2

Mathematical Simulation of Particle-Laden Gas Flows

2.1 Preliminary Remarks

In studying the processes of motion of disperse impurity in the form of solid particles and its inverse effect on the characteristics of turbulence of the carrier continuum, an important part is played by methods of mathematical simulation. Numerous modes of flow of gas suspension, an attempt at classifying which is described in Sect. 1.5, served a basis for the development of a large number of mathematical models of such flows. In constructing models of heterogeneous flows of the most diverse classes, investigators always face an alternative. On the one hand, it is necessary to take into account as many as possible physical processes occurring in heterogeneous flows, which often brings about an undue complication of mathematical formalization of the phenomena being treated. On the other hand, the detailing of a large number of processes the information about each one of these processes is not always indisputable may result in a lower reliability of the model being developed.

It is the objective of this chapter to describe the presently available methods of mathematical simulation of heterogeneous flows. The models of heterogeneous flows of the main types and the characteristic features of simulation of turbulent particle-laden flows of different classes are treated in Sect. 2.2. Section 2.3 is devoted to the description of the possibilities of studying the behavior of solid particles in a turbulent gas flow using two different approaches, namely, stochastic Lagrangian approach and Eulerian continuum approach. The characteristic features of mathematical simulation of gas flow in view of the inverse effect of particles on the flow characteristics are treated in Sect. 2.4.

2.2 Special Features of Simulation of Heterogeneous Flows of Different Types

The wide range of presently existing mathematical models of heterogeneous flows may be divided into two major classes (types). The models of the first class describe the motion of the carrier gas phase and the motion of a plurality of suspended particles and are based on the Eulerian continuum approach. The models of the other type are those based on the Eulerian–Lagrangian description of motion of heterogeneous medium, namely, the equations of motion of the gas phase are solved in the Eulerian formulation, while the motion of particles is described by Lagrangian equations which are integrated along their trajectories.

It is clear that attempts at making an adequate description of the entire diversity of heterogeneous flows using models of both types mentioned earlier are hardly justified. Therefore, for certain classes of flows (see Sect. 1.5), which are first of all characterized by the concentration of disperse impurity and its inertia (Stokes number), models of one or the other type must be preferred.

We will consider briefly the advantages and limitations of the Eulerian (two-fluid) and Eulerian–Lagrangian models of description of gas–solid flows [28, 52, 58].

The advantage of two-fluid models is the use of like equations for the description of the gas and dispersed phases. This enables one to utilize the rich experience of simulation of single-phase turbulent flows and apply the same numerical methods of solving the entire set of equations. The disadvantages of such models include some "loss" of information about the motion of individual particles, as well as the difficulties in the formulation of boundary conditions for the dispersed phase on surfaces which bound the flow.

We will now turn to the Eulerian–Lagrangian models. The advantage of these models consists in the possibility of obtaining detailed statistical information about the motion of individual particles as a result of integration of equations of motion (heat transfer) of particles in a known (pre-calculated) velocity (temperature) field of carrier gas. However, as the concentration of the dispersed phase increases, difficulties arise which are associated with the use of the Eulerian–Lagrangian models. Two aspects may be identified in this respect. First, the concentration increase leads to the inverse effect of particles on the carrier gas parameters, and the calculations need to be performed in several iterations; as a result, the computation procedure is complicated. Second, the concentration increase causes a rise of the probability of particle collisions with one another, which brings about entanglement of their trajectories. As the particle size decreases, the use of trajectory methods for the calculation of particle motion is also complicated. This is associated with the fact that it is necessary to take into account the interaction between the particles and turbulent eddies of ever smaller dimensions in order to obtain correct information about the averaged characteristics of the dispersed phase. The latter fact further complicates the computations.

Flows of two extreme classes exist (see Sect. 1.5), namely, flows with particles of extremely low inertia (the case of equilibrium flow) and flows with an extremely low concentration of the dispersed phase (the mode with single particles, in which their presence has no effect on the carrier gas flow). Simplified mathematical models may be employed for flows of these classes, namely, a one-velocity one-temperature diffusion model (Eulerian approach) for low-inertia particles and a single-particle approximation (Lagrangian approach) for a low-concentration flow.

In the case of increasing concentration and inertia of particles, it is not a simple problem to choose between two types of models of heterogeneous flows. Therefore, the types of heterogeneous flows which are most complex from the standpoint of mathematical simulation are flows of "intermediate" classes. According to the classification given earlier (see Sect. 1.5), such flows are nonequilibrium flows and flows with large particles at moderate values of volume concentration of the dispersed phase, when the presence of particles affects all (without exception) characteristics of carrier gas.

Treating the hydrodynamics of flows of a special class such as the flow past a stationary "frozen" particle (see Sect. 1.5), a peculiar analog of which is the flow of a single-phase liquid (gas) past tube bundles, falls outside of the scope of this monograph.

When one tries to use two-fluid models, the question arises first of all whether it is possible to use the methods of continuum dynamics to describe the motion of a plurality of particles [39]. A continual description for an ensemble of particles is possible in the case where a geometric scale may be indicated which, on the one hand, is negligible compared to the scale of variation of the flow parameters and, on the other hand, is large enough to contain a significant number of particles which permits a correct determination of their averaged parameters [10]. We will make the simplest estimates which enable one to determine such a geometric scale for a heterogeneous flow with particles of diameter d_{p} and volume concentration Φ. For this purpose, we will treat an element of flow in the form of a cube with edge a, which contains N_{p} particles. The expression for the volume concentration of particles will be written as:

$$\Phi = \frac{\pi d_{\mathrm{p}}^3 N_{\mathrm{p}}}{6a^3}. \tag{2.1}$$

We use (2.1) to find the formula for the ratio of the cube edge to the particle diameter,

$$\frac{a}{d_{\mathrm{p}}} = \sqrt[3]{\frac{\pi N_{\mathrm{p}}}{6\Phi}}. \tag{2.2}$$

The dependence of the relative dimension of the edge of a cube containing N_{p} particles on the value of the volume concentration of the dispersed phase, obtained by relation (2.2), is given in Fig. 2.1. The calculations were performed for two values of the number of particles in the flow volume of interest to us,

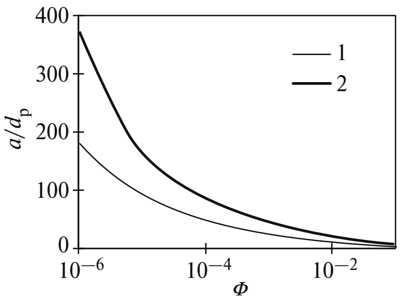

Fig. 2.1. The relative dimension of the edge of a cube as a function of the value of the volume concentration of particles: (1) $N_p = 10$, (2) $N_p = 100$

$N_p = 10$ and 100. Obviously, the relative fluctuation of distributed density of the dispersed phase in the volume being treated increases with decreasing number of particles and reaches several percent at $N_p = 100$. If the foregoing error in determining the particle density is inadequate, a plurality of particles cannot be regarded as a continuum on scales comparable to a or lower. In this case, the motion of particles cannot be described by the methods of continuum dynamics.

The data given in Fig. 2.1 indicate that the scale a increases with decreasing volume concentration of particles and with increasing particle size. For example, for particles 50 µm in diameter ($\Phi = 10^{-3}$), the scale is $a \approx 1.9$ mm, and for particles 100 µm in diameter ($\Phi = 10^{-4}$) – $a \approx 8$ mm.

Therefore, the general tendency is as follows: as the concentration of particles increases and their inertia decreases, the Eulerian continuum approach turns out to be preferable for use in describing the dynamics of disperse impurity.

2.3 Description of Motion of Solid Particles Suspended in Turbulent Flow

The motion of particles suspended in a turbulent gas flow may be calculated both within the frame of stochastic Lagrangian approach and using Eulerian continuum approach.

