Theory of holors

A generalization of tensors

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Historical introduction

The birth of the holor concept may be set at 1673, when John Wallis suggested the geometric representation of complex numbers by points in a plane.[1] A complex number is an entity consisting of two parts. It is usually written as

$$z = x + iy,$$

where x and y are real numbers and i emphasizes the fact that x and y are independent quantities. A more modern notation writes a two-element holor as an ordered number pair,

$$z = (x, y)$$

or, in index notation,

$$x^i = (x^1, x^2).$$

This constitutes the first step in a fascinating mathematical development that now includes vectors, matrices, tensors, and other holors.

0.01. Quaternions

Mathematicians expected that the extension from the complex number, with $n = 2$ to $n = 3$, would be child's play; but considerable time elapsed before they found that no such extension is possible without violating a rule of ordinary algebra.[2] After wrestling with the problem for 15 years, Sir William Rowan Hamilton in 1843 found a holor with four elements,

$$x^i = (x^1, x^2, x^3, x^4),$$

which he called a *quaternion*.[3] The ordinary rules of algebra hold for quaternions except that multiplication is noncommutative. Hamilton believed that quaternions constituted his greatest contribution to mathematics and that most of physics would eventually be written in quaternion form. Tait[4] devoted himself to that end. But physicists were not convinced. For instance, Maxwell[5] in his celebrated treatise (1873) mentioned quaternions several times but was careful not to use them.

0.02. Linear associative algebras

The work of Hamilton should certainly have suggested the possibility of a host of new algebras in which associative or commutative properties are abandoned. This possibility, however, was not recognized until 1870, when Benjamin Peirce[6] invented a whole new branch of mathematics – the study of *linear associative algebras.* The subject became popular, and hundreds of algebras* were developed with their associated holors.[7] One would expect that a few of these algebras would have had interesting practical applications, but such does not seem to be the case. The reason for this disappointing result will now be explained.

Consider holors such as

$$x^i = (x^1, x^2, \ldots, x^n),$$
$$y^i = (y^1, y^2, \ldots, y^n),$$

and two operations, which we shall call addition and multiplication. The universally accepted definition of addition is

$$x^i + y^i = (x^1 + y^1, x^2 + y^2, \ldots, x^n + y^n).$$

It appears that nothing is to be gained by changing the definition of addition. But multiplication is another story! We are at liberty to select a multiplication table, and each table determines a new algebra with its own peculiar properties.

A linear associative algebra is a closed system: a sum or product is always a holor of the same system. For instance, the product of two quaternions is always a quaternion, the product of two matrices is always a matrix. This is a delightful property, closely associated with groups, rings, and other aspects of modern algebra. But it is not the kind of behavior that is needed for most physical applications. Experience shows that the product of two vectors, for instance, must sometimes be a vector, sometimes a scalar, sometimes a matrix. And this variety cannot

* This includes algebras of Dedekind, Frobenius, Scheffers, Peirce, Kronecker, Weierstrass, Dickson, Sylvester, and Cartan.

be obtained with any of the linear associative algebras. Here is the reason that this branch of mathematics, once pursued with such enthusiasm, has proved to be of limited value as regards physical applications.

0.03. Matrices

We have been considering *univalent* holors, such as

$$x^i = (x^1, x^2, \ldots, x^n).$$

Also important are *bivalent* holors (matrices), for instance,

$$A^{ij} = \begin{bmatrix} A^{11} & A^{12} & \cdots & A^{1n} \\ A^{21} & A^{22} & \cdots & A^{2n} \\ \cdot & \cdot & \cdots & \cdot \\ A^{n1} & A^{n2} & \cdots & A^{nn} \end{bmatrix}.$$

The theory of matrices* was initiated by Arthur Cayley[8] in 1855. The only product ordinarily used is the Cayley product, which in index notation is

$$A^{ij} B_{jk} = C^i_k.$$

This is an associative but noncommutative product. Thus, matrix algebra[9] may be regarded as one of the linear associative algebras.

0.04. Grassmann

In the same year (1844) that Hamilton published his first paper on quaternions, a much more general treatment of holors was published by Herman Grassmann.[10] Not only did *Die Ausdehnungslehre* deal with n-space but, unlike the linear associative algebras, it was not limited to a single product per algebra. Thus it opened up new possibilities, particularly in geometric and physical applications of holors. Hamilton's comment was typical[11]: "I have recently been reading. . .more than a hundred pages of Grassmann's *Ausdehnungslehre,* with great admiration and interest. . . . But it is curious to see how narrowly, yet how completely, Grassmann failed to hit off the quaternions." As if Grassmann, with his immensely more general treatment, would be particularly interested in the special case of the quaternion!

