Preface

Objectives and Audience

In the past three decades, we have witnessed the phenomenal growth in the trading of financial derivatives and structured products in the financial markets around the globe and the surge in research on derivative pricing theory. Leading financial institutions are hiring graduates with a science background who can use advanced analytical and numerical techniques to price financial derivatives and manage portfolio risks, a phenomenon coined as *Rocket Science on Wall Street*. There are now more than a hundred Master level degreed programs in Financial Engineering/Quantitative Finance/Computational Finance in different continents. This book is written as an introductory textbook on derivative pricing theory for students enrolled in these degree programs. Another audience of the book may include practitioners in quantitative teams in financial institutions who would like to acquire the knowledge of option pricing techniques and explore the new development in pricing models of exotic structured derivatives. The level of mathematics in this book is tailored to readers with preparation at the advanced undergraduate level of science and engineering majors, in particular, basic proficiencies in probability and statistics, differential equations, numerical methods, and mathematical analysis. Advance knowledge in stochastic processes that are relevant to the martingale pricing theory, like stochastic differential calculus and theory of martingale, are introduced in this book.

The cornerstones of derivative pricing theory are the Black–Scholes–Merton pricing model and the martingale pricing theory of financial derivatives. The renowned risk neutral valuation principle states that the price of a derivative is given by the expectation of the discounted terminal payoff under the risk neutral measure, in accordance with the property that discounted security prices are martingales under this measure in the financial world of absence of arbitrage opportunities. This second edition presents a substantial revision of the first edition. The new edition presents the theory behind modeling derivatives, with a strong focus on the martingale pricing principle. The continuous time martingale pricing theory is motivated through the analysis of the underlying financial economics principles within a discrete time framework. A wide range of financial derivatives commonly traded in the equity and fixed income markets are analyzed, emphasizing on the aspects of pricing, hedging, and their risk management. Starting from the Black–Scholes–Merton formulation of the option pricing model, readers are guided through the book on the new advances in the state-of-the-art derivative pricing models and interest rate models. Both analytic techniques and numerical methods for solving various types of derivative pricing models are emphasized. A large collection of closed form price formulas of various exotic path dependent equity options (like barrier options, lookback options, Asian options, and American options) and fixed income derivatives are documented.

Guide to the Chapters

This book contains eight chapters, with each chapter being ended with a comprehensive set of well thought out exercises. These problems not only provide the stimulus for refreshing the concepts and knowledge acquired from the text, they also help lead the readers to new research results and concepts found scattered in recent journal articles on the pricing theory of financial derivatives.

The first chapter serves as an introduction to the basic derivative instruments, like the forward contracts, options, and swaps. Various definitions of terms in financial economics, say, self-financing strategy, arbitrage, hedging strategy are presented. We illustrate how to deduce the rational boundaries on option values without any distribution assumptions on the dynamics of the price of the underlying asset.

In Chap. 2, the theory of financial economics is used to show that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure under the discrete securities models. This important result is coined as the Fundamental Theorem of Asset Pricing. This leads to the risk neutral valuation principle, which states that the price of an attainable contingent claim is given by the expectation of the discounted value of the claim under a risk neutral measure. The concepts of attainable contingent claims, absence of arbitrage and risk neutrality form the cornerstones of the modern option pricing theory. Brownian processes and basic analytic tools in stochastic calculus are introduced. In particular, we discuss the Feynman–Kac representation, Radon–Nikodym derivative between two probability measures and the Girsanov theorem that effects the change of measure on an Ito process.

Some of the highlights of the book appear in Chap. 3, where the Black–Scholes–Merton formulation of the option pricing model and the martingale pricing approach of financial derivatives are introduced. We illustrate how to apply the pricing theory to obtain the price formulas of different types of European options. Various extensions of the Black–Scholes–Merton framework are discussed, including the transaction costs model, jump-diffusion model, and stochastic volatility model.

Path dependent options are options with payoff structures that are related to the path history of the asset price process during the option's life. The common examples are the barrier options with the knock-out feature, the Asian options with the averaging feature, and the lookback options whose payoff depends on the realized extremum value of the asset price process. In Chap. 4, we derive the price formulas of the various types of European path dependent options under the Geometric Brownian process assumption of the underlying asset price.

Chapter 5 is concerned with the pricing of American options. We present the characterization of the optimal exercise boundary associated with the American option models. In particular, we examine the behavior of the exercise boundary before and after a discrete dividend payment, and immediately prior to expiry. The two common pricing formulations of the American options, the linear complementarity formulation and the optimal stopping formulation, are discussed. We show how to express the early exercise premium in terms of the exercise boundary in the form of an integral representation. Since analytic price formulas are in general not available for American options, we present several analytic approximation methods for pricing American options. We also consider the pricing models for the American barrier options, the Russian option and the reset-strike options.

Since option models which have closed price formulas are rare, it is common to resort to numerical methods for valuation of option prices. The usual numerical approaches in option valuation are the lattice tree methods, finite difference algorithms, and Monte Carlo simulation. The primary essence of the lattice tree methods is the simulation of the continuous asset price process by a discrete random walk model. The finite difference approach seeks the discretization of the differential operators in the Black–Scholes equation. The Monte Carlo simulation method provides a probabilistic solution to the option pricing problems by simulating the random process of the asset price. An account of option pricing algorithms using these approaches is presented in Chap. 6.

Chapter 7 deals with the characterization of the various interest rate models and pricing of bonds. We start our discussion with the class of one-factor short rate models, and extend to multi-factor models. The Heath–Jarrow–Morton (HJM) approach of modeling the stochastic movement of the forward rates is discussed. The HJM methodologies provide a uniform approach to modeling the instantaneous interest rates. We also present the formulation of the forward LIBOR (London-Inter-Bank-Offered-Rate) process under the Gaussian HJM framework.

The last chapter provides an exposition on the pricing models of several commonly traded interest rate derivatives, like the bond options, range notes, interest rate caps, and swaptions. To facilitate the pricing of equity derivatives under stochastic interest rates, the technique of the forward measure is introduced. Under the forward measure, the bond price is used as the numeraire. In the pricing of the class of LIBOR derivative products, it is more effective to use the LIBORs as the underlying state variables in the pricing models. To each forward LIBOR process, the Lognormal LIBOR model assigns a forward measure defined with respect to the settlement date of the forward rate. Unlike the HJM approach which is based on the non-observable instantaneous forward rates, the Lognormal LIBOR models are based on the observable market interest rates. Similarly, the pricing of a swaption can be effectively performed under the Lognormal Swap Rate model, where an annuity (sum of bond prices) is used as the numeraire in the appropriate swap measure. Lastly, we consider the hedging and pricing of cross-currency interest rate swaps under an appropriate two-currency LIBOR model.

Acknowledgement

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Final Words on the Book Cover Design

One can find the Bank of China Tower in Hong Kong and the Hong Kong Legislative Council Building in the background underneath the usual yellow and blue colors on the book cover of this Springer text. The design serves as a compliment on the recent acute growth of the financial markets in Hong Kong, which benefits from the phenomenal economic development in China and the rule of law under the Hong Kong system.

1

Introduction to Derivative Instruments

The past few decades have witnessed a revolution in the trading of *derivative securities* in world financial markets. A financial derivative may be defined as a security whose value depends on the values of more basic underlying variables, like the prices of other traded securities, interest rates, commodity prices or stock indices. The three most basic derivative securities are forwards, options and swaps. A *forward contract* (called a *futures contract* if traded on an exchange) is an agreement between two parties that one party will purchase an asset from the counterparty on a certain date in the future for a predetermined price. An *option* gives the holder the *right* (but not the obligation) to buy or sell an asset by a certain date for a predetermined price. A *swap* is a financial contract between two parties to exchange cash flows in the future according to some prearranged format. There has been a great proliferation in the variety of derivative securities traded and new derivative products are being invented continually over the years. The development of pricing methodologies of new derivative securities has been a major challenge in the field of financial engineering. The theoretical studies on the use and risk management of financial derivatives have become commonly known as the *Rocket Science* on Wall Street.

In this book, we concentrate on the study of pricing models for financial derivatives. Derivatives trading is an integrated part in portfolio management in financial firms. Also, many financial strategies and decisions can be analyzed from the perspective of options. Throughout the book, we explore the characteristics of various types of financial derivatives and discuss the theoretical framework within which the fair prices of derivative instruments can be determined.

In Sect. 1.1, we discuss the payoff structures of forward contracts and options and present various definitions of terms commonly used in financial economics theory, such as self-financing strategy, arbitrage, hedging, etc. Also, we discuss various trading strategies associated with the use of options and their combinations. In Sect. 1.2, we deduce the rational boundaries on option values without any assumptions on the stochastic behavior of the prices of the underlying assets. We discuss how option values are affected if an early exercise feature is embedded in the option contract and dividend payments are paid by the underlying asset. In Sect. 1.3, we consider the pricing of forward contracts and analyze the relation between forward price and futures price under a constant interest rate. The product nature and uses of interest rate swaps and currency swaps are discussed in Sect. 1.4.

1.1 Financial Options and Their Trading Strategies

First, let us define the different terms in option trading. An option is classified either as a call option or a put option. A *call* (or *put*) option is a contract which gives its holder the *right* to buy (or sell) a prescribed asset, known as the *underlying asset*, by a certain date (*expiration date*) for a predetermined price (commonly called the *strike price* or *exercise price*). Since the holder is given the right but not the obligation to buy or sell the asset, he or she will make the decision depending on whether the deal is favorable to him or not. The option is said to be *exercised* when the holder chooses to buy or sell the asset. If the option can only be exercised on the expiration date, then the option is called a *European* option. Otherwise, if the exercise is allowed at any time prior to the expiration date, then the option is called an *American* option (these terms have nothing to do with their continental origins). The simple call and put options with no special features are commonly called *plain vanilla options*. Also, we have options coined with names like *Asian option*, *lookback option*, *barrier option*, etc. The precise definitions of these exotic types of options will be given in Chap. 4.

The counterparty to the holder of the option contract is called the *option writer*. The holder and writer are said to be, respectively, in the *long* and *short* positions of the option contract. Unlike the holder, the writer does have an obligation with regard to the option contract. For example, the writer of a call option must sell the asset if the holder chooses in his or her favor to buy the asset. This is a zero-sum game as the holder gains from the loss of the writer or vice versa.

An option is said to be *in-the-money* (*out-of-the-money*) if a positive (negative) payoff would result from exercising the option immediately. For example, a call option is in-the-money (out-of-the-money) when the current asset price is above (below) the strike price of the call. An *at-the-money* option refers to the situation where the payoff is zero when the option is exercised immediately, that is, the current asset price is exactly equal to the option's strike price.

Terminal Payoffs of Forwards and Options

The holder of a forward contract is obligated to buy the underlying asset at the forward price (also called delivery price) K on the expiration date of the contract. Let S_T denote the asset price at expiry T. Since the holder pays K dollars to buy an asset worth S_T, the terminal payoff to the holder (long position) is seen to be $S_T - K$. The seller (short position) of the forward faces the terminal payoff $K - S_T$, which is negative to that of the holder (by the zero-sum nature of the forward contract).