2.3.1 Lagrangian Approach

The study of regularities of the behavior of particles in the known velocity field of the carrier phase is of interest per se when calculating weakly dusty flows without the inverse effect of the dispersed phase on the characteristics of gas and may also be an integral part of the process of construction of complex mathematical models for the description of heterogeneous flows of most diverse classes.

The Lagrangian equation of instantaneous motion of a single solid particle in a turbulent gas flow has the form:

$$\rho_{\mathrm{p}} \frac{\pi d_{\mathrm{p}}^3}{6} \frac{\mathrm{d}v_i}{\mathrm{d}\tau} = \sum_i f_i(r_{\mathrm{p}}, \tau), \tag{2.3}$$

where $f_i(r_{\mathrm{p}}, \tau)$ denotes the external forces acting on the particle, and r_{p} is the particle coordinate.

The main force factors affecting the motion of the dispersed phase will be treated later.

Aerodynamic Drag Force

This force arises due to the difference between the velocity of gas and the velocity of a particle moving in this gas (see Fig. 2.2). The effect of the aerodynamic drag force causes the particle acceleration if $U > V$ and, on the contrary, the deceleration in the case of $U < V$. The expression for aerodynamic force has the form:

$$\overrightarrow{F}_{\mathrm{A}} = C_{\mathrm{D}} \rho \frac{\pi d_{\mathrm{p}}^2}{4} \frac{|\overrightarrow{U} - \overrightarrow{V}|(\overrightarrow{U} - \overrightarrow{V})}{2}, \tag{2.4}$$

where the particle drag coefficient in the case of incompressible flow is a function of the Reynolds number, i.e., $C_{\mathrm{D}} = C_{\mathrm{D}}(Re_{\mathrm{p}})$. The graph of this dependence is often referred to as standard drag curve. Numerous formulas are available in the literature, which approximate this curve for different ranges of the Reynolds number [39,46]. For low values of the Reynolds number ($Re_{\mathrm{p}} < 1$), the well-known Stokes formula is valid,

$$C_{\mathrm{D}} = \frac{24}{Re_{\mathrm{p}}}, \quad Re_{\mathrm{p}} = \frac{|\overrightarrow{U} - \overrightarrow{V}| d_{\mathrm{p}}}{\nu}. \tag{2.5}$$

The equation of averaged motion of a Stokesian particle has the form

$$\frac{\mathrm{d}V_i}{\mathrm{d}\tau} = \frac{U_i - V_i}{\tau_{\mathrm{p}0}} \tag{2.6}$$

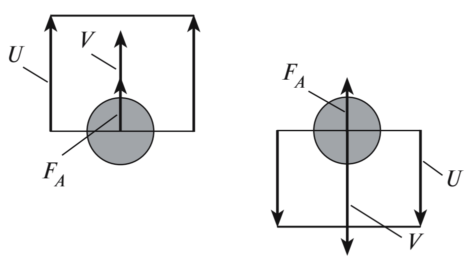

Fig. 2.2. A scheme of particle motion under the effect of the aerodynamic drag force

where τ_{p0} is the time of dynamic relaxation of the Stokesian particle (see Sect. 1.4).

As the Reynolds number increases ($Re_p \geq 1$), the value of the particle drag coefficient deviates from the Stokes law toward higher values, while the particle relaxation time, on the contrary, decreases. For taking this fact into account, the correction function $C = C(Re_p)$ is introduced. The values of this function are given in Sect. 1.4. Expression (2.6) takes the following form for a non-Stokesian particle:

$$\frac{\mathrm{d}V_i}{\mathrm{d}\tau} = \frac{U_i - V_i}{\tau_p}, \tag{2.7}$$

where $\tau_p = \tau_{p0}/C$.

Equation (2.7) of averaged motion of a non-Stokesian particle is very approximate, because it does not include the effect of turbulent fluctuations of the carrier phase.

Note that the standard curve describes the drag of single smooth spherical particles during their uniform motion in a laminar flow of liquid (gas). The problems associated with the inclusion of the effect made on the drag of the dispersed phase by the asphericity of particles, by the state of their surface, by the degree of flow turbulence, by the concentration and geometric constraint of motion, and by other factors, were treated in [39, 46].

Gravity Force

Along with the aerodynamic drag force, this force is one of the most important force factors defining the dynamics of particles. The expression for gravity force has the form:

$$\overrightarrow{F}_g = \rho_p \frac{\pi d_p^3}{6} \overrightarrow{g}. \tag{2.8}$$

The effect of gravity force on particle motion will be significant, and its inclusion is necessary in the case where the free-fall velocity of particles and the velocity of flow in which they are suspended are quantities of the same order of magnitude.

Saffman Force

This force arises because of the nonuniformity of the profile of averaged velocity of carrier gas. The difference between the relative velocities of flow past a particle on different sides results in the emergence of a pressure difference. The particle will move toward lower pressure (see Fig. 2.3). The value of the Saffman force acting on a particle during its motion in a laminar flow with a linear velocity profile is determined as follows [38]:

$$F_S = k_S \nu^{1/2} \rho d_p^2 (U_x - V_x) \left(\frac{\mathrm{d}U_x}{\mathrm{d}r} \right)^{1/2}. \tag{2.9}$$

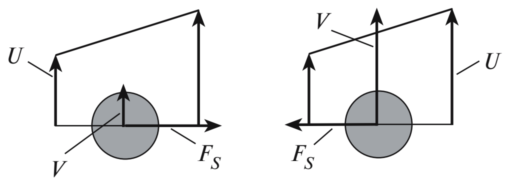

Fig. 2.3. A scheme of transverse migration of a particle in a nonuniform flow under the effect of the Saffman force

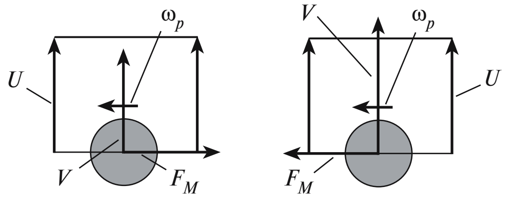

Fig. 2.4. A scheme of migration of a rotating particle under the effect of the Magnus force

In the case of $U_x/(\nu\, \mathrm{d}U_x/\mathrm{d}r)^{1/2} \ll 1$, the value of the coefficient in (2.9) is $k_\mathrm{S} = 1.61$.

The Saffman force may have a significant effect on the particle motion in the wall region where high gradients of averaged velocity of carrier gas are observed.

Magnus Force

Its emergence is due to the particle rotation. During their motion in a gas flow, particles of complex shape (aspherical) always rotate. As to spherical particles, they will also rotate in a flow with a nonuniform velocity profile. A rotating particle entrains the gas. As a result, the pressure on the side where the directions of flow past the particle and rotation of gas elements coincide becomes lower compared to the region in which these directions are opposite. Therefore, the particle will move toward lower pressure (see Fig. 2.4). The magnitude of the force acting on a particle during its rotation in a laminar flow with a uniform velocity profile at $Re_\mathrm{p} = |\overrightarrow{W}|d_\mathrm{p}/\nu \ll 1$ and $Re_\omega = |\overrightarrow{\omega}_\mathrm{p}|d_\mathrm{p}^2/\nu \ll 1$ is defined by the following expression [37]:

$$\overrightarrow{F}_\mathrm{M} = k_\mathrm{M}\rho\left(\frac{d_\mathrm{p}}{2}\right)^3(\overrightarrow{W} \times \overrightarrow{\omega}_\mathrm{p}). \qquad (2.10)$$

Here, ω_p is the rotational velocity of the particle. For the foregoing values of the Reynolds number, the coefficient in (2.10) is $k_\mathrm{M} = \pi$. For the other limiting case of high values of the Reynolds number ($Re_\mathrm{p} \to \infty, Re_\omega \to \infty$), this coefficient becomes $k_\mathrm{M} = 8\pi/3$ [34].