* The name "matrices" was first used by J. J. Sylvester, *Collected math. papers,* Vol. 1 (Cambridge University Press, Cambridge, 1904), p. 145.

0.05. Vector analysis

To other mathematicians and physicists at this time, the *Ausdehnungslehre* seemed quite incomprehensible; and quaternions, based on Hamilton's 800-page explanation, seemed hardly simpler. It was not until 40 years later that Williard Gibbs[12] (and independently, Oliver Heaviside)[13] worked out a special case of quaternions called vector analysis. This was based largely on Hamilton's pioneer work and even used Hamilton's terms *scalar* and *vector*, but it applied only to 3-space and was usually expressed in rectangular coordinates. It did have two distinct products, however, and it proved to be just what was needed in a large number of physical applications. As a consequence, vector analysis has been universally accepted and is widely used today.[14]

0.06. Invariance

If one reads an early treatment of vectors, such as that of Gibbs or Coffin,[15] he is impressed by the exclusive use of rectangular Cartesian coordinates. Even such concepts as divergence, curl, and scalar and vector Laplacians[16] are usually defined in rectangular coordinates. In a way, this procedure makes for simplicity. But it ignores the subject of invariance. Even with Gibbs, a vector was more than a set of numbers representing rectangular coordinates: it was a geometric entity that could be expressed equally well in other coordinate systems.[12] The merates changed when the coordinates were changed, but the geometric object maintained its identity.

Invariants were studied by Cayley[17] beginning in 1841. The subject received such enthusiastic support during the latter half of the nineteenth century that Sylvester[18] remarked in 1864, "As all roads lead to Rome, so I find in my own case at least that all algebraic inquiries, sooner or later, end at the Capitol of modern algebra over whose shining portal is inscribed the *Theory of Invariants.*" The subject is treated in a voluminous literature.[19]

An important application of invariant theory is differential geometry. Riemann[20] outlined an approach in his *Habilitationsschrift* (Göttingen, 1854). Felix Klein,[21] in his Erlanger program, stated a principle that affected the development of geometry for many years. Christoffel, Beltrami, Bianchi, and others made valuable contributions. Ricci[22] developed a new notation, which he called "the absolute differential calculus" but which is now called tensor* theory. Ricci wrote many papers, including a celebrated one with Levi-Civita[23] (1901).

Physicists in general were not interested in invariants and had never heard of Ricci's work. It was not until 1916 that Albert Einstein[24] applied tensors in formulating his general relativity. Because of the extraordinary popular appeal of this theory, tensors became famous overnight. Hundreds of books appeared on relativity and several on tensors.[25] And even the old subject of vectors took on new life when presented in tensor form.[14]

0.07. Tensors

Index notation was designed particularly to handle behavior under coordinate transformation. But the basic notation is applicable even when no coordinate transformations are involved. Thus, index notation can be employed advantageously to unify a great field of holors of various valences and dimensionalities, even where the question of invariance does not necessarily enter. We have seen how vector analysis originated with little thought for coordinate transformation. The same may be said for quaternions, matrices, and the holors treated in linear associative algebras.

Since 1916, however, the feeling seems to be prevalent that the only hypernumbers of any value are tensors. On the contrary, many valuable applications of holors do not need a consideration of coordinate transformation. The simplest application of holors specifies a collection of discrete objects – for instance, machine screws. Each merate gives the number of screws of a particular type. The complete holor designates the inventory of a particular tool room (as regards screws in stock). Transformation of coordinates is meaningless here. Another example has merates that specify the loop currents in a given l-loop electric network. Again, transformation does not enter. And there are examples in which transformation is possible but not of much interest. Thus, we have a host of practical applications of holors where the holor is definitely not a tensor. And we have, of course, a host of other applications where the holor is a tensor.

* The name *tensor* is often said to be the invention of Woldemar Voigt,[26] who used it to represent a bivector in crystal physics. The word, however, is much older, having been applied in connection with quaternions by Hamilton. See *The Mathematical papers of Sir William Rowan Hamilton* (Cambridge University Press), p. 237.

