

# *Theory of holors*

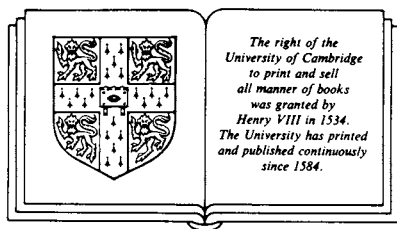
## *A generalization of tensors*

PARRY MOON

*Massachusetts Institute of Technology*

DOMINA EBERLE SPENCER

*University of Connecticut*



CAMBRIDGE UNIVERSITY PRESS

*Cambridge*

*London New York New Rochelle*

*Melbourne Sydney*

Published by the Press Syndicate of the University of Cambridge  
The Pitt Building, Trumpington Street, Cambridge CB2 1RP  
32 East 57th Street, New York, NY 10022, USA  
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

© Cambridge University Press 1986

First published 1986

*Library of Congress Cataloging in Publication Data*

Moon, Parry Hiram, 1898-

Theory of holors.

Bibliography: p.

Includes index.

1. Calculus of tensors. I. Spencer, Domina Eberle,

1920- II. Title. III. Title: Holors.

QA433.M66 1986 515'.63 85-6657

ISBN 0 521 24585 0

*British Library of Congress Cataloguing in Publication applied for.*

Transferred to digital printing 2003

# Contents

<i>Preface</i>	<i>page</i> xiii
<i>Nomenclature</i>	xv
<b>0 Historical introduction</b>	1
0.01 Quaternions	1
0.02 Linear associative algebras	2
0.03 Matrices	3
0.04 Grassmann	3
0.05 Vector analysis	4
0.06 Invariance	4
0.07 Tensors	5
0.08 Holor notation	6
0.09 Derivatives	7
0.10 Outline of the book	8
<b>I Holors</b>	
<b>1 Index notation</b>	11
1.01 Variety of holors	12
1.02 Notation	14
1.03 Conventions	15
1.04 The transpose	18
1.05 Symmetry and antisymmetry	19
1.06 Diagonal holors	21
1.07 Generalized Kronecker deltas	22
1.08 Summary	26
Problems	27

<b>2 Holor algebra</b>	30
2.01 Equality of holors	30
2.02 Addition of holors	32
2.03 The uncontracted product	36
2.04 The contracted product	40
2.05 Matrices	43
2.06 Index balance	45
2.07 Unit holors	47
2.08 The inverse	49
2.09 General conditions for an inverse	55
2.10 Null products	57
2.11 Cancellation rule for products	59
2.12 Summary	59
Problems	61
<b>3 Gamma products</b>	65
3.01 Change of index positions	66
3.02 Change of valence	67
3.03 Symmetric and antisymmetric matrices	69
3.04 Products of univalent holors	72
3.05 Nilvalent products	75
3.06 Univalent products	76
3.07 Alternating currents	78
3.08 Bivalent products	80
3.09 Other products	81
3.10 Summary	83
Problems	83
<b>II Transformations</b>	
<b>4 Tensors</b>	89
4.01 Linear transformations	89
4.02 Invariants	91
4.03 Groups	94
4.04 General transformations	95
4.05 Geometric objects	97
4.06 Univalent tensors	99
4.07 Other tensors	100
4.08 Tensor algebra	105
4.09 Tensor equations	108
4.10 Tensor character	111

4.11	Tensors with fixed merates	114
4.12	Summary	117
	Problems	119
<b>5</b>	<b>Akinetors</b>	<b>126</b>
5.01	Definitions	128
5.02	Algebra	129
5.03	The akinetor as generalization of the tensor	131
5.04	Combinations	134
5.05	Derivatives	136
5.06	Combinations of derivatives	139
5.07	Pseudodivergence	141
5.08	Pseudocurl	143
5.09	Other akinetors	145
5.10	Summary	148
	Problems	152
<b>6</b>	<b>Geometric spaces</b>	<b>156</b>
6.01	Arithmetic space	157
6.02	Invariants	158
6.03	Transformations	160
6.04	Affine space	162
6.05	Parting	165
6.06	Ratios of areas and of volumes	167
6.07	Second-degree surfaces	168
6.08	Axonometric space	170
6.09	Homogeneous coordinates	178
6.10	The projective transformation	180
6.11	Projective space	181
6.12	Perspective space	189
6.13	Inversions	193
6.14	Homogeneous coordinates in affine space	195
6.15	Bivalent tensors in affine space	197
6.16	Summary	203
	Problems	203