Next, we consider a European call option with strike price X. If $S_T > X$, then the holder of the call option will choose to exercise at expiry T since the holder can buy the asset, which is worth S_T dollars, at the cost of X dollars. The gain to the holder from the call option is then $S_T - X$. However, if $S_T \leq X$, then the holder will forfeit the right to exercise the option since he or she can buy the asset in the market at a cost less than or equal to the predetermined strike price X. The terminal payoff from the long position (holder's position) of a European call is then given by

$$\max(S_T - X, 0).$$

Similarly, the terminal payoff from the long position in a European put can be shown to be

$$\max(X - S_T, 0),$$

since the put will be exercised at expiry T only if $S_T < X$. The asset worth S_T can be sold by the put's holder at a higher price of X under the put option contract. In both call and put options, the terminal payoffs are guaranteed to be nonnegative. These properties reflect the very nature of options: they will not be exercised if a negative payoff results.

Option Premium

Since the writer of an option is exposed to potential liabilities in the future, he must be compensated with an up-front premium paid by the holder when they together enter into the option contract. An alternative viewpoint is that since the holder is guaranteed a nonnegative terminal payoff, he must pay a premium get into the option game. The natural question is: What should be the fair option premium (called the option price) so that the game is fair to both the writer and holder? Another but deeper question: What should be the optimal strategy to exercise prior to expiration date for an American option? At least, the option price is easily seen to depend on the strike price, time to expiry and current asset price. The less obvious factors involved in the pricing models are the prevailing *interest rate* and the degree of randomness of the asset price (characterized by the *volatility* of the stochastic asset price process).

Self-Financing Strategy

Suppose an investor holds a portfolio of securities, such as a combination of options, stocks and bonds. As time passes, the value of the portfolio changes because the prices of the securities change. Besides, the trading strategy of the investor affects the portfolio value by changing the proportions of the securities held in the portfolio, say, and adding or withdrawing funds from the portfolio. An investment strategy is said to be *self-financing* if no extra funds are added or withdrawn from the initial investment. The cost of acquiring more units of one security in the portfolio is completely financed by the sale of some units of other securities within the same portfolio.

Short Selling

Investors buy a stock when they expect the stock price to rise. How can an investor profit from a fall of stock price? This can be achieved by short selling the stock. Short selling refers to the trading practice of borrowing a stock and selling it immediately, buying the stock later and returning it to the borrower. The short seller hopes to profit from a price decline by selling the asset before the decline and buying it back afterwards. Usually, there are rules in stock exchanges that restrict the timing of the short selling and the use of the short sale proceeds. For example, an exchange may impose the rule that short selling of a security is allowed only when the most recent movement in the security price is an uptick. When the stock pays dividends, the short seller has to compensate the lender of the stock with the same amount of dividends.

No Arbitrage Principle

One of the fundamental concepts in the theory of option pricing is the absence of arbitrage opportunities, is called the *no arbitrage principle*. As an illustrative example of an arbitrage opportunity, suppose the prices of a given stock in Exchanges A and B are listed at \$99 and \$101, respectively. Assuming there is no transaction cost, one can lock in a riskless profit of \$2 per share by buying at \$99 in Exchange A and selling at \$101 in Exchange B. The trader who engages in such a transaction is called an *arbitrageur*. If the financial market functions properly, such an arbitrage opportunity cannot occur since traders are well aware of the differential in stock prices and they immediately compete away the opportunity. However, when there is transaction cost, which is a common form of market friction, the small difference in prices may persist. For example, if the transaction costs for buying and selling per share in Exchanges A and B are both \$1.50, then the total transaction costs of \$3 per share will discourage arbitrageurs.

More precisely, an *arbitrage opportunity* can be defined as a self-financing trading strategy requiring no initial investment, having zero probability of negative value at expiration, and yet having some possibility of a positive terminal payoff. More detailed discussions on the "no arbitrage principle" are given in Sect. 2.1.

No Arbitrage Price of a Forward

Here we discuss how the no arbitrage principle can be used to price a forward contract on an underlying asset that provides the asset holder no income in the form of dividends. The forward price is the price the holder of the forward pays to acquire the underlying asset on the expiration date. In the absence of arbitrage opportunities, the forward price F on a nondividend paying asset with spot price S is given by

$$F = Se^{r\tau}, \tag{1.1.1}$$

where r is the constant riskless interest rate and τ is the time to expiry of the forward contract. Here, $e^{r\tau}$ is the growth factor of cash deposit that earns continuously compounded interest over the period τ.

It can be shown that when either $F > Se^{r\tau}$ or $F < Se^{r\tau}$, an arbitrageur can lock in a risk-free profit. First, suppose $F > Se^{r\tau}$, the arbitrage strategy is to borrow S dollars from a bank and use the borrowed cash to buy the asset, and also take up a short position in the forward contract. The loan with loan period τ will grow to $Se^{r\tau}$. At expiry, the arbitrageur will receive F dollars by selling the asset under the forward contract. After paying back the loan amount of $Se^{r\tau}$, the riskless profit is then $F - Se^{r\tau} > 0$. Otherwise, suppose $F < Se^{r\tau}$, the above arbitrage strategy is reversed, that is, short selling the asset and depositing the proceeds into a bank, and taking up a long position in the forward contract. At expiry, the arbitrageur acquires the asset by paying F dollars under the forward contract and closing out the short selling position by returning the asset. The riskless profit now becomes $Se^{r\tau} - F > 0$. Both cases represent arbitrage opportunities. By virtue of the no arbitrage principle, the forward price formula (1.1.1) follows.

One may expect that the forward price should be set equal to the expectation of the terminal asset price S_T at expiry T. However, this expectation approach does not enforce the forward price since the expectation value depends on the forward holder's view on the stochastic movement of the underlying asset's price. The above no arbitrage argument shows that the forward price can be enforced by adopting a certain trading strategy. If the forward price deviates from this no arbitrage price, then arbitrage opportunities arise and the market soon adjusts to trade at the "no arbitrage price".

Volatile Nature of Options

Option prices are known to respond in an exaggerated scale to changes in the underlying asset price. To illustrate this claim, we consider a call option that is near the time of expiration and the strike price is $100. Suppose the current asset price is $98, then the call price is close to zero since it is quite unlikely for the asset price to increase beyond $100 within a short period of time. However, when the asset price is $102, then the call price near expiry is about $2. Though the asset price differs by a small amount, between $98 to $102, the relative change in the option price can be very significant. Hence, the option price is seen to be more volatile than the underlying asset price. In other words, the trading of options leads to more price action per dollar of investment than the trading of the underlying asset. A precise analysis of the elasticity of the option price relative to the asset price requires detailed knowledge of the relevant pricing model for the option (see Sect. 3.3).

Hedging

If the writer of a call does not simultaneously own a certain amount of the underlying asset, then he or she is said to be in a *naked position* since he or she has no protection if the asset price rises sharply. However, if the call writer owns some units of the underlying asset, the loss in the short position of the call when the asset's price rises can be compensated by the gain in the long position of the underlying asset. This strategy is called *hedging*, where the risk in a portfolio is monitored by taking opposite directions in two securities which are highly negatively correlated. In a *perfect hedge* situation, the *hedger* combines a risky option and the corresponding underlying asset in an appropriate proportion to form a riskless portfolio. In Sect. 3.1, we examine how the *riskless hedging principle* is employed to formulate the option pricing theory.

1.1.1 Trading Strategies Involving Options

We have seen in the above simple hedging example how the combined use of an option and the underlying asset can monitor risk exposure. Now, we would like to examine the various strategies of portfolio management using options and the underlying asset as the basic financial instruments. Here, we confine our discussion of portfolio strategies to the use of European vanilla call and put options. We also assume that the underlying asset does not pay dividends within the investment time horizon.

The simplest way to analyze a portfolio strategy is to construct a corresponding *terminal profit diagram*. This shows the profit on the expiration date from holding the options and the underlying asset as a function of the terminal asset price. This simplified analysis is applicable only to a portfolio that contains options all with the same date of expiration and on the same underlying asset.

Covered Calls and Protective Puts

Consider a portfolio that consists of a short position (writer) in one call option plus a long holding of one unit of the underlying asset. This investment strategy is known as *writing a covered call*. Let c denote the premium received by the writer when selling the call and S_0 denote the asset price at initiation of the option contract [note that $S_0 > c$, see (1.2.12)]. The initial value of the portfolio is then $S_0 - c$. Recall that the terminal payoff for the call is $\max(S_T - X, 0)$, where S_T is the asset price at expiry and X is the strike price. Assuming the underlying asset to be nondividend paying, the portfolio value at expiry is $S_T - \max(S_T - X, 0)$, so the profit of a covered call at expiry is given by

$$
\begin{aligned}
&S_T - \max(S_T - X, 0) - (S_0 - c) \\
&= \begin{cases} (c - S_0) + X & \text{when } S_T \geq X \\ (c - S_0) + S_T & \text{when } S_T < X. \end{cases}
\end{aligned}
\tag{1.1.2}
$$

Observe that when $S_T \geq X$, the profit is capped at the constant value $(c - S_0) + X$, and when $S_T < X$, the profit grows linearly with S_T. The corresponding terminal profit diagram for a covered call is illustrated in Fig. 1.1. Readers may wonder why $c - S_0 + X > 0$? For hints, see (1.2.3a).

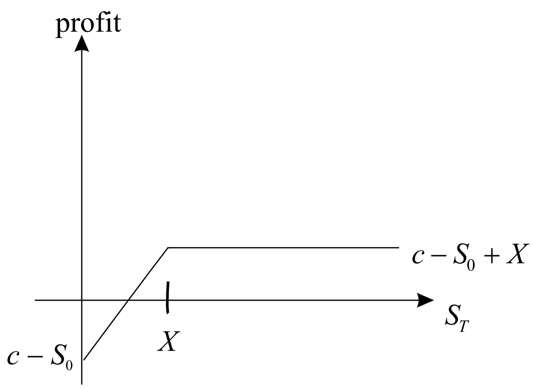


Fig. 1.1. Terminal profit diagram of a covered call.

The investment portfolio that involves a long position in one put option and one unit of the underlying asset is called a *protective put*. Let p denote the premium paid for the acquisition of the put. It can be shown similarly that the profit of the protective put at expiry is given by

$$\begin{aligned} &S_T + \max(X - S_T, 0) - (p + S_0) \\ &= \begin{cases} -(p + S_0) + S_T & \text{when } S_T \geq X \\ -(p + S_0) + X & \text{when } S_T < X. \end{cases} \end{aligned} \tag{1.1.3}$$

Do we always have $X - (p + S_0) < 0$?

Is it meaningful to create a portfolio that involves the long holding of a put and short selling of the asset? This portfolio strategy will have no hedging effect because both positions in the put option and the underlying asset are in the same direction in risk exposure—both positions lose when the asset price increases.