For the range of moderate values of the Reynolds number, the following expression may be recommended for the calculation of the coefficient [55]:

$$k_{\mathrm{M}} = 0.534 Re_{\omega}^{-0.64} Re_{\mathrm{p}}^{0.715}. \tag{2.11}$$

The use of relation (2.11) enables one to describe the majority of available calculation and experimental data in the Reynolds number range of $590 < Re_{\omega} < 45,000$ and $360 < Re_{\mathrm{p}} < 13,500$.

Shraiber et al. [39] analyzed the effect of the Magnus force on the particle motion. They showed the Magnus force to be almost always less than the Saffman force. Nevertheless, it is wrong to ignore the transverse shift of particles due to the effect of the Magnus force in high-velocity flows in which high gradients of gas velocity are realized and, consequently, high rotational velocities of particles.

Turbophoresis Force

This force arises because of the nonuniformity of the profile of fluctuation velocity of carrier gas. The gradient of the profile of the transverse component of fluctuation velocity of gas (see Sect. 1.3) leads to a directional shift of a particle toward decreasing intensity of fluctuations (see Fig. 2.5). The expression for the turbophoresis force acting on a particle has the form [31]

$$F_{\mathrm{Tu}} = -\frac{1}{2}\rho_{\mathrm{p}} \frac{\pi d_{\mathrm{p}}^3}{6} \frac{\partial \overline{u_r'^2}}{\partial r}. \tag{2.12}$$

This force may bring about a significant transverse displacement of a particle during its motion in the wall region.

Thermophoresis Force

This force arises as a result of the nonuniformity of the temperature profile of carrier gas. The gas molecules make a more intense force effect on a particle on its higher-temperature side. Therefore, the particle tends to move from

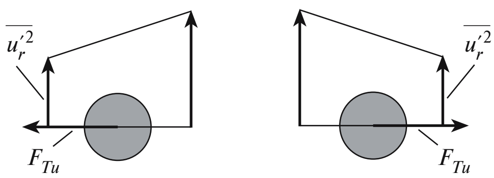

Fig. 2.5. A scheme of displacement of a particle in a nonuniform field of fluctuation velocity of gas under the effect of the turbophoresis force

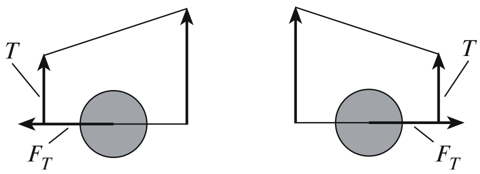

Fig. 2.6. A scheme of motion of a particle in a nonuniform temperature field under the effect of the thermophoresis force

the more heated to less heated regions (see Fig. 2.6). The expression for the thermophoresis force acting on a particle of low thermal conductivity has the form [19]:

$$F_{\mathrm{T}} = -\frac{4.5\rho\nu^2 d_{\mathrm{p}}\lambda}{T(2\lambda + \lambda_{\mathrm{p}})}\frac{\partial T}{\partial r}. \tag{2.13}$$

More theoretical formulas have been suggested for determining the value of thermophoretic force. The most complete inventory of the available relations is found in [31, 43].

Note a very important point. One must know the instantaneous values of forces in order to calculate the actual velocity of particles in accordance with (2.3). The foregoing formulas make it possible to determine only some averaged values of the force factors acting on the particles, because they fully ignore the turbulent fluctuations of gas velocity (temperature). The question of the effect of turbulence of the dispersed phase on magnitude of the forces remains open.

Shraiber et al. [39] and Gavin and Shraiber [22] tried to determine the fluctuation values of forces by applying the Reynolds procedure and using the thus derived expressions to construct equations of fluctuation motion and heat transfer of particles. However, the expressions obtained for the averaged and fluctuation values of forces are, in my opinion, too cumbersome and cannot be recommended for use.

Lagrangian Equations of Fluctuation Motion and Heat Transfer for Particles

For the case where the main effect on the particle motion is made by the aerodynamic drag and gravity forces, the Lagrangian equations of motion and heat transfer have the form

$$\frac{\mathrm{d}v_i}{d\tau} = \frac{u_i - v_i}{\tau_{\mathrm{p}}} \pm g, \tag{2.14}$$

$$\frac{\mathrm{d}t_{\mathrm{p}}}{d\tau} = \frac{t - t_{\mathrm{p}}}{\tau_{\mathrm{t}}}. \tag{2.15}$$

We will derive equations of fluctuation motion and heat transfer for inertial particles. The difficulties associated with the construction of such equations for the case of nonlinear law of aerodynamic drag were treated in detail by Shraiber et al. [39]. The developed approach to the derivation of fluctuation equations for the dispersed phase is based on the application of the Reynolds procedure to actual Lagrangian equations for particles. The results given later were borrowed from [48, 51], where the method described earlier was used to derive and analyze the approximate one-dimensional equations for fluctuations of velocity and temperature of the dispersed phase during the realization of heterogeneous flows of different classes.

We will make the following assumptions for analysis (1) the case of weakly dusty flows is treated, where the particles have little effect on one another; (2) the particles have a spherical shape; (3) the particle motion is defined by the effect of only two force factors, namely, the aerodynamic drag and gravity forces; (4) the fluctuations of the physical properties of carrier gas are ignored; (5) assumption is made of the additivity of the averaged and fluctuation dynamic slip between the phases in determining the instantaneous value of the particle drag coefficient; (6) the heat transfer between the particles and the carrier phase is defined by the convection component alone; and (7) the temperature gradient within a particle is negligible.

We will rewrite the equations of one-dimensional motion and heat transfer of a particle (2.14) and (2.15) in instantaneous (actual) variables as:

$$\frac{\mathrm{d}v_x}{\mathrm{d}\tau} = \frac{u_x - v_x}{\tau_\mathrm{p}} \pm g, \tag{2.16}$$

$$\frac{\mathrm{d}t_\mathrm{p}}{\mathrm{d}\tau} = \frac{t - t_\mathrm{p}}{\tau_\mathrm{t}}, \tag{2.17}$$

where

$$\tau_\mathrm{p} = \frac{\tau_{\mathrm{p}0}}{C} = \frac{\rho_\mathrm{p} d_\mathrm{p}^2}{18\mu C},\ C = 1 + \frac{1}{6}\tilde{R}e_\mathrm{p}^{2/3},$$
$$\tilde{R}e_\mathrm{p} = \frac{|u_x - v_x| d_\mathrm{p}}{\nu},\ \tau_\mathrm{t} = \frac{\tau_{\mathrm{t}0}}{C_1} = \frac{C_\mathrm{p}\rho_\mathrm{p} d_\mathrm{p}^2}{12\lambda C_1},$$
$$C_1 = 1 + 0.3\,\tilde{R}e_\mathrm{p}^{1/2} Pr^{1/3}, \tilde{R}e_\mathrm{p} \le 10^3.$$

We will represent the actual velocities and temperatures of the particle and carrier gas in the form of sums of respective averaged and fluctuation components,

$$v_x = V_x + v'_x, \tag{2.18}$$

$$u_x = U_x + u'_x, \tag{2.19}$$

$$t_\mathrm{p} = T_\mathrm{p} + t'_\mathrm{p}, \tag{2.20}$$

$$t = T + t'. \tag{2.21}$$

We will treat the instantaneous Reynolds number of the particle similarly,

$$\tilde{Re}_{\mathrm{p}} = Re_{\mathrm{p}} + Re'_{\mathrm{p}}, \tag{2.22}$$

where $Re_{\mathrm{p}} = \frac{|U_x - V_x| d_{\mathrm{p}}}{\nu}$ and $Re'_{\mathrm{p}} = \frac{|u'_x - v'_x| d_{\mathrm{p}}}{\nu}$.