### III Holor calculus

<b>7</b>	<b>The linear connection</b>	<b>211</b>
7.01	The linear connection	213
7.02	The covariant derivatives of a contravariant vector	215
7.03	The covariant derivatives of a covariant vector	218

7.04	The covariant derivatives of bivalent tensors	219
7.05	Covariant derivatives of tensors of valence $m$	222
7.06	Contracted covariant derivatives	229
7.07	Derivatives with respect to a scalar parameter	232
7.08	The covariant derivatives of akinetors	234
7.09	Summary	238
	Problems	239
<b>8</b>	<b>The Riemann–Christoffel tensors</b>	242
8.01	Second covariant derivatives of scalars	242
8.02	Second covariant derivatives of contravariant vectors	243
8.03	Second covariant derivatives of covariant vectors	246
8.04	Second covariant derivatives of tensors	246
8.05	Second covariant derivatives of nilvalent akinetors	248
8.06	Second covariant derivatives of akinetors	249
8.07	Integrability of scalar fields	251
8.08	Integrability of contravariant vector fields	253
8.09	Integrability of covariant vector fields	256
8.10	Integrability of tensor fields	258
8.11	Integrability of nilvalent akinetor fields	260
8.12	Integrability of akinetor fields	262
8.13	Properties of the Riemann–Christoffel tensors	264
8.14	The Bianchi identities	266
8.15	Summary	268
	Problems	270
<b>IV Space structure</b>		
<b>9</b>	<b>Non-Riemannian spaces</b>	275
9.01	The absolute differential	275
9.02	Homeomorphic displacements	279
9.03	The infinitesimal pentagon	282
9.04	Linearly connected spaces	285
9.05	Equations of paths	288
9.06	Parameter transformation	291
9.07	Weyl space	292
9.08	Projectively connected space	294
9.09	Projective invariants	296
9.10	Summary	297
	Problems	297

<b>10 Riemannian space</b>	300
10.01 The metric tensor	301
10.02 Modifications	304
10.03 The linear connection	308
10.04 Geodesics	314
10.05 Curvature	318
10.06 Homeomorphic transport of a vector	320
10.07 Volume and area	324
10.08 Scalar and vector products	326
10.09 Gradient, divergence, and curl	331
10.10 Theorems of Gauss and Stokes	336
10.11 The scalar and vector Laplacians	337
10.12 The contracted covariant derivative of a bivalent tensor	340
10.13 Summary	341
Problems	342
<b>11 Euclidean space</b>	346
11.01 Definition of Euclidean space	347
11.02 Metric coefficients	348
11.03 The linear connection	351
11.04 Merates and components	351
11.05 Euclidean $n$ -space	353
11.06 Skew coordinates	354
11.07 Euclidean 3-space	358
11.08 Index notation versus vector notation	360
11.09 Sliding vectors	361
11.10 Sums of sliding vectors	365
11.11 Summary	369
Problems	370
<i>References</i>	376
<i>Index</i>	387

# 0

## *Historical introduction*

The birth of the holor concept may be set at 1673, when John Wallis suggested the geometric representation of complex numbers by points in a plane.<sup>1</sup> A complex number is an entity consisting of two parts. It is usually written as

$$z = x + iy,$$

where  $x$  and  $y$  are real numbers and  $i$  emphasizes the fact that  $x$  and  $y$  are independent quantities. A more modern notation writes a two-element holor as an ordered number pair,

$$z = (x, y)$$

or, in index notation,

$$x^i = (x^1, x^2).$$

This constitutes the first step in a fascinating mathematical development that now includes vectors, matrices, tensors, and other holors.