Spreads

A spread strategy refers to a portfolio which consists of options of the same type (that is, two or more calls, or two or more puts) with some options in the long position and others in the short position in order to achieve a certain level of hedging effect. The two most basic spread strategies are the price spread and the calendar spread. In a *price spread*, one option is bought while another is sold, both on the same underlying asset and the same date of expiration but with different strike prices. A *calendar spread* is similar to a price spread except that the strike prices of the options are the same but the dates of expiration are different.

Price Spreads

Price spreads can be classified as either bullish or bearish. The term *bullish* (*bearish*) means the holder of the spread benefits from an increase (decrease) in the asset price. A bullish price spread can be created by forming a portfolio which consists of a call option in the long position and another call option with a higher strike price in the short position. Since the call price is a decreasing function of the strike price [see (1.2.6a)], the portfolio requires an up-front premium for its creation. Let X_1 and X_2 $(X_2 > X_1)$ be the strike prices of the calls and c_1 and c_2 $(c_2 < c_1)$ be their respective premiums. The sum of terminal payoffs from the two calls is shown to be

$$\begin{aligned} &\max(S_T - X_1, 0) - \max(S_T - X_2, 0) \\ &= \begin{cases} 0 & S_T < X_1 \\ S_T - X_1 & X_1 \leq S_T \leq X_2 \\ X_2 - X_1 & S_T > X_2. \end{cases} \end{aligned} \tag{1.1.4}$$

The terminal payoff stays at the zero value until S_T reaches X_1, it then grows linearly with S_T when $X_1 \leq S_T \leq X_2$ and it is capped at the constant value $X_2 - X_1$ when $S_T > X_2$. The bullish price spread has its maximum gain at expiry when both calls expire in-the-money. When both calls expire out-of-the-money, corresponding to $S_T < X_1$, the overall loss would be the initial set up cost for the bullish spread.

Suppose we form a new portfolio with two calls, where the call bought has a higher strike price than the call sold, both with the same date of expiration, then a bearish price spread is created. Unlike its bullish counterpart, the bearish price spread leads to an up front positive cash flow to the investor. The terminal profit of a bearish price spread using two calls of different strike prices is exactly negative to that of its bullish counterpart. Note that the bullish and bearish price spreads can also be created by portfolios of puts.

Butterfly Spreads
Consider a portfolio created by buying a call option at strike price X_1 and another call option at strike price X_3 (say, $X_3 > X_1$) and selling two call options at strike price $X_2 = \frac{X_1+X_3}{2}$. This is called a *butterfly spread*, which can be considered as the combination of one bullish price spread and one bearish price spread. The creation of the butterfly spread requires the set up premium of $c_1 + c_3 - 2c_2$, where c_i denotes the price of the call option with strike price X_i, $i = 1, 2, 3$. Since the call price is a convex function of the strike price [see (1.2.13a)], we have $2c_2 < c_1 + c_3$. Hence, the butterfly spread requires a positive set-up cost. The sum of payoffs from the four call options at expiry is found to be

$$\max(S_T - X_1, 0) + \max(S_T - X_3, 0) - 2\max(S_T - X_2, 0)$$
$$= \begin{cases} 0 & S_T \leq X_1 \\ S_T - X_1 & X_1 < S_T \leq X_2 \\ X_3 - S_T & X_2 < S_T \leq X_3 \\ 0 & S_T > X_3 \end{cases}. \tag{1.1.5}$$

The terminal payoff attains the maximum value at $S_T = X_2$ and declines linearly on both sides of X_2 until it reaches the zero value at $S_T = X_1$ or $S_T = X_3$. Beyond the interval (X_1, X_3), the payoff of the butterfly spread becomes zero. By subtracting the initial set-up cost of $c_1 + c_3 - 2c_2$ from the terminal payoff, we get the terminal profit diagram of the butterfly spread shown in Fig. 1.2.

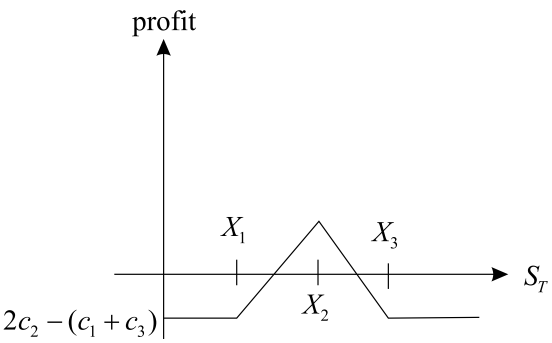


Fig. 1.2. Terminal profit diagram of a butterfly spread with four calls.

The butterfly spread is an appropriate strategy for an investor who believes that large asset price movements during the life of the spread are unlikely. Note that the terminal payoff of a butterfly spread with a wider interval (X_1, X_3) dominates that of the counterpart with a narrower interval. Using the no arbitrage argument, one deduces that the initial set-up cost of the butterfly spread increases with the width of the interval (X_1, X_3). If otherwise, an arbitrageur can lock in riskless profit by buying the presumably cheaper butterfly spread with the wider interval and selling the more expensive butterfly spread with the narrower interval. The strategy guarantees a non-negative terminal payoff while having the possibility of a positive terminal payoff.

Calendar Spreads
Consider a calendar spread that consists of two calls with the same strike price but different dates of expiration T_1 and T_2 ($T_2 > T_1$), where the shorter-lived and longer-lived options are in the short and long positions, respectively. Since the longer-lived call is normally more expensive,[1] an up-front set-up cost for the calendar spread is required. In our subsequent discussion, we consider the usual situation where the longer-lived call is more expensive. The two calls with different expiration dates decrease in value at different rates, with the shorter-lived call decreasing in value at a faster rate. Also, the rate of decrease is higher when the asset price is closer to the strike price (see Sect. 3.3). The gain from holding the calendar spread comes from the difference between the rates of decrease in value of the shorter-lived call and longer-lived call. When the asset price at T_1 (expiry date of the shorter-lived call) comes closer to the common strike price of the two calls, a higher gain of the calendar spread at T_1 is realized because the rates of decrease in call value are higher when the call options come closer to being at-the-money. The profit at T_1 is given by this gain minus the initial set-up cost. In other words, the profit of the calendar spread at T_1 becomes higher when the asset price at T_1 comes closer to the common strike price.

Combinations

Combinations are portfolios that contain options of different types but on the same underlying asset. A popular example is a *bottom straddle*, which involves buying a call and a put with the same strike price X and expiration time T. The payoff at expiry from the bottom straddle is given by

$$\begin{aligned} &\max(S_T - X, 0) + \max(X - S_T, 0) \\ &= \begin{cases} X - S_T & \text{when } S_T \leq X \\ S_T - X & \text{when } S_T > X. \end{cases} \end{aligned} \tag{1.1.6}$$

Since both options are in the long position, an up-front premium of $c + p$ is required for the creation of the bottom straddle, where c and p are the option premium of the European call and put. As revealed from the terminal payoff as stated in (1.1.6), the

[1] Longer-lived European call may become less expensive than the shorter-lived counterpart only when the underlying asset is paying dividend and the call option is sufficiently deep-in-the-money (see Sect. 3.3).

terminal profit diagram of the bottom straddle resembles the letter "V". The terminal profit achieves its lowest value of $-(c+p)$ at $S_T = X$ (negative profit value actually means loss). The bottom straddle holder loses when S_T stays close to X at expiry, but receives substantial gain when S_T moves further away from X in either direction.

The other popular examples of combinations include *strip, strap, strangle, box spread*, etc. Readers are invited to explore the characteristics of their terminal profits through Problems 1.1–1.4.

There are many other possibilities to create spread positions and combinations that approximate a desired pattern of payoff at expiry. Indeed, this is one of the major advantages of trading options rather than the underlying asset alone. In particular, the terminal payoff of a butterfly spread resembles a triangular "spike" so one can approximate the payoff according to an investor's preference by forming an appropriate combination of these spikes. As a reminder, the terminal profit diagrams presented above show the profits of these portfolio strategies when the positions of the options are held to expiration. Prior to expiration, the profit diagrams are more complicated and relevant option valuation models are required to find the value of the portfolio at a particular instant.

1.2 Rational Boundaries for Option Values

In this section, we establish some rational boundaries for the values of options with respect to the price of the underlying asset. At this point, we do not specify the probability distribution of the asset price process so we cannot derive the *fair* option value. Rather, we attempt to deduce reasonable limits between which any acceptable equilibrium price falls. The basic assumptions are that investors prefer more wealth to less and there are no arbitrage opportunities.

First, we present the rational boundaries for the values of both European and American options on an underlying asset that pays no dividend. We derive mathematical properties of the option values as functions of the strike price X, asset price S and time to expiry τ. Next, we study the impact of dividends on these rational boundaries for the option values. The optimal early exercise policies of American options on a non-dividend paying asset can be inferred from the analysis of these bounds on option values. The relations between put and call prices (called the *put-call parity relations*) are also deduced. As an illustrative and important example, we extend the analysis of rational boundaries and put-call parity relations to foreign currency options.

Here, we introduce the concept of time value of cash. It is common sense that \$1 at present is worth more than \$1 at a later instant since the cash can earn positive interest, or conversely, an amount less than \$1 will eventually grow to \$1 after a sufficiently long interest-earning period. In the simplest form of a bond with zero coupon, the bond contract promises to pay the par value at maturity to the bondholder, provided that the bond issuer does not default prior to maturity. Let $B(\tau)$ be the current price of a zero coupon default-free bond with the par value of \$1 at maturity, where τ is the time to maturity (we commonly use "maturity" for bonds and "expiry" for options). When the riskless interest rate r is taken to be constant and interest is compounded continuously, the bond value $B(\tau)$ is given by $e^{-r\tau}$. When r is nonconstant but a deterministic function of τ, $B(\tau)$ is found to be $e^{-\int_0^\tau r(u)\,du}$. The formula for $B(\tau)$ becomes more complicated when the interest rate is assumed to be stochastic (see Sect. 7.2). The bond price $B(\tau)$ can be interpreted as the discount factor over the τ-period.

Throughout this book, we adopt the notation where capitalized letters C and P denote American call and put values, respectively, and small letters c and p for their European counterparts.

Nonnegativity of Option Prices

All option prices are nonnegative, that is,

$$C \geq 0, \quad P \geq 0, \quad c \geq 0, \quad p \geq 0. \tag{1.2.1}$$

These relations are derived from the nonnegativity of the payoff structure of option contracts. If the price of an option were negative, this would mean an option buyer receives cash up front while being guaranteed a nonnegative terminal payoff. In this way, he can always lock in a riskless profit.