We will substitute (2.18)–(2.22) into (2.16) and (2.17) and perform the averaging procedure on the resultant equations. The equations of averaged motion and heat transfer of the dispersed phase will take the form:

$$\begin{aligned}\frac{\mathrm{d}V_x}{\mathrm{d}\tau} &= \frac{U_x - V_x}{\tau_{\mathrm{p0}}} + \frac{1}{6\tau_{\mathrm{p0}}}\Big[(U_x - V_x)\overline{(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{2/3}} \\ &\quad + \overline{(u'_x - v'_x)(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{2/3}}\Big] \pm g, \end{aligned} \tag{2.23}$$

$$\begin{aligned}\frac{\mathrm{d}T_{\mathrm{p}}}{\mathrm{d}\tau} &= \frac{T - T_{\mathrm{p}}}{\tau_{\mathrm{t0}}} + \frac{0.3Pr^{1/3}}{\tau_{\mathrm{t0}}}\Big[(T - T_{\mathrm{p}})\overline{(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{1/2}} \\ &\quad + \overline{(t' - t'_{\mathrm{p}})(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{1/2}}\Big]. \end{aligned} \tag{2.24}$$

We will subtract (2.23) and (2.24) term-by-term from (2.16) and (2.17), respectively, in view of the substitution of (2.18)–(2.22) into the latter equations, to derive equations of fluctuation motion and fluctuation heat transfer for particles,

$$\begin{aligned}\frac{\mathrm{d}v'_x}{\mathrm{d}\tau} &= \frac{u'_x - v'_x}{\tau_{\mathrm{p0}}} + \frac{U_x - V_x}{6\tau_{\mathrm{p0}}}\Big[(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{2/3} - \overline{(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{2/3}}\Big] \\ &\quad + \frac{1}{6\tau_{\mathrm{p0}}}\Big[(u'_x - v'_x)(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{2/3} - \overline{(u'_x - v'_x)(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{2/3}}\Big], \end{aligned} \tag{2.25}$$

$$\begin{aligned}\frac{\mathrm{d}t'_{\mathrm{p}}}{\mathrm{d}\tau} &= \frac{t' - t'_{\mathrm{p}}}{\tau_{\mathrm{t0}}} + \frac{0.3Pr^{1/3}}{\tau_{\mathrm{t0}}}\Bigg\{(T - T_{\mathrm{p}})\Big[(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{1/2} - \overline{(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{1/2}}\Big] \\ &\quad + \Big[(t' - t'_{\mathrm{p}})(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{1/2} - \overline{(t' - t'_{\mathrm{p}})(Re_{\mathrm{p}} + Re'_{\mathrm{p}})^{1/2}}\Big]\Bigg\}. \end{aligned} \tag{2.26}$$

It is difficult to use the resultant equations of fluctuation motion and heat transfer for particles (2.25) and (2.26), as well as the respective averaged equations (2.23) and (2.24), for calculations by virtue of indeterminacy of the correlation terms. In [48, 51], (2.25) and (2.26) for particle-laden flows of different classes were analyzed (see Sect. 1.5). The results obtained in [48, 51] will be given later.

Quasiequilibrium flow. We will treat two possible versions of realization of quasiequilibrium flow. The first version involves a flow with a low fluctuation slip of particles ($Re'_{\mathrm{p}} < 1$). In this case, the drag of particles obeys the Stokes law. The second version involves a flow with a relatively high slip of the dispersed phase in fluctuation motion ($1 \leq Re'_{\mathrm{p}} < 1{,}000$). For this case, the correction to the Stokes law of resistance must be taken into account.

In view of the fact that, in the case of quasiequilibrium flow, the averaged dynamic and thermal slip is zero ($Re_{\text{p}} = 0, T - T_{\text{p}} = 0$), (2.25) and (2.26) may yield, for the case of Stokesian particles [48, 51],

$$\frac{\mathrm{d}v'_x}{\mathrm{d}\tau} = \frac{u'_x - v'_x}{\tau_{\text{p0}}}, \tag{2.27}$$

$$\frac{\mathrm{d}t'_{\text{p}}}{\mathrm{d}\tau} = \frac{t' - t'_{\text{p}}}{\tau_{\text{t0}}}. \tag{2.28}$$

Approximate equations of fluctuation motion and heat transfer for particles for the case where the fluctuation slip is significant have the form [48, 51]

$$\frac{\mathrm{d}v'_x}{\mathrm{d}\tau} = \frac{u'_x - v'_x}{\tau_{\text{p0}}} \left(1 + \frac{1}{6}\, Re'_{\text{p}}{}^{2/3}\right), \tag{2.29}$$

$$\frac{\mathrm{d}t'_{\text{p}}}{\mathrm{d}\tau} = \frac{t' - t'_{\text{p}}}{\tau_{\text{t0}}} \left(1 + 0.3\, Re'_{\text{p}}{}^{2/3} Pr^{1/3}\right). \tag{2.30}$$

Nonequilibrium flow. In this case, it does not appear possible to ignore the interphase slip in averaged or fluctuation motion, because the values of slip in these motions often turn out to be of the same order of magnitude, i.e., $O(Re'_{\text{p}}/Re_{\text{p}}) = 1$.

In view of assumptions made in [48, 51] and in order to simplify analysis of the correlation terms, the approximate equations of fluctuation motion and heat transfer (2.25) and (2.26) for nonequilibrium flow take the form:

$$\frac{\mathrm{d}v'_x}{\mathrm{d}\tau} = \frac{u'_x - v'_x}{\tau_{\text{p0}}} \left[1 + \frac{1}{6}\, (Re_{\text{p}} + Re'_{\text{p}})^{2/3}\right], \tag{2.31}$$

$$\frac{\mathrm{d}t'_{\text{p}}}{\mathrm{d}\tau} = \frac{t' - t'_{\text{p}}}{\tau_{\text{t0}}} \left[1 + 0.3\, (Re_{\text{p}} + Re'_{\text{p}})^{1/2} Pr^{1/3}\right]. \tag{2.32}$$

It follows from (2.31) and (2.32) that the averaged slip causes an increase in the fluctuation velocity and temperature of particles.

Flow with large particles. Under conditions of this flow, the averaged slip between the phases is far beyond the fluctuation slip, i.e., $Re'_{\text{p}}/Re_{\text{p}} \to 0$. In this case, the inertia of particles is so high that they hardly take part either in fluctuation motion ($v'_x = 0$) or in fluctuation heat transfer ($t'_{\text{p}} = 0$). The following trivial notation of (2.25) and (2.26) was obtained in [48, 51], using some assumptions, for flows of this class:

$$\frac{\mathrm{d}v'_x}{\mathrm{d}\tau} = 0, \tag{2.33}$$

$$\frac{\mathrm{d}t'_{\text{p}}}{\mathrm{d}\tau} = 0. \tag{2.34}$$

The foregoing approximate equations of fluctuation motion and fluctuation heat transfer for particles are of interest per se and may be used to determine the fluctuation velocity and temperature of particles. For this purpose, the resultant equations are integrated with respect to time. This time is the minimal of three times [49, 54, 56], namely, (1) the time of dynamic (thermal) relaxation of particles, (2) the time of interaction between particles and energy-carrying turbulent eddies of carrier gas, and (3) the lifetime of turbulent eddy.

At first glance it would seem that the obtained relations may also be employed to construct equations for correlations associated with the dispersed phase. Such correlations are present in equations which describe the carrier gas motion (see Sect. 2.4). It is necessary to calculate these correlations for assessing the inverse effect of particles on the parameters of gas flow. However, equations of motion of the carrier medium are written using the Eulerian continuum approach. Consequently, the correlations appearing in these equations must also be derived using Euler's method [39]. As to the method of constructing equations of fluctuation motion and heat transfer for particles, which is described earlier, it is purely Lagrangian; therefore, the resultant equations cannot be used to study the inverse effect of particles within the Eulerian approach.

The possibilities of using the Lagrangian trajectory method for studying the behavior of particles in turbulent gas flows may be well illustrated by studies [40–42].