### **0.01. Quaternions**

Mathematicians expected that the extension from the complex number, with  $n = 2$  to  $n = 3$ , would be child's play; but considerable time elapsed before they found that no such extension is possible without violating a rule of ordinary algebra.<sup>2</sup> After wrestling with the problem for 15 years, Sir William Rowan Hamilton in 1843 found a holor with four elements,

$$x^i = (x^1, x^2, x^3, x^4),$$

which he called a *quaternion*.<sup>3</sup> The ordinary rules of algebra hold for quaternions except that multiplication is noncommutative. Hamilton



believed that quaternions constituted his greatest contribution to mathematics and that most of physics would eventually be written in quaternion form. Tait<sup>4</sup> devoted himself to that end. But physicists were not convinced. For instance, Maxwell<sup>5</sup> in his celebrated treatise (1873) mentioned quaternions several times but was careful not to use them.

### 0.02. Linear associative algebras

The work of Hamilton should certainly have suggested the possibility of a host of new algebras in which associative or commutative properties are abandoned. This possibility, however, was not recognized until 1870, when Benjamin Peirce<sup>6</sup> invented a whole new branch of mathematics – the study of *linear associative algebras*. The subject became popular, and hundreds of algebras\* were developed with their associated honors.<sup>7</sup> One would expect that a few of these algebras would have had interesting practical applications, but such does not seem to be the case. The reason for this disappointing result will now be explained.

Consider holors such as

$$x^i = (x^1, x^2, \dots, x^n),$$

$$y^i = (y^1, y^2, \dots, y^n),$$

and two operations, which we shall call addition and multiplication. The universally accepted definition of addition is

$$x^i + y^i = (x^1 + y^1, x^2 + y^2, \dots, x^n + y^n).$$

It appears that nothing is to be gained by changing the definition of addition. But multiplication is another story! We are at liberty to select a multiplication table, and each table determines a new algebra with its own peculiar properties.

A linear associative algebra is a closed system: a sum or product is always a holor of the same system. For instance, the product of two quaternions is always a quaternion, the product of two matrices is always a matrix. This is a delightful property, closely associated with groups, rings, and other aspects of modern algebra. But it is not the kind of behavior that is needed for most physical applications. Experience shows that the product of two vectors, for instance, must sometimes be a vector, sometimes a scalar, sometimes a matrix. And this variety cannot

\* This includes algebras of Dedekind, Frobenius, Scheffers, Peirce, Kronecker, Weierstrass, Dickson, Sylvester, and Cartan.

be obtained with any of the linear associative algebras. Here is the reason that this branch of mathematics, once pursued with such enthusiasm, has proved to be of limited value as regards physical applications.

### 0.03. Matrices

We have been considering *univalent* holors, such as

$$x^i = (x^1, x^2, \dots, x^n).$$

Also important are *bivalent* holors (matrices), for instance,

$$A^{ij} = \begin{bmatrix} A^{11} & A^{12} & \dots & A^{1n} \\ A^{21} & A^{22} & \dots & A^{2n} \\ \cdot & \cdot & \dots & \cdot \\ A^{n1} & A^{n2} & \dots & A^{nn} \end{bmatrix}.$$

The theory of matrices\* was initiated by Arthur Cayley<sup>8</sup> in 1855. The only product ordinarily used is the Cayley product, which in index notation is

$$A^{ij} B_{jk} = C_k^i.$$

This is an associative but noncommutative product. Thus, matrix algebra<sup>9</sup> may be regarded as one of the linear associative algebras.