Intrinsic Values

Let $C(S, \tau; X)$ denote the price function of an American call option with current asset price S, time to expiry τ and strike price X; similar notation will be used for other American option price functions. At expiry time $\tau = 0$, the terminal payoffs are

$$C(S, 0; X) = c(S, 0; X) = \max(S - X, 0) \tag{1.2.2a}$$

$$P(S, 0; X) = p(S, 0; X) = \max(X - S, 0). \tag{1.2.2b}$$

The quantities $\max(S - X, 0)$ and $\max(X - S, 0)$ are commonly called the *intrinsic value* of a call and a put, respectively. One argues that since American options can be exercised at any time before expiration, their values must be worth at least their intrinsic values, that is,

$$C(S, \tau; X) \geq \max(S - X, 0) \tag{1.2.3a}$$

$$P(S, \tau; X) \geq \max(X - S, 0). \tag{1.2.3b}$$

Since $C \geq 0$, it suffices to consider the case $S > X$, where the American call is in-the-money. Suppose C is less than $S - X$ when $S > X$, then an arbitrageur can lock in a riskless profit by borrowing $C + X$ dollars to purchase the American call and exercise it immediately to receive the asset worth S. The riskless profit would be $S - X - C > 0$. The same no arbitrage argument can be used to show condition (1.2.3b).

However, as there is no early exercise privilege for European options, conditions (1.2.3a,b) do not necessarily hold for European calls and puts, respectively. Indeed, the European put value can be below the intrinsic value $X - S$ at sufficiently low asset value and the value of a European call on a dividend paying asset can be below the intrinsic value $S - X$ at sufficiently high asset value.

American Options Are Worth at Least Their European Counterparts

An American option confers all the rights of its European counterpart plus the privilege of early exercise. Obviously, the additional privilege cannot have negative value. Therefore, American options must be worth at least their European counterparts, that is,

$$C(S, \tau; X) \geq c(S, \tau; X) \tag{1.2.4a}$$

$$P(S, \tau; X) \geq p(S, \tau; X). \tag{1.2.4b}$$

Values of Options with Different Dates of Expiration

Consider two American options with different times to expiry τ_2 and τ_1 $(\tau_2 > \tau_1)$, the one with the longer time to expiry must be worth at least that of the shorter-lived counterpart since the longer-lived option has the additional right to exercise between the two expiration dates. This additional right should have a positive value; so we have

$$C(S, \tau_2; X) > C(S, \tau_1; X), \quad \tau_2 > \tau_1, \tag{1.2.5a}$$

$$P(S, \tau_2; X) > P(S, \tau_1; X), \quad \tau_2 > \tau_1. \tag{1.2.5b}$$

The above argument cannot be applied to European options because the early exercise privilege is absent.

Values of Options with Different Strike Prices

Consider two call options, either European or American, the one with the higher strike price has a lower expected profit than the one with the lower strike. This is because the call option with the higher strike has strictly less opportunity to exercise a positive payoff, and even when exercised, it induces a smaller cash inflow. Hence, the call option price functions are decreasing functions of their strike prices, that is,

$$c(S, \tau; X_2) < c(S, \tau; X_1), \quad X_1 < X_2, \tag{1.2.6a}$$

$$C(S, \tau; X_2) < C(S, \tau; X_1), \quad X_1 < X_2. \tag{1.2.6b}$$

By reversing the above argument, the European and American put price functions are increasing functions of their strike prices, that is,

$$p(S, \tau; X_2) > p(S, \tau; X_1), \quad X_1 < X_2, \tag{1.2.7a}$$

$$P(S, \tau; X_2) > P(S, \tau; X_1), \quad X_1 < X_2. \tag{1.2.7b}$$

Values of Options at Different Asset Price Levels

For a call (put) option, either European or American, when the current asset price is higher, it has a strictly higher (lower) chance to be exercised and when exercised it induces higher (lower) cash inflow. Therefore, the call (put) option price functions are increasing (decreasing) functions of the asset price, that is,

$$c(S_2, \tau; X) > c(S_1, \tau; X), \quad S_2 > S_1, \tag{1.2.8a}$$

$$C(S_2, \tau; X) > C(S_1, \tau; X), \quad S_2 > S_1; \tag{1.2.8b}$$

and

$$p(S_2, \tau; X) < p(S_1, \tau; X), \quad S_2 > S_1, \tag{1.2.9a}$$

$$P(S_2, \tau; X) < P(S_1, \tau; X), \quad S_2 > S_1. \tag{1.2.9b}$$

Upper Bounds on Call and Put Values

A call option is said to be a *perpetual call* if its date of expiration is infinitely far away. The asset itself can be considered an American perpetual call with zero strike price plus additional privileges such as voting rights and receipt of dividends, so we deduce that $S \geq C(S, \infty; 0)$. By applying conditions (1.2.4a) and (1.2.5a), we can establish

$$S \geq C(S, \infty; 0) \geq C(S, \tau; X) \geq c(S, \tau; X). \tag{1.2.10}$$

Hence, American and European call values are bounded above by the asset value. Furthermore, by setting $S = 0$ in condition (1.2.10) and applying the nonnegativity property of option prices, we obtain

$$0 = C(0, \tau; X) = c(0, \tau; X),$$

that is, call values become zero at zero asset value.

The price of an American put equals its strike price when the asset value is zero; otherwise, it is bounded above by the strike price. Together with condition (1.2.4b), we have

$$X \geq P(S, \tau; X) \geq p(S, \tau; X). \tag{1.2.11}$$

Lower Bounds on Values of Call Options on a Nondividend Paying Asset

A lower bound on the value of a European call on a nondividend paying asset is found to be at least equal to or above the underlying asset value minus the present value of the strike price. To illustrate the claim, we compare the values of two portfolios, A and B. Portfolio A consists of a European call on a nondividend paying asset plus a discount bond with a par value of X whose date of maturity coincides with the expiration date of the call. Portfolio B contains one unit of the underlying asset. Table 1.1 lists the payoffs at expiry of the two portfolios under the two scenarios $S_T < X$ and $S_T \geq X$, where S_T is the asset price at expiry.

At expiry, the value of Portfolio A, denoted by V_A, is either greater than or at least equal to the value of Portfolio B, denoted by V_B. Portfolio A is said to be dominant

Table 1.1. Payoffs at expiry of Portfolios A and B

Asset value at expiry	$S_T < X$	$S_T \geq X$
Portfolio A	X	$(S_T - X) + X = S_T$
Portfolio B	S_T	S_T
Result of comparison	$V_A > V_B$	$V_A = V_B$

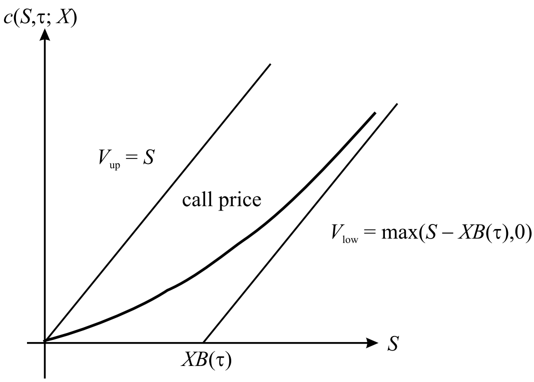


Fig. 1.3. The upper and lower bounds of the option value of a European call on a nondividend paying asset are $V_{up} = S$ and $V_{low} = \max(S - XB(\tau), 0)$, respectively.

over Portfolio B. The present value of Portfolio A (dominant portfolio) must be equal to or greater than that of Portfolio B (dominated portfolio). If otherwise, arbitrage opportunity can be secured by buying Portfolio A and selling Portfolio B. The above result can be represented by

$$c(S, \tau; X) + XB(\tau) \geq S.$$

Together with the nonnegativity property of option value, the lower bound on the value of the European call is found to be

$$c(S, \tau; X) \geq \max(S - XB(\tau), 0).$$

Combining with condition (1.2.10), the upper and lower bounds of the value of a European call on a nondividend paying asset are given by (see Fig. 1.3)

$$S \geq c(S, \tau; X) \geq \max(S - XB(\tau), 0). \tag{1.2.12}$$

Furthermore, as deduced from condition (1.2.10) again, the above lower and upper bounds are also valid for the value of an American call on a nondividend paying asset. The above results on the rational boundaries of European option values have to be modified when the underlying asset pays dividends [see (1.2.14), (1.2.23)].

Early Exercise Polices of American Options

First, we consider an American call on a nondividend paying asset. An American call is exercised only if it is in-the-money, where $S > X$. At any moment when an American call is exercised, its exercise payoff becomes $S - X$, which ought to be positive. However, the exercise value is less than $\max(S - XB(\tau), 0)$, the lower bound of the call value given that the call remains alive. Thus the act of exercising prior to expiry causes a decline in value of the American call. To the benefit of the holder, an American call on a nondividend paying asset will not be exercised prior to expiry. Since the early exercise privilege is forfeited, the American and European call values should be the same.

When the underlying asset pays dividends, the early exercise of an American call prior to expiry may become optimal when the asset value is very high and the dividends are sizable. Under these circumstances, it then becomes more attractive for the investor to acquire the asset through early exercise rather than holding the option. When the American call is deep-in-the-money, $S \gg X$, the chance of regret of early exercise (loss of insurance protection against downside move of the asset price) is low. On the other hand, the earlier acquisition of the underlying asset allows receipt of the dividends paid by the asset. For American puts, irrespective whether the asset is paying dividends or not, it can be shown [see (1.2.16)] that it is always optimal to exercise prior to expiry when the asset value is low enough. More details on the effects of dividends on the early exercise policies of American options will be discussed later in this section.

Convexity Properties of the Option Price Functions

The call prices are convex functions of the strike price. Write $X_2 = \lambda X_3 + (1-\lambda)X_1$ where $0 \leq \lambda \leq 1$, $X_1 \leq X_2 \leq X_3$. Mathematically, the convexity properties are depicted by the following inequalities:

$$c(S, \tau; X_2) \leq \lambda c(S, \tau; X_3) + (1 - \lambda)c(S, \tau; X_1) \tag{1.2.13a}$$

$$C(S, \tau; X_2) \leq \lambda C(S, \tau; X_3) + (1 - \lambda)C(S, \tau; X_1). \tag{1.2.13b}$$

Figure 1.4 gives a graphical representation of the above inequalities.

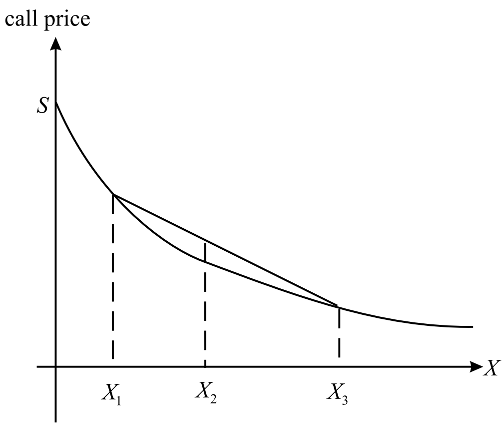


Fig. 1.4. The call price is a convex function of the strike price X. The call price equals S when $X = 0$ and tends to zero at large value of X.