2.3.2 Eulerian Continuum Approach

We will now consider the presently existing approaches to the construction of continuum equations of particle motion and analyze the singularities of the description of behavior of the dispersed phase for heterogeneous flows of different classes.

Equations describing the averaged motion and heat transfer of particles are written by analogy with the equations for gas (1.6)–(1.8). The set of equations for the dispersed phase also turns out to be nonclosed, because the equations contain second moments for the fluctuations of velocity $\overline{v'_i v'_j}$, as well as of velocity and temperature $\overline{v'_j t'_\mathrm{p}}$ of particles, similar to Reynolds stresses and turbulent heat flux in gas. Based on the experience of studying single-phase flows, various models are used for closing the set of averaged equations of motion and heat transfer for particles. The best known are the algebraic and differential models.

Two basic approaches exist to determining the correlations of velocity of the dispersed phase within the algebraic models. According to the first approach, the correlation moments are expressed directly in terms of Reynolds stresses of the carrier flow [23, 26],

$$\overline{v'_i v'_j} = A\overline{u'_i u'_j}, \tag{2.35}$$

where A is the function of involvement of particles in the fluctuation motion of gas.

Expression (2.35) is valid for relatively small particles (quasiequilibrium flow) under conditions of uniform distribution of the averaged velocity of the dispersed phase in the flow.

The second method of determining turbulent stresses in the dispersed phase is by using gradient relations of the Boussinesq type for single-phase flow [11],

$$\overline{v_i' v_j'} = -\nu_\mathrm{p} \left(\frac{\partial V_i}{\partial x_j} \right), \tag{2.36}$$

or in the form [32, 52]

$$\overline{v_i' v_j'} = -\nu_\mathrm{p} \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} - \frac{2}{3} \frac{\partial V_k}{\partial x_k} \delta_{ij} \right) + \frac{2}{3} k_\mathrm{p} \delta_{ij} , \tag{2.37}$$

where ν_p is the coefficient of turbulent viscosity of the dispersed phase. Various methods of determining ν_p are described in the literature [32, 52].

Along with the algebraic models, the differential models are extensively employed at present to describe the turbulent momentum and heat transfer in the dispersed phase. These models are based on the use of equations of energy balance of fluctuations of the dispersed phase or of the second moments of fluctuations of particle velocity and temperature.

A consistent method of constructing Euler's equations of motion and heat transfer for the dispersed phase in a turbulent flow is the method based on the use of a kinetic equation for the probability density function (PDF) of particle velocity and temperature [12, 13, 35, 57]. According to this approach, the probability density of particle distribution by coordinates $\vec{x}$, velocities $\vec{v}$, and temperatures t_p is introduced for making a transition from stochastic equations of the Langevin type (such as equations of instantaneous motion and heat transfer for a single particle) to a kinetic equation for a plurality of particles,

$$P(\vec{x}, \vec{v}, t_\mathrm{p}, \tau) = \overline{\delta(\vec{x} - \vec{r}_\mathrm{p}(\tau)) \delta(\vec{v} - \vec{v}_\mathrm{p}(\tau)) \delta(t - t_\mathrm{p}(\tau))} , \tag{2.38}$$

where averaging is performed over realizations of random fields of velocity and temperature of carrier gas. Then, the differentiation of (2.38) with respect to time in view of representation of the gas velocity and temperature in the instantaneous equations of motion and heat transfer for particles in the form of sums of averaged and fluctuation components is used to derive the equation for probability density. Then, the equation for the PDF of particle distribution by coordinates, velocities, and temperatures is used to construct equations for averaged concentration, velocity, and temperature of particles, which have the form [52]:

$$\frac{\partial \Phi}{\partial \tau} + \sum_j \frac{\partial \Phi V_j}{\partial x_j} = 0, \tag{2.39}$$

$$\frac{\partial V_i}{\partial \tau} + \sum_j V_j \frac{\partial V_i}{\partial x_j} = -\sum_j \frac{\partial \overline{v_i' v_j'}}{\partial x_j} + \frac{U_i - V_i}{\tau_\mathrm{p}} - \sum_j \frac{D_{\mathrm{p}ij}}{\tau_\mathrm{p}} \frac{\partial \ln \Phi}{\partial x_j}, \quad (2.40)$$

$$\frac{\partial T_\mathrm{p}}{\partial \tau} + \sum_j V_j \frac{\partial T_\mathrm{p}}{\partial x_j} = -\sum_j \frac{\partial \overline{\nu_j' t_\mathrm{p}'}}{\partial x_j} + \frac{T - T_\mathrm{p}}{\tau_\mathrm{t}} - \sum_j \frac{D^\mathrm{t}_{\mathrm{p}j}}{\tau_\mathrm{t}} \frac{\partial \ln \Phi}{\partial x_j}, \quad (2.41)$$

where

$$\overline{v_i' v_j'} = \frac{1}{\Phi} \iint v_i' v_j' P \mathrm{d}\nu \mathrm{d}t_\mathrm{p}, \quad \overline{v_j' t_\mathrm{p}'} = \frac{1}{\Phi} \iint v_j' t_\mathrm{p}' P \mathrm{d}\nu \mathrm{d}t_\mathrm{p},$$

$$D_{\mathrm{p}ij} = \tau_\mathrm{p}(\overline{v_i' v_j'} + g_\mathrm{p} \overline{u_i' u_j'}), \quad D^\mathrm{t}_{\mathrm{p}j} = \tau_\mathrm{t} \overline{v_j' t_\mathrm{p}'} + \tau_\mathrm{p} g_\mathrm{pt} \overline{u_j' t'},$$

$$g_\mathrm{p} = \frac{T_\mathrm{pL}}{\tau_\mathrm{p}} - 1 + \exp(-T_\mathrm{pL}/\tau_\mathrm{p}),$$

$$g_\mathrm{pt} = \frac{T_\mathrm{pLt}}{\tau_\mathrm{p}} - 1 + \exp(-T_\mathrm{pLt}/\tau_\mathrm{p}).$$

Here, T_pL and T_pLt denote the time of interaction of particles with energy-intensive fluctuations of velocity and temperature, respectively. For an inertialess impurity,

$$T_\mathrm{pL} = T_\mathrm{L}, \quad T_\mathrm{pLt} = T_\mathrm{Lt}, \quad (2.42)$$

where T_L and T_Lt are the time scales of fluctuations of velocity and temperature of gas, respectively.

In the case of nonequilibrium flow, where the averaged and dynamic slips between the gas and particles become significant, the times of interaction with fluctuations of the carrier flow may differ significantly from the respective scales of fluctuations of the carrier phase.

The set of (2.39)–(2.41) is not closed, because the equations include the turbulent stresses $\overline{v_i' v_j'}$ and the turbulent heat flux $\overline{v_j' t_\mathrm{p}'}$ in the dispersed phase, associated with the involvement of particles in the fluctuation motion, as well as turbulent diffusion fluxes of momentum and heat arising because of the nonuniformity of the particle concentration.

Volkov et al. [52] developed a mathematical description of the processes of momentum and heat transfer in the dispersed phase of different levels of detail. A closed set of equations is given on the level for the third moments. In this case, the fourth moments of fluctuation characteristics, which appear in equations for the third moments, are expressed approximately in terms of the sum of products of the second moments [52]. Triple correlations must be determined in order to describe the hydrodynamics and heat transfer of the dispersed phase on the level of equations for the second moments. For this purpose, Volkov et al. [52] further used equations for the third moments; the simplification of these equations by ignoring small terms enables one to find algebraic relations for triple correlations which contain only the second moments. The computational scheme may be further simplified by replacing the equations for the second moments of velocity fluctuations by a single differential equation for the energy of fluctuations of the dispersed phase, which has the following form [52]:

$$\frac{\partial k_{\mathrm{p}}}{\partial \tau}+\sum_j V_j \frac{\partial k_{\mathrm{p}}}{\partial x_j}=-\frac{1}{\Phi}\sum_j \frac{\partial \Phi \overline{v_i' v_i' v_j'}}{2\partial x_j}-\sum_j\sum_i \overline{v_i' v_j'}\frac{\partial V_i}{\partial x_j}+\frac{2}{\tau_{\mathrm{p}}}(f_u k-k_{\mathrm{p}}),\quad (2.43)$$

where $k_{\mathrm{p}}=\frac{1}{2}\sum\limits_i \overline{v_i' v_i'}$ is the energy of fluctuations of particle velocity.

