and Joachim deal with foundational matters and with the construction of E_∞ structures.

- Tony Elmendorf's article "The development of structured ring spectra" [5] gives background material to the evolution of the symmetric monoidal category of spectra of S-modules of [8]. His second paper "Compromises forced by Lewis's theorem" [6] describes problems one necessarily meets when constructing such a symmetric monoidal category which models the stable homotopy category. There is a list of natural properties such a category might reasonably be required to have and Lewis's theorem shows that these are inconsistent. The third paper [7] in this series represents joint work of Tony Elmendorf and Mike Mandell. That paper sketches how one might use permutative categories to model connective spectra. They construct a symmetric monoidal product on permutative categories and give evidence why this should model the smash-product of spectra. A by-product is a short proof for the fact that bipermutative categories give rise to commutative spectra by forming the associated K-theory spectrum.

- Stefan Schwede's paper "Morita theory in abelian, derived and stable model categories" discusses an enlarged version of Morita equivalence. The subject of the paper [21] is to transfer the well-known notion for Morita equivalence of rings to the derived category of a ring and further to modules over a ring spectrum.
- In "Higher coherences for equivariant K-theory" Michael Joachim gives an E_∞ model for G-equivariant K-theory, for G a compact Lie group and he proves that the G-equivariant analog of the Atiyah-Bott-Shapiro orientation from Spin^c-bordism to K-theory has an E_∞-model as well.

The second part of this book is concerned with obstruction theories for algebra structures on spectra.

- We start with a survey called "(Co-)Homology theories for commutative (S-)algebras" where we describe some of the cohomology theories for commutative algebras and discuss some comparison results for these.
- Alan Robinson gives an overview over his obstruction theories for A_∞ and E_∞ structures on spectra in his paper "Classical obstructions and S-algebras". Besides this he proves, that there are no obstructions to upgrading a homotopy unit to a strict unit for an A_∞ (resp. E_∞) spectrum.
- The paper by Paul Goerss and Mike Hopkins "Moduli spaces of commutative ring spectra" is a summary of the obstruction theory using methods of Dwyer, Kan and Stover; in particular it discusses the example of the Lubin-Tate spectra.
- Andrey Lazarev's paper "Cohomology theories for highly structured ring spectra" describes the approach to obstruction theory using topological André-Quillen homology, which was defined by Basterra, and its counterpart in the theory for associative algebra structures, called *topological derivations*.

Andrew Baker   Birgit Richter

References

[1] A. Baker, *A_∞ structures on some spectra related to Morava K-theory*, Quart. J. Math. Oxf. **42**, 2 (1991), 403–419.
[2] M. Basterra, *André-Quillen cohomology of commutative S-algebras*, J. Pure Appl. Algebra **144**, 2 (1999), 111–143.
[3] M. Basterra & B. Richter, *Homology theories for commutative (S-)algebras*, this volume
[4] J.M. Boardman & R.M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, **347** Springer-Verlag, Berlin-New York (1973) x+257 pp.

[5] A.D. Elmendorf, *The development of structured ring spectra*, this volume
[6] A.D. Elmendorf, *Compromises forced by Lewis's theorem*, this volume
[7] A.D. Elmendorf & M.A. Mandell, *Permutative categories as a model of connective stable homotopy theory*, this volume
[8] A.D. Elmendorf, I. Kriz, M.A. Mandell & J.P. May, *Rings, modules, and algebras in stable homotopy theory*, with an appendix by M. Cole, Mathematical Surveys and Monographs, **47**, AMS, Providence, RI (1997)
[9] P.G. Goerss, *Associative MU-algebras*, (http://www.math.nwu.edu/~pgoerss/)
[10] P.G. Goerss & M.J. Hopkins, *Moduli Spaces of Commutative Ring Spectra*, this volume
[11] M. Hovey, B. Shipley & J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13**, 1 (2000), 149–208
[12] A. Lazarev, *Cohomology theories for highly structured ring spectra*, this volume
[13] M. Lydakis, *Smash products and* Γ*-spaces*, Math. Proc. Cambridge Philos. Soc. **126**, 2 (1999), 311–328.
[14] J.P. May, E_∞ *ring spaces and* E_∞ *ring spectra*, with contributions by F. Quinn, N. Ray and J. Tornehave, Lecture Notes in Mathematics **577** (1977).
[15] M. A. Mandell, J. P. May, S. Schwede and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. **82** (2001), 441-512.
[16] M.A. Mandell, J.P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159**, 755 (2002), x+108 pp.
[17] C. Rezk, *Notes on the Hopkins-Miller theorem*, Contemp. Math., **220** (1998) 313–366.
[18] A. Robinson, *Obstruction theory and the strict associativity of Morava K-theories*, in 'Advances in Homotopy Theory', eds. S. Salamon, B. Steer, W. Sutherland, London Mathematical Society Lecture Note Series **139** (1989) 143–52
[19] A. Robinson, *Gamma homology, Lie representations and* E_∞ *multiplications*, Invent. Math. **152**, 2 (2003), 331–348
[20] A. Robinson, *Classical obstructions and S-algebras*, this volume
[21] S. Schwede, *Morita theory in abelian, derived and stable model categories*, this volume