Table 1.2. Payoff at expiry of Portfolios C and D

Asset value at expiry	$S_T \leq X_1$	$X_1 \leq S_T \leq X_2$	$X_2 \leq S_T \leq X_3$	$X_3 \leq S_T$
Portfolio C	0	$(1-\lambda)(S_T - X_1)$	$(1-\lambda)(S_T - X_1)$	$\lambda(S_T - X_3)+$ $(1-\lambda)(S_T - X_1)$
Portfolio D	0	0	$S_T - X_2$	$S_T - X_2$
Result of comparison	$V_C = V_D$	$V_C \geq V_D$	$V_C \geq V_D$	$V_C = V_D$

To show that inequality (1.2.13a) holds for European calls, we consider the payoffs of the following two portfolios at expiry. Portfolio C contains λ units of call with strike price X_3 and $(1-\lambda)$ units of call with strike price X_1, and Portfolio D contains one call with strike price X_2. In Table 1.2, we list the payoffs of the two portfolios at expiry for all possible values of S_T.

Since $V_C \geq V_D$ for all possible values of S_T, Portfolio C is dominant over Portfolio D. Therefore, the present value of Portfolio C must be equal to or greater than that of Portfolio D; so this leads to inequality (1.2.13a). In the above argument, there is no factor involving τ, so the result also holds even when the calls in the two portfolios are allowed to be exercised prematurely. Hence, the convexity property also holds for American calls. By changing the call options in the above two portfolios to the corresponding put options, it can be shown by a similar argument that European and American put prices are also convex functions of the strike price.

Furthermore, by using the linear homogeneity property of the call and put option functions with respect to the asset price and strike price, one can show that the call and put prices (both European and American) are convex functions of the asset price (see Problem 1.7).

1.2.1 Effects of Dividend Payments

Now we examine the effects of dividends on the rational boundaries for option values. In the forthcoming discussion, we assume the size and payment date of the dividends to be known. One important result is that the early exercise of an American call option may become optimal if dividends are paid during the life of the option.

First, we consider the impact of dividends on the asset price. When an asset pays a certain amount of dividend, no arbitrage argument dictates that the asset price is expected to fall by the same amount (assuming there exist no other factors affecting the income proceeds, like taxation and transaction costs). Suppose the asset price falls by an amount less than the dividend, an arbitrageur can lock in a riskless profit by borrowing money to buy the asset right before the dividend date, selling the asset right after the dividend payment and returning the loan. The net gain to the arbitrageur is the amount that the dividend income exceeds the loss caused by the difference in the asset price in the buying and selling transactions. If the asset price falls by an amount greater than the dividend, then the above strategical transactions are reversed in order to catch the arbitrage profit.

Let $D_1, D_2, \cdots, D_n$ be the dividend amount paid at $\tau_1, \tau_2, \cdots, \tau_n$ periods from the current time. Let D denote the present value of all known discrete dividends paid between now and the expiration date. Assuming constant interest rate, we then have

$$D = D_1 e^{-r\tau_1} + D_2 e^{-r\tau_2} + \cdots + D_n e^{-r\tau_n},$$

where r is the riskless interest rate and $e^{-r\tau_1}, e^{-r\tau_2}, \cdots, e^{-r\tau_n}$ are the respective discount factors. We examine the impact of dividends on the lower bound on the European call value and the early exercise feature of an American call option, with dependence on the lumped dividend D. Similar to the two portfolios shown in Table 1.1, we modify Portfolio B to contain one unit of the underlying asset and a loan of D dollars (in the form of a portfolio of bonds with par value D_i and time to expiry τ_i, $i = 1, 2, \cdots, n$). At expiry, the value of Portfolio B will always become S_T since the loan of D will be paid back during the life of the option using the dividends received. One observes again $V_A \geq V_B$ at expiry so that the present value of Portfolio A must be at least as much as that of Portfolio B. Together with the nonnegativity property of option values, we obtain

$$c(S, \tau; X, D) \geq \max(S - XB(\tau) - D, 0). \tag{1.2.14}$$

This gives us the new lower bound on the price of a European call option on a dividend paying asset. Since the call price becomes lower due to the dividends of the underlying asset, it may be possible that the call price falls below the intrinsic value $S - X$ when the lumped dividend D is deep enough. Accordingly, the condition on D such that $c(S, \tau; X, D)$ may fall below the intrinsic value $S - X$ is given by

$$S - X > S - XB(\tau) - D \text{ or } D > X[1 - B(\tau)]. \tag{1.2.15}$$

If D does not satisfy the above condition, it is never optimal to exercise the American call prematurely. In addition to the necessary condition (1.2.15) on the size of D, the American call must be sufficiently deep in-the-money so that the chance of regret on early exercise is low (see Sect. 5.1). Since there will be an expected decline in asset price right after a discrete dividend payment, the optimal strategy is to exercise right before the dividend payment so as to capture the dividend paid by the asset. The behavior of the American call price right before and after the dividend dates are examined in detail in Sect. 5.1.

Unlike holding a call, the holder of a put option gains when the asset price drops after a discrete dividend is paid because put value is a decreasing function of the asset price. Using an argument similar to that above (considering two portfolios), the bounds for American and European puts can be shown as

$$P(S, \tau; X, D) \geq p(S, \tau; X, D) \geq \max(XB(\tau) + D - S, 0). \tag{1.2.16}$$

Even without dividend ($D = 0$), the lower bound $XB(\tau) - S$ may become less than the intrinsic value $X - S$ when the put is sufficiently deep in-the-money (corresponding to a low value for S). Since the holder of an American put option would not tolerate the value falling below the intrinsic value, the American put should be exercised prematurely. The presence of dividends makes the early exercise of an American put option less likely since the holder loses the future dividends when the asset is sold upon exercising the put. Using an argument similar to that used in (1.2.15), one can show that when $D \geq X[1 - B(\tau)]$, the American put should never be exercised prematurely. The effects of dividends on the early exercise policies of American puts are in general more complicated than those for American calls (see Sect. 5.1).

The underlying asset may incur a *cost of carry* for the holder, like the storage and spoilage costs for holding a physical commodity. The effect of the cost of carry on the early exercise policies of American options appears to be opposite to that of dividends received through holding the asset.

1.2.2 Put-Call Parity Relations

Put-call parity states the relation between the prices of a pair of call and put options. For a pair of European put and call options on the same underlying asset and with the same expiration date and strike price, we have

$$p = c - S + D + XB(\tau). \tag{1.2.17}$$

When the underlying asset is nondividend paying, we set $D = 0$.

The proof of the above put-call parity relation is quite straightforward. We consider the following two portfolios. The first portfolio involves long holding of a European call, a portfolio of bonds: τ_1-maturity discount bond with par $D_1, \cdots, \tau_n$-maturity discount bond with par D_n and τ-maturity discount bond with par X, and short selling of one unit of the asset. The second portfolio contains only one European put. The sum of the present values of the bonds in the first portfolio is

$$D_1 B(\tau_1) + \cdots + D_n B(\tau_n) + XB(\tau) = D + XB(\tau).$$

The bond par values are taken to match with the sizes of the dividends and they are used to compensate the dividends due to the short position of one unit of the asset. At expiry, both portfolios have the same value $\max(X - S_T, 0)$. Since both European options cannot be exercised prior to expiry, both portfolios have the same value throughout the life of the options. By equating the values of the two portfolios, we obtain the parity relation (1.2.17).

The above parity relation cannot be applied to a pair of American call and put options due to their early exercise feature. However, we can deduce the lower and upper bounds on the difference of the prices of American call and put options. First, we assume the underlying asset is nondividend paying. Since $P > p$ and $C = c$, we deduce from (1.2.17) (putting $D = 0$) that

$$C - P < S - XB(\tau),$$

giving the upper bound on $C - P$. Let us consider the following two portfolios: one contains a European call plus cash of amount X, and the other contains an American put together with one unit of underlying asset. The first portfolio can be shown to be dominant over the second portfolio, so we have

$$c + X > P + S.$$

Further, since $c = C$ when the asset does not pay dividends, the lower bound on $C - P$ is given by

$$S - X < C - P.$$

Combining the two bounds, the difference of the American call and put option values on a nondividend paying asset is bounded by

$$S - X < C - P < S - XB(\tau). \tag{1.2.18}$$

The right side inequality, $C - P < S - XB(\tau)$, also holds for options on a dividend paying asset since dividends decrease call value and increase put value. However, the left side inequality has to be modified as $S - D - X < C - P$ (see Problem 1.8). Combining the results, the difference of the American call and put option values on a dividend paying asset is bounded by

$$S - D - X < C - P < S - XB(\tau). \tag{1.2.19}$$

1.2.3 Foreign Currency Options

The above techniques of analysis are now extended to foreign currency options. Here, the underlying asset is a foreign currency and all prices are denominated in domestic currency. As an illustration, we take the domestic currency to be the U.S. dollar and the foreign currency to be the Japanese yen. In this case, the spot domestic currency price S of one unit of foreign currency refers to the spot value of one Japanese yen in U.S. dollars, say, ¥ 1 for U.S.$0.01. Now both domestic and foreign interest rates are involved. Let $B_f(\tau)$ denote the foreign currency price of a default-free zero coupon bond, which has unit par and time to maturity τ. Since the underlying asset, which is a foreign currency, earns the riskless foreign interest rate r_f continuously, it is analogous to an asset that pays continuous dividend yield. The rational boundaries for the European and American foreign currency option values have to be modified accordingly.

Lower and Upper Bounds on Foreign Currency Call and Put Values
First, we consider the lower bound on the value of a European foreign currency call. Consider the following two portfolios: Portfolio A contains the European foreign currency call with strike price X and a domestic discount bond with par value of X whose maturity date coincides with the expiration date of the call. Portfolio B contains a foreign discount bond with par value of unity in the foreign currency, which also matures on the expiration date of the call. Portfolio B is worth the foreign currency price of $B_f(\tau)$, so the domestic currency price of $SB_f(\tau)$. On expiry of the call, Portfolio B becomes one unit of foreign currency and this equals S_T in domestic currency. The value of Portfolio A equals $\max(S_T, X)$ in domestic currency, thus Portfolio A is dominant over Portfolio B. Together with the nonnegativity property of option value, we obtain

$$c \geq \max(SB_f(\tau) - XB(\tau), 0).$$

As mentioned earlier, premature exercise of the American call on a dividend paying asset may become optimal. Recall that a necessary (but not sufficient) condition for optimal early exercise is that the lower bound $SB_f(\tau) - XB(\tau)$ is less than the intrinsic value $S - X$. In the present context, the necessary condition is seen to be

$$SB_f(\tau) - XB(\tau) < S - X \quad \text{or} \quad S > X\frac{1 - B(\tau)}{1 - B_f(\tau)}. \tag{1.2.20}$$

When condition (1.2.20) is not satisfied, we then have $C > S - X$. The premature early exercise of the American foreign currency call would give $C = S - X$, resulting in a drop in value. Therefore, it is not optimal to exercise the American foreign currency call prematurely. In summary, the lower and upper bounds for the American and European foreign currency call values are given by

$$S \geq C \geq c \geq \max(SB_f(\tau) - XB(\tau), 0). \tag{1.2.21}$$

Using similar arguments, the necessary condition for the optimal early exercise of an American foreign currency put option is given by

$$S < X\frac{1 - B(\tau)}{1 - B_f(\tau)}. \tag{1.2.22}$$

The lower and upper bounds on the values of American and European foreign currency put options can be shown to be

$$X \geq P \geq p \geq \max(XB(\tau) - SB_f(\tau), 0). \tag{1.2.23}$$

The corresponding put-call parity relation for the European foreign currency put and call options is given by

$$p = c - SB_f(\tau) + XB(\tau), \tag{1.2.24}$$

and the bounds on the difference of the prices of American call and put options on a foreign currency are given by (see Problem 1.11)

$$SB_f(\tau) - X < C - P < S - XB(\tau). \tag{1.2.25}$$

In conclusion, we have deduced the rational boundaries for the option values of calls and puts and their put-call parity relations. The impact of the early exercise privilege and dividend payment on option values have also been analyzed. An important result is that it is never optimal to exercise prematurely an American call option on a nondividend paying asset. More comprehensive discussion of the analytic properties of option price functions can be found in the seminal paper by Merton (1973) and the review article by Smith (1976).