In a steady-state uniform flow or for small particles (quasiequilibrium flow), (2.43) yields $k_{\mathrm{p}}=f_u k$, where $f_u=(1+Stk_{\mathrm{L}})^{-1}$. In this case, (2.39) and (2.40) in view of relation (2.37) give a description of momentum transfer in the dispersed phase on the level of equations for the first moments.

2.4 Description of Motion of Gas Carrying Solid Particles

We will treat the motion of gas in the presence of particles when the particles start making an inverse effect on the gas characteristics. The equations of continuity, motion, and energy for the gas phase with a relatively low content of particles ($\varphi \ll 1$) in the absence of external mass forces have the form:

$$\sum_j \frac{\partial u_j}{\partial x_j}=0, \quad (2.44)$$

$$\frac{\partial u_i}{\partial \tau}+\sum_j u_j \frac{\partial u_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\sum_j \frac{\partial^2 u_i}{\partial x_j \partial x_j}-\frac{\rho_{\mathrm{p}}\varphi}{\rho}\frac{(u_i-v_i)}{\tau_{\mathrm{p}}}, \quad (2.45)$$

$$\frac{\partial t}{\partial \tau}+\sum_j u_j \frac{\partial t}{\partial x_j}=a\sum_j \frac{\partial^2 t}{\partial x_j \partial x_j}-\frac{C_{\mathrm{p_p}}\rho_{\mathrm{p}}\varphi}{C_{\mathrm{p}}\rho}\frac{(t-t_{\mathrm{p}})}{\tau_{\mathrm{t}}}. \quad (2.46)$$

The continuity equation (2.44) has a similar form as (1.1) for a single-phase flow. Equations (2.45) and (2.46) differ from the respective equations of motion and energy for a single-phase gas (1.2) and (1.3) by the presence in their right-hand parts of terms which take into account the dynamic and thermal effect of the dispersed phase on the carrier flow.

We will average (2.44)–(2.46) over time. In so doing, we will follow the well-known method of averaging in the theory of single- phase flows of variable density [25], as well as the PDF-based method of constructing equations for the dispersed phase [52], and assume $\overline{\varphi' v_i'}=\overline{\varphi' t_{\mathrm{p}}'}=0$. The averaged equations of continuity, motion, and energy have the form:

$$\sum_j \frac{\partial U_j}{\partial x_j}=0, \quad (2.47)$$

$$\frac{\partial U_i}{\partial \tau} + \sum_j U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \sum_j \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \sum_j \frac{\partial(\overline{u'_i u'_j})}{\partial x_j} - \frac{\rho_\mathrm{p} \mathit{\Phi}}{\rho} \frac{(U_i - V_i)}{\tau_\mathrm{p}} - \frac{\rho_\mathrm{p} \overline{\varphi' u'_i}}{\rho \tau_\mathrm{p}}, \tag{2.48}$$

$$\frac{\partial T}{\partial \tau} + \sum_j U_j \frac{\partial T}{\partial x_j} = a \sum_j \frac{\partial^2 T}{\partial x_j \partial x_j} - \sum_j \frac{\partial(\overline{u'_j t'})}{\partial x_j} - \frac{C_{\mathrm{p_p}} \rho_\mathrm{p} \mathit{\Phi}}{C_\mathrm{p} \rho} \frac{(T - T_\mathrm{p})}{\tau_\mathrm{t}} - \frac{C_{\mathrm{p_p}} \rho_\mathrm{p} \overline{\varphi' t'}}{C_\mathrm{p} \rho \tau_\mathrm{t}}. \tag{2.49}$$

Equations (2.48) and (2.49) indicate that the inverse effect of particles on the motion and heat transfer of carrier gas is defined by the averaged dynamic and thermal slip of the dispersed phase, as well as by the fluctuations of the particle concentration. Note that the contribution made by the penultimate and last terms of the right-hand parts of (2.48) and (2.49) will be determining for the case of flow with large particles and quasiequilibrium heterogeneous flow, respectively (see Sect. 1.5). In the case of nonequilibrium heterogeneous flow, where the averaged and fluctuation dynamic and thermal slip occurs between the phases, it is necessary to take into account the contribution by all of the above-identified terms of equations of motion and energy.

We will treat the case where the distributions of averaged velocities and concentrations of the dispersed phase are known. In order to close the set of averaged equations, one must know the turbulent stresses of gas $\overline{u'_i u'_j}$ and the turbulent heat flux $\overline{u'_j t'}$, as well as the correlations of the fluctuations of particle concentration with the fluctuations of gas velocity and temperature $\overline{\varphi' u'_i}$ and $\overline{\varphi' t'}$ which may be represented as follows [14, 15]:

$$\overline{\varphi' u'_i} = -\tau_\mathrm{p} g_\mathrm{p} \overline{u'_i u'_j} \frac{\partial \mathit{\Phi}}{\partial x_j}, \tag{2.50}$$

$$\overline{\varphi' t'} = -\tau_\mathrm{p} g_\mathrm{pt} \overline{u'_j t'} \frac{\partial \mathit{\Phi}}{\partial x_j}, \tag{2.51}$$

where

$$g_\mathrm{p} = T_\mathrm{pL}/\tau_\mathrm{p} - 1 + \exp(-T_\mathrm{pL}/\tau_\mathrm{p}), \quad g_\mathrm{pt} = T_\mathrm{pLt}/\tau_\mathrm{p} - 1 + \exp(-T_\mathrm{pLt}/\tau_\mathrm{p}).$$

One can subtract (2.47)–(2.49) from (2.44)–(2.46), respectively, and derive the fluctuation equations of continuity, motion, and energy of the gas phase in the presence of particles,

$$\sum_j \frac{\partial u'_j}{\partial x_j} = 0, \tag{2.52}$$

$$\frac{\partial u_i'}{\partial \tau} + \sum_j \left[u_j' \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i'}{\partial x_j} + \frac{\partial (u_i' u_j')}{\partial x_j} \right] = -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \sum_j \frac{\partial^2 u_i'}{\partial x_j \partial x_j}$$
$$+ \sum_j \frac{\partial (\overline{u_i' u_j'})}{\partial x_j} - \frac{\rho_\mathrm{p} \varPhi}{\rho} \frac{(u_i' - v_i')}{\tau_\mathrm{p}} - \frac{\rho_\mathrm{p} \varphi'}{\rho} \frac{[(U_i - V_i) + (u_i' - v_i')]}{\tau_\mathrm{p}} + \frac{\rho_\mathrm{p} \overline{\varphi' u_i'}}{\rho \tau_\mathrm{p}}, \tag{2.53}$$

$$\frac{\partial t'}{\partial \tau} + \sum_j \left[u_j' \frac{\partial T}{\partial x_j} + U_j \frac{\partial t'}{\partial x_j} + \frac{\partial (u_j' t')}{\partial x_j} \right] = a \sum_j \frac{\partial^2 t'}{\partial x_j \partial x_j} + \sum_j \frac{\partial (\overline{u_j' t'})}{\partial x_j}$$
$$- \frac{C_{\mathrm{p_p}} \rho_\mathrm{p} \varPhi}{C_\mathrm{p} \rho} \frac{(t' - t_\mathrm{p}')}{\tau_t} - \frac{C_{\mathrm{p_p}} \rho_\mathrm{p} \varphi'}{C_\mathrm{p} \rho} \frac{[(T - T_\mathrm{p}) + (t' - t_\mathrm{p}')]}{\tau_\mathrm{t}} + \frac{C_{\mathrm{p_p}} \rho_\mathrm{p} \overline{\varphi' t'}}{C_\mathrm{p} \rho \tau_\mathrm{t}}. \tag{2.54}$$

The fluctuation equation of continuity (2.50) has a form similar to that of the respective (1.9) for a single-phase flow. Equations (2.53) and (2.54) differ from analogous equations of motion and energy for single-phase gas (1.10) and (1.11) by the presence in their right-hand parts of terms which take into account the dynamic and thermal effect of the dispersed phase on the carrier flow. These equations indicate that the inverse effect of particles on the fluctuation motion and heat transfer of carrier gas is defined by the fluctuation and averaged dynamic and thermal slip of the dispersed phase, as well as by the fluctuations of the particle concentration. Note that the contribution made by the penultimate terms of the right-hand parts of (2.53) and (2.54) will be determining for the case of flow with large particles characterized by a significant difference of the averaged velocities and temperatures between the phases.