0.08. Holor notation

Heaviside[13] suggested boldfaced type to distinguish vectors from scalars, and this convention has been widely accepted. Some means of distinguishing other holors is definitely needed. In particular, a distinctive symbol should be established for matrices, though no such standard seems to have been fixed. Schouten[27] invented a complicated notation for holors in 1914. Struik[28] developed another ingenious system in 1922.

A quite different nomenclature for holors developed from invariance theory. A quadratic form is usually written as

$$A = \sum_{1}^{n} a_{ij} x^i x^j,$$

where $x^1, x^2, \ldots, x^n$ are real numbers, x^i is a vector in n-space, and a_{ij} is a matrix. Such expressions were used by Riemann in his *Habilitationsschrift* (1854) and were incorporated into Ricci's work. Thus, tensor analysis was printed in ordinary type, and the kind of quantity was distinguished by the number and position of the indices. The scheme was further developed by Levi-Civita, Schouten and Struik,[29] Veblen, Weyl, Cartan, Eisenhart, T. Y. Thomas, and many others, so that it may now be considered as a universal notation for all holors.*

Einstein introduced the summation convention (1916), which has become an important characteristic of modern index notation. Whereas Ricci always wrote a summation with the customary Σ, as

$$\Sigma_{rs}\, a_{rs}\, dx_r\, dx_s.$$

Einstein omitted the summation sign and wrote

$$a_{rs}\, dx^r\, dx^s.$$

He says,[24] "A glance at the equations of this paragraph shows that there is always a summation with respect to the indices which occur twice under a sign of summation...and only with respect to indices which occur twice. It is therefore possible, without loss of clearness, to omit the sign of summation. In its place we introduce the convention: If an index occurs twice in one term of an expression, it is always to be summed unless the contrary is expressly stated." This apparently trivial change had an astonishing effect on the simplicity and power of index notation.†

* Note that both Schouten and Struik abandoned their "direct" representations after 1922 and employed the standard tensor notation.

† The quotation ignores the important distinction between subscripts and superscripts. A more modern treatment is given in Chapter 2.

0.09. Derivatives

Derivatives of holors are frequently encountered. For example, a contravariant vector transforms as

$$v^{i'} = \frac{\partial x^{i'}}{\partial x^i} v^i$$

and a covariant vector as

$$u_{i'} = \frac{\partial x^i}{\partial x^{i'}} u_i.$$

For a metric space, Riemann (1854) wrote

$$(ds)^2 = g_{ij}\, dx^i\, dx^j.$$

He also used geodesics defined by the second-order differential equation,

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

Christoffel[30] (1869) introduced the Christoffel symbols which were defined in terms of the metric coefficients g_{ij}.

In 1917, Levi-Civita[31] extended the idea of parallelism to a Riemannian n-space. The covariant derivative appeared, and the limitations of Riemann and Christoffel were removed to give a more general treatment. The linear connection Γ^i_{jk} is defined as a holor whose transformation equation is[30]

$$\Gamma^i_{jk} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{k'}}{\partial x^k} \Gamma^{i'}_{j'k'} + \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x^{i'}}.$$

The linear connection may be considered as the sum of a symmetric and an antisymmetric part:

$$\Gamma^i_{jk} = S^i_{jk} + \Omega^i_{jk}.$$

The covariant derivative employs a symbolism used by Schouten,[32]

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} + \Gamma^i_{jk} v^k.$$

Also noteworthy are the Riemann–Christoffel tensor,

$$R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^m_{jl} \Gamma^k_{im} - \Gamma^m_{il} \Gamma^k_{jm},$$

and the Ricci tensors,

$$R_{ij} = R^l_{ijl} \quad \text{and} \quad R_{ij} = R^l_{lij}.$$

These curvature tensors are very important in Einstein's general relativity.

0.10. Outline of the book

Our book begins with three chapters on holor notation and holor algebra. At this stage, we are interested in index notation but not at all in transformation properties. Thus, we consider addition of holors as well as uncontracted and contracted multiplication.

Part II introduces coordinate transformations and defines tensors, akinetors, and geometric objects.

Part III introduces holor calculus in a very general form in which the linear connection is entirely arbitrary except for its transformation equation. Note that the keystone of tensor calculus is a linear connection holor that is itself not a tensor.

Part IV depends on the introduction of a metric. It thus includes the important subjects of Riemannian spaces and the special case of Euclidean space.