1.3 Forward and Futures Contracts

Recall that a forward contract is an agreement between two parties that the holder agrees to buy an asset from the writer at the delivery time T in the future for a predetermined delivery price K. Unlike an option contract where the holder pays the writer an up-front option premium, no up-front payment is involved when a forward contract is transacted. The delivery price of a forward is chosen so that the value of the forward contract to both parties is zero at the time when the contract is initiated. The *forward price* is defined as the delivery price which makes the initial value of the forward contract zero. The forward price in a new forward contract is liable to change due to the subsequent fluctuation of the price of the underlying asset while the delivery price of the already transacted forward contract is held fixed.

Suppose that on July 1 the forward price of silver with maturity date on October 31 is quoted at \$30. This means that \$30 is the price (paid upon delivery) at which the person in long (short) position of the forward contract agrees to buy (sell) the contracted amount and quality of silver on the maturity date. A week later (July 8), the quoted forward price of silver for October 31 delivery changes to a new value due to price fluctuation of silver during the week. Say, the forward price moves up to \$35. The forward contract entered on July 1 earlier now has positive *value* since the delivery price has been fixed at \$30 while the new forward price for the same maturity date has been increased to \$35. Imagine that while holding the earlier forward, the holder can short another forward on the same commodity and maturity date. The opposite positions of the two forward contracts will be exactly canceled off on the October 31 delivery date. The holder will pay \$30 to buy the asset but will receive \$35 from selling the asset. Hence, the holder will be secured to receive $\$35 - \$30 = \$5$ on the delivery date. Recall that the holder pays nothing on both July 1 and July 8 when the two forward contracts are transacted. Obviously, there is some value associated with the holding of the earlier forward contract. This value is related to the spot forward price and the fixed delivery price. While we have been using the terms "price" and "value" interchangeably for options, but "forward price" and "forward value" are different quantities for forward contracts.

1.3.1 Values and Prices of Forward Contracts

We would like to consider the pricing formulas for forward contracts under three separate cases of dividend behaviors of the underlying asset, namely, no dividend, known discrete dividends and known continuous dividend yields.

Nondividend Paying Asset

Let $f(S,\tau)$ and $F(S,\tau)$ denote, respectively, the value and the price of a forward contract with current asset value S and time to maturity τ, and let r denote the constant riskless interest rate. Consider a portfolio that contains one long forward contract and a bond with the same maturity date and par value as the delivery price. The bond price is $Ke^{-r\tau}$, where K is the delivery price at maturity. The other portfolio contains one unit of the underlying asset. At maturity, the par received from holding the bond could be used to pay for the purchase of one unit of the asset to honor the forward contract. Both portfolios become one unit of the asset at maturity. Assuming that the asset does not pay any dividend, by the principle of no arbitrage, both portfolios should have the same value at all times prior to maturity. It then follows that the value of the forward contract is given by

$$f = S - Ke^{-r\tau}. \tag{1.3.1}$$

Recall that we have defined the forward price to be the delivery price which makes the value of the forward contract zero. In (1.3.1), the value of K which makes $f = 0$ is given by $K = Se^{r\tau}$. The forward price is then $F = Se^{r\tau}$, which agrees with formula (1.1.1). Together with the put-call parity relation for a pair of European call and put options, we obtain

$$f = (F - K)e^{-r\tau} = c(S, \tau; K) - p(S, \tau; K), \tag{1.3.2}$$

where the strike prices of the call and put options are set equal to the delivery price of the forward contract. The put-call parity relation reveals that holding a call is equivalent to holding a put and a forward.

Discrete Dividend Paying Asset

Now, suppose the asset pays discrete dividends to the holder during the life of the forward contract. Let D denote the present value of all dividends paid by the asset within the life of the forward. To find the value of the forward contract, we modify the above second portfolio to contain one unit of the asset plus borrowing of D dollars. At maturity, the second portfolio again becomes worth one unit of the asset since the loan of D dollars will be repaid by the dividends received by holding the asset. Hence, the value of the forward contract on a discrete dividend paying asset is found to be

$$f = S - D - Ke^{-r\tau}.$$

By finding the value of K such that $f = 0$, we obtain the forward price to be given by

$$F = (S - D)e^{r\tau}. \tag{1.3.3}$$

Continuous Dividend Paying Asset

Next, suppose the asset pays a continuous dividend yield at the rate q. The dividend is paid continuously throughout the whole time period and the dividend amount over a differential time interval dt is $qS\,dt$, where S is the spot asset price. Under this dividend behavior, we choose the second portfolio to contain $e^{-q\tau}$ units of asset with all dividends being reinvested to acquire additional units of asset. At maturity, the second portfolio will be worth one unit of the asset since the number of units of asset can be considered to have the continuous compounded growth at the rate q. It follows from the equality of the values of the two portfolios that the value of the forward contract on a continuous dividend paying asset is

$$f = Se^{-q\tau} - Ke^{-r\tau},$$

and the corresponding forward price is

$$F = Se^{(r-q)\tau}. \tag{1.3.4}$$

Since an investor is not entitled to receive any dividends through holding a put, call or forward, the put-call parity relation (1.3.2) also holds for put, call and forward on assets that pay either discrete dividends or continuous dividend yield.

Interest Rate Parity Relation

When we consider forward contracts on foreign currencies, the value of the underlying asset S is the price in the domestic currency of one unit of the foreign currency. The foreign currency considered as an asset can earn interest at the foreign riskless rate r_f. This is equivalent to a continuous dividend yield at the rate r_f. Therefore, the delivery price of a forward contract on the domestic currency price of one unit of foreign currency is given by

$$F = Se^{(r-r_f)\tau}. \tag{1.3.5}$$

Equation (1.3.5) is called the *Interest Rate Parity Relation*.

Cost of Carry and Convenience Yield

For commodities like grain and livestock, there may be additional costs to hold the assets such as storage, insurance, spoilage, etc. In simple terms, these additional costs can be considered as negative dividends paid by the asset. Suppose we let U denote the present value of all additional costs that will be incurred during the life of the forward contract, then the forward price can be obtained by replacing $-D$ in (1.3.3) by U. The forward price is then given by

$$F = (S + U)e^{r\tau}. \tag{1.3.6}$$

If the additional holding costs incurred at any time is proportional to the price of the commodity, they can be considered as negative dividend yield. If u denotes the cost per annum as a proportion of the spot commodity price, then the forward price is

$$F = Se^{(r+u)\tau}, \tag{1.3.7}$$

which is obtained by replacing $-q$ in (1.3.4) by u.

We may interpret $r + u$ as the *cost of carry* that must be incurred to maintain the commodity inventory. The cost consists of two parts: one part is the cost of funds tied up in the asset which requires interest for borrowing and the other part is the holding costs due to storage, insurance, spoilage, etc. It is convenient to denote the cost of carry by b. When the underlying asset pays a continuous dividend yield at the rate q, then $b = r - q$. In general, the forward price is given by

$$F = Se^{b\tau}. \tag{1.3.8}$$

There may be some advantages to users who hold the commodity, like the avoidance of temporary shortages of supply and the ensurance of production process running. These holding advantages may be visualized as negative holding costs. Suppose the market forward price F is below the cost of ownership of the commodity

$Se^{(r+u)\tau}$, the difference gives a measure of the benefits realized from actual ownership. We define the convenience yield y (benefit per annum) as a proportion of the spot commodity price. In this way, y has the effect negative to that of u. By netting the costs and benefits, the forward price is then given by

$$F = Se^{(r+u-y)\tau}. \tag{1.3.9}$$

With the presence of convenience yield, F is seen to be less than $Se^{(r+u)\tau}$. This is due to the multiplicative factor $e^{-y\tau}$, whose magnitude is less than one.

1.3.2 Relation between Forward and Futures Prices

Forward contracts and futures are much alike, except that the former are traded over-the-counter and the latter are traded in exchanges. Since the exchanges would like to organize trading such that contract defaults are minimized, an investor who buy a futures in an exchange is requested to deposit funds in a *margin account* to safeguard against the possibility of default (the futures agreement is not honored at maturity). At the end of each trading day, the futures holder will pay to or receive from the writer the full amount of the change in the futures price from the previous day through the margin account. This process is called *marking to market the account*. Therefore, the payment required on the maturity date to buy the underlying asset is simply the spot price at that time. However, for a forward contract traded outside the exchanges, no money changes hands initially or during the life-time of the contract. Cash transactions occur only on the maturity date. Such difference in the payment schedules may lead to differences in the prices of a forward contract and a futures on the same underlying asset and date of maturity. This is attributed to the possibility of different interest rates applied on the intermediate payments. In Sect. 8.1, we show how the forward price and futures price differ when the interest rate is stochastic and exhibiting positive correlation with the underlying asset price process.

Here, we present the argument to illustrate the equality of the two prices when the interest rate is constant. First, consider one forward contract and one futures which both last for n days. Let F_i and G_i denote the forward price and the futures price at the end of the ith day, respectively, $i = 0, 1, \cdots, n$. We would like to show that $F_0 = G_0$. Let S_n denote the asset price at maturity. Let the constant interest rate per day be δ. Suppose we initiate the long position of one unit of the futures on day 0. The gain/loss of the futures on the ith day is $(G_i - G_{i-1})$ and this amount grows to the dollar value $(G_i - G_{i-1})\, e^{\delta(n-i)}$ at maturity, which is the end of the nth day $(n \geq i)$. Therefore, the value of this *one* long futures position at the end of the nth day is the summation of $(G_i - G_{i-1})\, e^{\delta(n-i)}$, where i runs from 1 to n. The sum can be expressed as

$$\sum_{i=1}^{n} (G_i - G_{i-1})\, e^{\delta(n-i)}.$$

The summation of gain/loss of each day reflects the daily settlement nature of a futures.