We will derive the equation for the second moments of fluctuations of velocity of the carrier phase in the presence of particles by analogy with the case of single-phase flow in Sect. 1.2. We will first replace j by k in (2.53) for u_i' and multiply both parts of the resultant equation by u_j',

$$u_j' \frac{\partial u_i'}{\partial \tau} + \sum_k \left[u_j' u_k' \frac{\partial U_i}{\partial x_k} + u_j' U_k \frac{\partial u_i'}{\partial x_k} + u_j' \frac{\partial (u_i' u_k')}{\partial x_k} \right]$$
$$= -u_j' \frac{1}{\rho} \frac{\partial p'}{\partial x_i} + u_j' \nu \sum_k \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + u_j' \sum_k \frac{\partial (\overline{u_i' u_k'})}{\partial x_k} - \frac{\rho_\mathrm{p} \varPhi}{\rho} \frac{u_j' (u_i' - v_i')}{\tau_\mathrm{p}}$$
$$- \frac{\rho_\mathrm{p} \varphi'}{\rho} \frac{u_j' [(U_i - V_i) + (u_i' - v_i')]}{\tau_\mathrm{p}} + \frac{\rho_\mathrm{p} u_j' \overline{\varphi' u_i'}}{\rho \tau_\mathrm{p}}. \tag{2.55}$$

We will write a similar equation for u'_j and multiply both its parts by u'_i,

$$u'_i \frac{\partial u'_j}{\partial \tau} + \sum_k \left[u'_i u'_k \frac{\partial U_j}{\partial x_k} + u'_i U_k \frac{\partial u'_j}{\partial x_k} + u'_i \frac{\partial (u'_j u'_k)}{\partial x_k} \right]$$

$$= -u'_i \frac{1}{\rho} \frac{\partial p'}{\partial x_j} + u'_i \nu \sum_k \frac{\partial^2 u'_j}{\partial x_k \partial x_k} + u'_i \sum_k \frac{\partial (\overline{u'_j u'_k})}{\partial x_k}$$

$$- \frac{\rho_\mathrm{p} \Phi}{\rho} \frac{u'_i (u'_j - v'_j)}{\tau_\mathrm{p}} - \frac{\rho_\mathrm{p} \varphi'}{\rho} \frac{u'_i [(U_j - V_j) + (u'_j - v'_j)]}{\tau_\mathrm{p}} + \frac{\rho_\mathrm{p} u'_i \overline{\varphi' u'_j}}{\rho \tau_\mathrm{p}}. \quad (2.56)$$

We will combine (2.55) and (2.56) term-by-term and perform averaging. As a result, the equation of transport of turbulent stresses of gas in the presence of particles takes the form:

$$\frac{\partial (\overline{u'_i u'_j})}{\partial \tau} + \sum_k U_k \frac{\partial (\overline{u'_i u'_j})}{\partial x_k} = \sum_k \frac{\partial}{\partial x_k} \left[\nu \frac{\partial (\overline{u'_i u'_j})}{\partial x_k} - \overline{u'_i u'_j u'_k} \right]$$

$$- \sum_k \left[(\overline{u'_j u'_k}) \frac{\partial U_i}{\partial x_k} + (\overline{u'_i u'_k}) \frac{\partial U_j}{\partial x_k} \right] - \frac{1}{\rho} \left(\overline{u'_i \frac{\partial p'}{\partial x_j}} + \overline{u'_j \frac{\partial p'}{\partial x_i}} \right) - 2\nu \sum_k \overline{\frac{\partial u'_i \partial u'_j}{\partial x_k \partial x_k}}$$

$$- \frac{\rho_\mathrm{p}}{\rho \tau_\mathrm{p}} \Big[\Phi (2\overline{u'_i u'_j} - \overline{v'_i u'_j} - \overline{u'_i v'_j}) + \overline{\varphi' u'_j} (U_i - V_i)$$

$$+ \overline{\varphi' u'_i} (U_j - V_j) + (2\overline{\varphi' u'_i u'_j} - \overline{\varphi' v'_i u'_j} - \overline{\varphi' u'_i v'_j}) \Big]. \quad (2.57)$$

Equation (2.57) differs from the similar equation for single-phase gas (1.14) by the presence in the right-hand part of the last group of terms which take into account the dynamic effect of the dispersed phase on the carrier flow. The inverse effect of particles on the balance of Reynolds stresses of carrier gas is caused by the fluctuation and averaged slip of the dispersed phase, as well as by the fluctuations of particle concentration.

The set of (2.47), (2.48), (2.50), and (2.57) turns out to be nonclosed, because (2.57) includes unknown triple correlations of fluctuations of the carrier phase velocities, as well as the correlations associated with the fluctuations of concentration and velocity of the dispersed phase. Various models are used to derive the closed set of equations describing the averaged motion of gas in the presence of particles. Most extensively employed (similar to the theory of turbulent single-phase flows) are algebraic, one-parameter, and two-parameter models.

2.4.1 Algebraic Models

The concepts of the Prandtl semiempirical theory of turbulence are usually used in models of this type (see Sect. 1.2). In his pioneering study, Abramovich

[1] used the mixing length theory to determine the fluctuation velocities of gas and particles. The thus developed model is based on the equation of conservation of momentum of turbulent eddy and particles moving in this eddy, as well as on the equation of fluctuation motion of particles within the eddy. It is assumed that that low-inertia particles are entrained in the fluctuation motion by turbulent eddies of the carrier phase; as a result, the fluctuation velocity of gas decreases. The obtained values of fluctuation velocities of gas and particles are used to find correlations by multiplying together the respective fluctuation quantities, which makes this method very approximate. Models of this type were developed further in [2–5, 24, 27, 29, 50, 59].

2.4.2 One-Parameter Models

The widest acceptance (similar to the case of single-phase flow) was received by the model based on the equation for turbulent energy.