Instead of holding one unit of futures throughout the whole period, the investor now keeps changing the amount of futures to be held on each day. Suppose he holds α_i units at the end of the $(i-1)$th day, $i = 1, 2, \cdots, n$, α_i to be determined. Since there is no cost incurred when a futures is transacted, the investor's portfolio value at the end of the nth day becomes

$$\sum_{i=1}^{n} \alpha_i (G_i - G_{i-1})\, e^{\delta(n-i)}.$$

On the other hand, since the holder of one unit of the forward contract initiated on day 0 can purchase the underlying asset which is worth S_n using F_0 dollars at maturity, the value of the long position of one forward at maturity is $S_n - F_0$. Now, we consider the following two portfolios:

Portfolio A : long position of a bond with par value F_0 maturing on the nth day
long position of one unit of forward contract

Portfolio B : long position of a bond with par value G_0 maturing on the nth day
long position of $e^{-\delta(n-i)}$ units of futures held at the end of the $(i-1)$th day, $i = 1, 2, \cdots n$.

At maturity (end of the nth day), the values of the bond and the forward contract in Portfolio A become F_0 and $S_n - F_0$, respectively, so that the total value of the portfolio is S_n. For Portfolio B, the bond value is G_0 at maturity. The value of the long position of the futures (number of units of futures held is kept changing on each day) is obtained by setting $\alpha_i = e^{-\delta(n-i)}$. This gives

$$\sum_{i=1}^{n} e^{-\delta(n-i)}\, (G_i - G_{i-1})\, e^{\delta(n-i)} = \sum_{i=1}^{n} (G_i - G_{i-1}) = G_n - G_0.$$

Hence, the total value of Portfolio B at maturity is $G_0 + (G_n - G_0) = G_n$. Since the futures price must be equal to the asset price S_n at maturity, we have $G_n = S_n$. The two portfolios have the same value at maturity, while Portfolio A and Portfolio B require an initial investment of $F_0 e^{-\delta n}$ and $G_0 e^{-\delta n}$ dollars, respectively. In the absence of arbitrage opportunities, the initial values of the two portfolios must be the same. We then obtain $F_0 = G_0$, that is, the current forward and futures prices are equal.

1.4 Swap Contracts

A swap is a financial contract between two counterparties who agree to exchange one cash flow stream for another according to some prearranged rules. Two important types of swaps are considered in this section: interest rate swaps and currency swaps. Interest rate swaps have the effect of transforming a floating-rate loan into a fixed-rate loan or vice versa. A currency swap can be used to transform a loan in one currency into a loan in another currency. One may regard a swap as a *package of forward contracts*. It would be interesting to examine how two firms may benefit by entering into a swap and the financial rationales that determine the prearranged rules for the exchange of cash flows.

1.4.1 Interest Rate Swaps

The most common form of an interest rate swap is a fixed-for-floating swap, where a series of payments, calculated by applying a fixed rate of interest to a notional principal amount, are exchanged for a stream of payments calculated using a floating rate of interest. The exchange of cash flows in net amount occurs on designated swap dates during the life of the swap contract. In the simplest form, all payments are made in the same currency. The principal amount is said to be notional since no exchange of principal will occur and the principal is used only to compute the actual cash amounts to be exchanged periodically on the swap dates.

The floating rate in an interest rate swap is chosen from one of the money market rates, like the London interbank offer rate (LIBOR), Treasury bill rate, federal funds rate, etc. Among them, the most common choice is the LIBOR. It is the interest rate at which prime banks offer to pay on Eurodollar deposits available to other prime banks for a given maturity. A Eurodollar is a U.S. dollar deposited in a U.S. or foreign bank outside the United States. The LIBOR comes with different maturities, say, one-month LIBOR is the rate offered on one-month deposits, etc. In the floating-for-floating interest rate swaps, two different reference floating rates are used to calculate the exchange payments.

As an example, consider a five-year fixed-for-floating interest rate swap. The fixed rate payer agrees to pay 8% per year (quoted with semi-annual compounding) to the counterparty while the floating rate payer agrees to pay in return six-month LIBOR. Assume that payments are exchanged every six months throughout the life of the swap and the notional amount is $10 million. This means for every six months, the fixed rate payer pays out fixed rate interest of amount $\$10$ million $\times$ $8\% \div 2 =$ $\$0.4$ million but receives floating rate interest of amount that equals $10 million times half of the six-month LIBOR prevailing six months before the payment. For example, suppose April 1, 2008, is the initiation date of the swap and the prevailing six-month LIBOR on that date is 6.2%. The floating rate interest payment on the first swap date (scheduled on October 1, 2008) will be $\$10$ million $\times$ $6.2\% \div 2 = \$0.31$ million. In this way, the fixed rate payer will pay a net amount of ($\$0.4 - \$0.31$) million $= \$0.09$ million to the floating rate payer on the first swap date.

The interest payments paid by the floating rate payer resemble those of a floating rate loan, where the interest rate is set at the beginning of the period over which the rate will be applied and the interest amount is paid at the end of the period. This class of swaps is known as plain vanilla interest rate swaps. Assuming no default of either swap counterparty, a plain vanilla interest rate swap can be characterized as the difference between a fixed rate bond and a floating rate bond. This property naturally leads to an efficient valuation approach for plain vanilla interest rate swaps.

Valuation of Plain Vanilla Interest Rate Swaps

Consider the fixed rate payer of the above five-year fixed-for-floating plain vanilla interest rate swap. The fixed rate payer will receive floating rate interest payments semi-annually according to the six-month LIBOR. This cash stream of interest payments will be identical to those generated by a floating rate bond having the same maturity, par value and reference interest rate as those of the swap. Unlike the holder of the floating rate bond, the fixed rate payer will not receive the notional principal on the maturity date of the swap. On the other hand, he or she will pay out, fixed rate interest rate payment semi-annually, like the issuer of a fixed rate bond with the same fixed interest rate, maturity and par value as those of the swap.

We observe that the position of the fixed rate payer of the plain vanilla interest swap can be replicated by long holding of the floating rate bond underlying the swap and short holding of the fixed rate bond underlying the swap. The two underlying bonds have the same maturity, par value and corresponding reference interest rates as those of the swap. Hence, the value of the swap to the fixed rate payer is the value of the underlying floating rate bond minus the value of the underlying fixed rate bond. Since the position of the floating rate payer of the fixed-for-floating swap is exactly opposite to that of the fixed rate payer, so the value of the swap to the floating rate payer is negative that of the fixed rate payer. In summary, we have

$$
\begin{aligned}
V_{fix} &= B_{f\ell} - B_{fix} \\
V_{f\ell} &= B_{fix} - B_{f\ell},
\end{aligned}
$$

where V_{fix} and $V_{f\ell}$ denote the value of the interest rate swap to the fixed rate payer and floating rate payer, respectively; B_{fix} and $B_{f\ell}$ denote the value of the underlying fixed rate bond and floating rate bond, respectively.

Uses of Interest Rate Swaps in Asset and Liability Management

Financial institutions often use an interest rate swap to alter the cash flow characteristics of their assets or liabilities to meet certain management goals or lock in a spread. As an example, suppose a bank is holding an asset (say, a loan or a bond) that earns semi-annually a fixed rate of interest of 8% (annual rate). To fund the holding of this asset, the bank issues six-month certificates of deposit that pay six-month LIBOR plus 60 basis points (1 basis point = 0.01%). How can the bank lock in a spread over the cost of its funds? This can be achieved by converting the fixed rate interests generated from the asset into floating rate interest incomes. This type of transaction is called an *asset swap*, which consists of a simultaneous asset purchase and entry into an interest rate swap. Suppose that the following interest rate swap is available to the bank:

> Every six months the bank pays 7% (annual rate) and receives LIBOR.

By entering into this interest rate swap, for every six months, the bank receives $8\% - 7\% = 1\%$ of net fixed rate interest payments and pays (LIBOR + 60 bps) − LIBOR = 0.6% of net floating rate interest payments. In this way, the bank can lock in a spread of 40 basis points over the funding costs.

On the other hand, suppose the bank has issued a fixed rate loan that pays every six months at the annual rate of 7%. Through a *liability swap*, the bank can transform this fixed rate liability into a floating rate liability by serving as the floating rate payer in an interest rate swap. Say, for every six months, the bank pays LIBOR + 50 basis points and receives 7.2% (annual rate). Now, the bank then applies the loan capital to purchase a floating rate bond so that the floating rate coupons received may be used to cover the floating rate interest payments under the interest rate swap. Through simple calculations, if the floating coupon rate is higher than LIBOR + 30 basis points, then the bank again locks in a positive spread on funding costs.

1.4.2 Currency Swaps

A currency swap is used to transform a loan in one currency into a loan of another currency. Suppose a U.S. company wishes to borrow British sterling to finance a project in the United Kingdom. On the other hand, a British company wants to raise U.S. dollars. Both companies would suffer comparative disadvantages in raising foreign capitals as compared to raising domestic capitals in their own country. As an example, we consider the following fixed borrowing rates for the two companies on the two currencies.

The above borrowing rates indicate that the U.S. company has better creditworthiness so that it enjoys lower borrowing rates at both currencies as compared to the UK company. Note that the difference between the borrowing rates in U.S. dollars is 2% while that in UK sterling is only 1.2%. With a spread of $2\% - 1.2\% = 0.8\%$ on the borrowing rates in the two currencies, it seems possible to construct a currency swap so that both companies receive the desired types of capital and take advantage of the lower borrowing rates on their domestic currencies.

Let the current exchange rate be £1 = \$1.4, and assume the notional principals to be £1 million and \$1.4 million. First, both companies borrow the principals from their domestic borrowers in their respective currencies. That is, the U.S. company enters into a loan of \$1.4 million at the borrowing rate of 9.0% while the UK company enters into a loan of £1 million at the borrowing rate of 13.6%. Next, a currency swap is structured as follows. At initiation of the swap, the U.S. company exchanges the capital of \$1.4 million for £1 million with the UK company. In this way, both companies obtain the types of capital that they desire. Within the swap period, the

Table 1.3. Borrowing at fixed rates for the U.S. and UK companies

	U.S. dollars	UK sterling
U.S. company	9.0%	12.4%
UK company	11.0%	13.6%

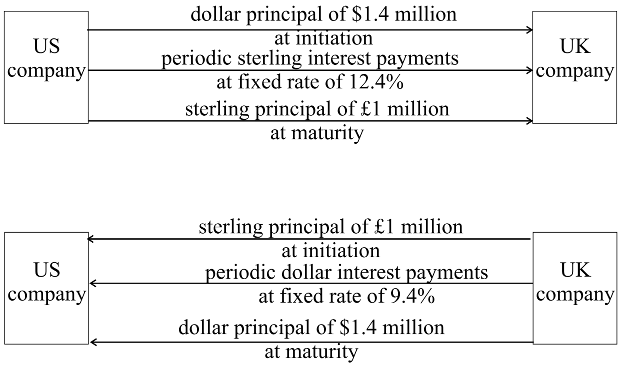


Fig. 1.5. Cash flow streams between the two counterparties in a currency swap.

US company pays periodically fixed sterling interest rate of 12.4% to the UK company, and in return, receives fixed dollar interest rate of 9.4% from the UK company. At maturity of the currency swap, the U.S. company returns the loan capital of £1 million to the UK company and receives $1.4 million back from the UK company. Both companies can pay back the loans to their respective domestic borrowers. The cash flow streams between the two companies are summarized in Fig. 1.5.

What would be the gains to both counterparties in the above currency swap? The U.S. company pays the same fixed sterling interest rate of 12.4%, but gains $9.4\% - 9.0\% = 0.4\%$ on the dollar interest rate. This is because it pays 9.0% to the domestic borrower but receives 9.4% from the UK company through the currency swap. On the other hand, the UK company pays only 9.4% on the dollar interest rate instead of 11.0%. This represents a gain on the spread of $11.0\% - 9.4\% = 1.6\%$ on the dollar interest rate, though it loses $13.6\% - 12.4\% = 1.2\%$ on the spread in the sterling interest rate. Note that the net gains and losses on the interest payments are in different currencies, so the parties in a currency swap face the exchange rate exposure.

1.5 Problems

1.1 How can we construct the portfolio of a butterfly spread that involves put options with different strike prices but the same date of expiration and on the same underlying asset? Draw the corresponding profit diagram of the spread at expiry.

1.2 A *strip* is a portfolio created by buying one call and writing two puts with the same strike price and expiration date. A *strap* is similar to a strip except it involves long holding of two calls and short selling of one put instead. Sketch the terminal profit diagrams for the strip and the strap and comment on their roles in monitoring risk exposure. How are they compared to a bottom straddle?

1.3 A *strangle* is a trading strategy where an investor buys a call and a put with the same expiration date but different strike prices. The strike price of the call may be higher or lower than that of the put (when the strike prices are equal, it reduces to a straddle). Sketch the terminal profit diagrams for both cases and discuss the characteristics of the payoffs at expiry.

1.4 A *box spread* is a combination of a bullish call spread with strike prices X_1 and X_2 and a bearish put spread with the same strike price. All four options are on the same underlying asset and have the same date of expiration. Discuss the characteristics of a box spread.

1.5 Suppose the strike prices X_1 and X_2 satisfy $X_2 > X_1$, show that for European calls on a nondividend paying asset, the difference in the call values satisfies

$$-B(\tau)(X_2 - X_1) \leq c(S, \tau; X_2) - c(S, \tau; X_1) \leq 0,$$

where $B(\tau)$ is the value of a pure discount bond with par value of unity and time to maturity τ. Furthermore, deduce that

$$-B(\tau) \leq \frac{\partial c}{\partial X}(S, \tau; X) \leq 0.$$

In other words, suppose the call price can be expressed as a differentiable function of the strike price, then the derivative must be nonpositive and not greater in absolute value than the price of a pure discount bond of the same maturity. Do the above results also hold for European/American calls on a dividend paying asset?

1.6 Show that a portfolio of holding various single-asset options with the same date of expiration is worth at least as much as a single option on the portfolio of the same number of units of each of the underlying assets. The single option is called a *basket option*. In mathematical representation, say for European call options, we have

$$\sum_{i=1}^{N} n_i c_i(S_i, \tau; X_i) \geq c\left(\sum_{i=1}^{N} n_i S_i, \tau; \sum_{i=1}^{N} n_i X_i\right), \quad n_i > 0,$$

where N is the total number of options in the portfolio, and n_i is the number of units of asset i in the basket.

1.7 Show that the put prices (European and American) are convex functions of the asset price, that is,

$$p(\lambda S_1 + (1 - \lambda)S_2, X) \leq \lambda p(S_1, X) + (1 - \lambda)p(S_2, X), \quad 0 \leq \lambda \leq 1,$$

where S_1 and S_2 denote the asset prices and X denotes the strike price.

Hint: Let $S_1 = h_1 X$ and $S_2 = h_2 X$, and note that the put price function is homogeneous of degree one in the asset price and the strike price, the above inequality can be expressed as

$$[\lambda h_1 + (1-\lambda)h_2]p\left(X, \frac{X}{\lambda h_1 + (1-\lambda)h_2}\right)$$
$$\leq \lambda h_1 p\left(X, \frac{X}{h_1}\right) + (1-\lambda)h_2 p\left(X, \frac{X}{h_2}\right).$$

Apply the property that the put prices are convex functions of the strike price.

1.8 Consider the following two portfolios:

Portfolio A: One European call option plus X dollars of money market account.

Portfolio B: One American put option, one unit of the underlying asset and borrowing of loan amount D. The loan is in the form of a portfolio of bonds whose par values and dates of maturity match with the sizes and dates of the discrete dividends.

Assume the underlying asset pays dividends and D denotes the present value of the dividends paid by the underlying asset during the life of the option. Show that if the American put is not exercised early, Portfolio B is worth $\max(S_T, X)$, which is less than the value of Portfolio A. Even when the American put is exercised prior to expiry, show that Portfolio A is always worth more than Portfolio B at the moment of exercise. Hence, deduce that

$$S - D - X < C - P.$$

Hint: $c < C$ for calls on a dividend paying asset and the loan (bond) value in Portfolio A grows with time.

1.9 Deduce from the put-call parity relation that the price of a European put on a nondividend paying asset is bounded above by

$$p \leq XB(\tau).$$

Then deduce that the value of a perpetual European put option is zero. When does equality hold in the above inequality?

1.10 Consider a European call option on a foreign currency. Show that

$$c(S, \tau) \sim SB_f(\tau) - XB(\tau) \quad \text{as } S \to \infty.$$

Give a financial interpretation of the result. Deduce the conditions under which the value of a shorter-lived European foreign currency call option is worth more than that of the longer-lived counterpart.

Hint: Use the put-call parity relation (1.2.24). At exceedingly high exchange rates, the European call is almost sure to be in-the-money at expiry.

1.11 Show that the lower and upper bounds on the difference between the prices of the American call and put options on a foreign currency are given by

$$SB_f(\tau) - X < C - P < S - XB(\tau),$$

where $B_f(\tau)$ and $B(\tau)$ are bond prices in the foreign and domestic currencies, respectively, both with par value of unity in the respective currency and time to maturity τ, S is the spot domestic currency price of one unit of foreign currency.
Hint: To show the left inequality, consider the values of the following two portfolios: the first one contains a European currency call option plus X dollars of domestic currency, the second portfolio contains an American currency put option plus $B_f(\tau)$ units of foreign currency. To show the right inequality, we choose the first portfolio to contain an American currency call option plus $XB(\tau)$ dollars of domestic currency, and the second portfolio to contain a European currency put option plus one unit of the foreign currency.

1.12 Suppose the strike price is growing at the riskless interest rate, show that the price of an American put option is the same as that of the corresponding European counterpart.
Hint: Show that the early exercise privilege of the American put is rendered useless.

1.13 Consider a forward contract whose underlying asset has a holding cost of c_j paid at time t_j, $j = 1, 2, \cdots, M-1$, where time t_M is taken to be the maturity date of the forward. For notational simplicity, we take the initiation date of the swap contract to be time t_0. Assume that the asset can be sold short. Let S denote the spot price of the asset at the initiation date, and we use d_j to denote the discount factor at time t_j for cash received on the expiration date. Show that the forward price F of this forward contract is given by

$$F = \frac{S}{d_0} + \sum_{j=1}^{M-1} \frac{c_j}{d_j}.$$

1.14 Consider a one-year forward contract whose underlying asset is a coupon paying bond with maturity date beyond the forward's expiration date. Assume the bond pays coupon semi-annually at the coupon rate of 8%, and the face value of the bond is \$100 (that is, each coupon payment is \$4). The current market price of the bond is \$94.6, and the previous coupon has just been paid. Taking the riskless interest rate to be at the constant value of 10% per annum, find the forward price of this bond forward.
Hint: The coupon payments may be considered as negative costs of carry.

1.15 Consider an interest rate swap of notional principal \$1 million and remaining life of nine months, the terms of the swap specify that six-month LIBOR is exchanged for the fixed rate of 10% per annum (quoted with semi-annual compounding). The market prices of unit par zero coupon bonds with maturity dates three months and nine months from now are \$0.972 and \$0.918, respectively, while the market price of unit par floating rate bond with maturity date three months from now is \$0.992. Find the value of the interest rate swap to the fixed-rate payer, assuming no default risk of the swap counterparty.

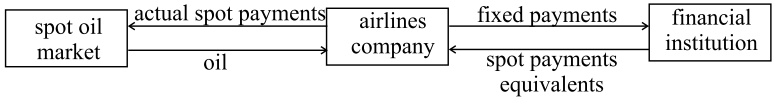
1.16 A financial institution X has entered into a five-year currency swap with another institution Y. The swap specifies that X receives fixed interest rate at 4% per annum in euros and pays fixed interest rate at 6% per annum in U.S. dollars. The principal amounts are 10 million U.S. dollars and 13 million euros, and interest payments are exchanged semi-annually. Suppose that Y defaults at the end of Year 3 after the initiation of the swap. Find the replacement cost to the counterparty X. Assume that the exchange rate at the time of default is \$1.32 per euro and the prevailing interest rates for all maturities for U.S. dollars and euros are 5.5% and 3.2%, respectively.

1.17 Suppose two financial institutions X and Y are faced with the following borrowing rates

	X	Y
U.S. dollars floating rate	LIBOR + 2.5%	LIBOR + 4.0%
British sterling fixed rate	4.0%	5.0%

Suppose X wants to borrow British sterling at a fixed rate and Y wants to borrow U.S. dollars at a floating rate. How can a currency swape be arranged that benefits both parties.

1.18 Consider an airlines company that has to purchase oil regularly (say, every three months) for its operations. To avoid the fluctuation of oil prices on the spot market, the company may wish to enter into a *commodity swap* with a financial institution. The following schematic diagram shows the flows of payment in the commodity swap:



Under the terms of the commodity swap, the airline company receives spot price for a certain number units of oil at each swap date while paying a fixed amount K per unit. Let $t_i, i = 1, 2, \cdots, M$, denote the swap dates and d_i be the discount factor at the swap initiation date for cash received at t_i. Let F_i denote the forward price of one unit of oil to be received at time t_i, and K be the fixed payment per unit paid by the airline company to the swap counterparty. Suppose K is chosen such that the initial value of the commodity swap is zero, show that

$$\frac{K = \sum_{i=1}^{M} d_i F_i}{\sum_{i=1}^{M} d_i \,.}$$

That is, the fixed rate is a weighted average of the prices of the forward contracts maturing on the swap dates with the corresponding discount factors as weights.

1.19 This problem examines the role of a *financial intermediary* in arranging two separate interest rate swaps with two companies that would like to transform a floating rate loan into a fixed rate loan and vice versa. Consider the following situation:

> Company A aims at transforming a fixed rate loan paying 6.2% per annum into a floating rate loan paying LIBOR + 0.2%.
> Company B aims at transforming a floating rate loan paying LIBOR + 2.2% into a fixed rate loan paying 8.4% per annum.

Instead of having these two companies getting in touch directly to arrange an interest rate swap, how can a financial intermediary design separate interest swaps with the two companies and secure a profit on the spread of the borrowing rates?