In order to construct the equation of transport of turbulent energy of gas in the presence of particles, the equation of fluctuation motion (2.53) must be multiplied by u_i', summed over i, and then averaged. The resultant equation will have the form

$$\begin{aligned}\frac{\partial k}{\partial \tau} + \sum_j U_j \frac{\partial k}{\partial x_j} = {} & \sum_j \frac{\partial}{\partial x_j}\left[\nu \frac{\partial k}{\partial x_j} - \overline{u_j'\left(\frac{1}{2}\sum_i u_i'^2 + \frac{p'}{\rho}\right)}\right] \\ & - \sum_j \sum_i \overline{u_i' u_j'} \frac{\partial U_i}{\partial x_j} - \nu \sum_j \sum_i \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} \\ & - \sum_i \frac{\rho_\mathrm{p}}{\rho \tau_\mathrm{p}}\Big[\Phi(\overline{u_i' u_i'} - \overline{u_i' v_i'}) + \overline{\varphi' u_i'}(U_i - V_i) \\ & + (\overline{\varphi' u_i' u_i'} - \overline{\varphi' u_i' v_i'})\Big] . \end{aligned} \tag{2.58}$$

In accordance with the equation of transport of turbulent energy of single-phase gas (1.24), (2.58) may also be rewritten in a condensed form,

$$\frac{Dk}{D\tau} = D + P - \varepsilon - \varepsilon_\mathrm{p}, \tag{2.59}$$

where the additional dissipation ε_p caused by the presence of particles has the form:

$$\varepsilon_\mathrm{p} = \sum_i \frac{\rho_\mathrm{p}}{\rho\tau_\mathrm{p}}\Big[\Phi(\overline{u_i' u_i'} - \overline{u_i' v_i'}) + \overline{\varphi' u_i'}(U_i - V_i) + (\overline{\varphi' u_i' u_i'} - \overline{\varphi' u_i' v_i'})\Big] . \tag{2.60}$$

The terms on the right-hand side of (2.60) are responsible for the dissipation of turbulent energy caused by the fluctuation interphase slip, the correlation of the fluctuations of particle concentration with the fluctuation velocity of carrier gas, and the presence of averaged dynamic slip, as well as by the correlations of the fluctuations of particle concentration and the fluctuation velocities of the phases, respectively.

The authors of a number of studies (for example, [17, 20, 21]) tried to estimate the terms in the right-hand part of (2.60) for particle-laden flows of different types. It was demonstrated that, in flows with relatively inertial particles ($Stk_{\mathrm{L}} \geq 1$), the fluctuations of concentration of the dispersed phase do not correlate with the field of fluctuation velocity of gas. This implies the smallness of the second and third terms of the right-hand part of (2.60) compared to its first term. Therefore, in the case of the quasiequilibrium and nonequilibrium flows (see Table 1.1), the first term on the right-hand side of (2.60) will play the determining part in the process of dissipation of turbulence. In the case of a flow with large particles which are not entrained in the fluctuation motion by energy-carrying eddies of the carrier phase, the expression for ε_{p} may be written as:

$$\varepsilon_{\mathrm{p}} = \sum_i \frac{\rho_{\mathrm{p}} \Phi}{\rho \tau_{\mathrm{p}}} \overline{u_i' u_i'} = \frac{2Mk}{\tau_{\mathrm{p}}}. \tag{2.61}$$

Note that, in the case of a flow with large particles whose relaxation time is significant, the value of additional dissipation of turbulent energy will be negligible compared to other terms of (2.58).

As was demonstrated by the experimental results, the presence of large particles in the flow may cause additional generation (production) of turbulence of the carrier gas. This mechanism is in no way taken into account in writing (2.45). We will write (2.59) as:

$$\frac{Dk}{D\tau} = D + P - \varepsilon + P_{\mathrm{p}} - \varepsilon_{\mathrm{P}}, \tag{2.62}$$

where P_{p} is the term responsible for the additional production of turbulent energy because of the presence of the dispersed phase. Therefore, the inclusion of the modification of turbulence in heterogeneous flows presumes a correct description of the terms of (2.62) responsible for the generation (P_{p}) and dissipation (ε_{p}).

A mathematical model is given in Sect. 4.3, which describes the processes of additional dissipation of turbulence by low-inertia particles (quasiequilibrium flow) and of additional generation of turbulence in wakes behind moving particles (flow with large particles). Analysis was performed in a diffusionless (algebraic) approximation, i.e., disregarding the contribution by the diffusion term D to (2.62). The effect of particles on the steady-state hydrodynamically developed pipe flow is treated; for this flow, the left-hand part of (2.62) goes to zero. In addition, for the purpose of deriving simple analytical relations, analysis was made for moderate values of particle concentration, when the effect of particles on the distribution of averaged velocity of the carrier gas was minor. As a result, expressions for the source terms ε_{p} and P_{p} were derived and used to find two complexes of physical parameters responsible for the dissipation and generation of turbulent energy of the carrier gas under conditions of quasiequilibrium flow and flow with large particles, respectively.

Good agreement between the calculation results and the available experimental results leads one to expect the efficiency of the model in the case of a nonequilibrium flow, when the joint action is possible of both mechanisms (laminarizing and turbulizing) of the effect of particles on turbulence.

2.4.3 Two-Parameter Models

As in studying single-phase turbulent flows, the most generally employed model is the two-parameter k–ε model of turbulence with the equation for the rate of dissipation used as the second equation.

By analogy with (1.28) for a single-phase flow, we have, in the case of a particle-laden flow,

$$\frac{D\varepsilon}{D\tau} = D_\varepsilon + P_\varepsilon - \varepsilon_\varepsilon - \varepsilon_{\varepsilon \mathrm{p}}, \tag{2.63}$$

where $\varepsilon_{\varepsilon \mathrm{p}}$ is the decrease in dissipation because of the presence of particles.

The expression for $\varepsilon_{\varepsilon \mathrm{p}}$ is most commonly represented in the form [18, 36]

$$\varepsilon_{\varepsilon \mathrm{p}} = C_{\varepsilon 3} \frac{\varepsilon}{k} \varepsilon_{\mathrm{p}}, \tag{2.64}$$

where the constant $C_{\varepsilon 3}$ may take the following values: $C_{\varepsilon 3} = 1.0$ [33], $C_{\varepsilon 3} = 1.2$ [18], and $C_{\varepsilon 3} = 1.9$ [7].

2.4.4 Methods of Direct Numerical Simulation

In conclusion, we must dwell briefly on the methods of direct numerical simulation (DNS) which are rapidly developing in recent years. A method of direct numerical simulation is the solution of nonstationary Navier–Stokes equations for instantaneous values without involving additional closing relations or equations, i.e., actually without the simulation of turbulence. The well-known limitation of such a method is the impossibility of using it at moderate or high values of the Reynolds number. A variety of this method is the method of large eddy simulation (LES) which involves the treatment of only large energy-carrying eddies [30]. In this manner, an attempt is made at obviating the disadvantage identified earlier and extending the range of application of the method.

In the overwhelming majority of early investigations of particle-laden two-phase flows [16, 44, 53], these two methods were used to simulate the motion of single particles; in accordance with the classification of heterogeneous flows developed in Sect. 1.5, this corresponds to the case of a weakly dusty flow without the inverse effect of particles on the carrier gas parameters. These investigations were performed to study the behavior of particles. For this purpose, the trajectories of a large ensemble of particles introduced into a turbulent flow were calculated, which was followed by the averaging of the obtained spatial characteristics of particle motion. Note that the spatial resolution was much less than the particle size proper. In performing the calculations, it was not intended to determine the parameters of gas flow about a particle. This was not necessary, because the particle motion is calculated in the usual way, i.e., using the law of resistance of the dispersed phase. The particle drag is defined by the Reynolds number; for determining the value of this number, one needs to know the carrier gas velocity rather than the distribution of this velocity over the particle contour. The foregoing restriction in the calculation of particle motion is valid only when describing the behavior of very fine particles whose size is less than the size of the smallest turbulent eddies (Kolmogorov scale).

In more recent investigations [6, 8, 9, 45, 47], the methods of direct numerical simulation were used as advantage for the calculation of weakly dusty flows with the inverse effect of particles on the characteristics of flow of the carrier phase. In this case, the calculations are performed in several iteration steps. First, the parameters of motion of “pure” gas are calculated. For this purpose, it is usually assumed that the fluctuations of gas velocity obey the normal law. In the known field of gas velocities, the trajectories of particles are calculated by integrating the equations of their motion. Then, given a fairly representative ensemble of particles, one finds the averaged characteristics of the dispersed phase which are then used to calculate the gas phase flow in the next stage. The thus obtained “new” field of gas velocities serves a basis for the calculations of particle trajectories at the next iteration step, and so on. The calculations are performed until the difference between the obtained characteristics of motion of both phases of heterogeneous flow at the previous and subsequent iteration steps is within the preassigned error.