### 0.04. Grassmann

In the same year (1844) that Hamilton published his first paper on quaternions, a much more general treatment of holors was published by Herman Grassmann.<sup>10</sup> Not only did *Die Ausdehnungslehre* deal with  $n$ -space but, unlike the linear associative algebras, it was not limited to a single product per algebra. Thus it opened up new possibilities, particularly in geometric and physical applications of holors. Hamilton's comment was typical<sup>11</sup>: "I have recently been reading... more than a hundred pages of Grassmann's *Ausdehnungslehre*, with great admiration and interest... But it is curious to see how narrowly, yet how completely, Grassmann failed to hit off the quaternions." As if Grassmann, with his immensely more general treatment, would be particularly interested in the special case of the quaternion!

\* The name "matrices" was first used by J. J. Sylvester, *Collected math. papers*, Vol. 1 (Cambridge University Press, Cambridge, 1904), p. 145.

### 0.05. Vector analysis

To other mathematicians and physicists at this time, the *Ausdehnungslehre* seemed quite incomprehensible; and quaternions, based on Hamilton's 800-page explanation, seemed hardly simpler. It was not until 40 years later that Williard Gibbs<sup>12</sup> (and independently, Oliver Heaviside)<sup>13</sup> worked out a special case of quaternions called vector analysis. This was based largely on Hamilton's pioneer work and even used Hamilton's terms *scalar* and *vector*, but it applied only to 3-space and was usually expressed in rectangular coordinates. It did have two distinct products, however, and it proved to be just what was needed in a large number of physical applications. As a consequence, vector analysis has been universally accepted and is widely used today.<sup>14</sup>

### 0.06. Invariance

If one reads an early treatment of vectors, such as that of Gibbs or Coffin,<sup>15</sup> he is impressed by the exclusive use of rectangular Cartesian coordinates. Even such concepts as divergence, curl, and scalar and vector Laplacians<sup>16</sup> are usually defined in rectangular coordinates. In a way, this procedure makes for simplicity. But it ignores the subject of invariance. Even with Gibbs, a vector was more than a set of numbers representing rectangular coordinates: it was a geometric entity that could be expressed equally well in other coordinate systems.<sup>12</sup> The merates changed when the coordinates were changed, but the geometric object maintained its identity.

Invariants were studied by Cayley<sup>17</sup> beginning in 1841. The subject received such enthusiastic support during the latter half of the nineteenth century that Sylvester<sup>18</sup> remarked in 1864, "As all roads lead to Rome, so I find in my own case at least that all algebraic inquiries, sooner or later, end at the Capitol of modern algebra over whose shining portal is inscribed the *Theory of Invariants*." The subject is treated in a voluminous literature.<sup>19</sup>

An important application of invariant theory is differential geometry. Riemann<sup>20</sup> outlined an approach in his *Habilitationsschrift* (Göttingen, 1854). Felix Klein,<sup>21</sup> in his Erlanger program, stated a principle that affected the development of geometry for many years. Christoffel, Beltrami, Bianchi, and others made valuable contributions. Ricci<sup>22</sup> developed a new notation, which he called "the absolute differential calculus" but which

is now called tensor\* theory. Ricci wrote many papers, including a celebrated one with Levi-Civita<sup>23</sup> (1901).

Physicists in general were not interested in invariants and had never heard of Ricci's work. It was not until 1916 that Albert Einstein<sup>24</sup> applied tensors in formulating his general relativity. Because of the extraordinary popular appeal of this theory, tensors became famous overnight. Hundreds of books appeared on relativity and several on tensors.<sup>25</sup> And even the old subject of vectors took on new life when presented in tensor form.<sup>14</sup>

### 0.07. Tensors

Index notation was designed particularly to handle behavior under coordinate transformation. But the basic notation is applicable even when no coordinate transformations are involved. Thus, index notation can be employed advantageously to unify a great field of holors of various valences and dimensionalities, even where the question of invariance does not necessarily enter. We have seen how vector analysis originated with little thought for coordinate transformation. The same may be said for quaternions, matrices, and the holors treated in linear associative algebras.

Since 1916, however, the feeling seems to be prevalent that the only hypernumbers of any value are tensors. On the contrary, many valuable applications of holors do not need a consideration of coordinate transformation. The simplest application of holors specifies a collection of discrete objects – for instance, machine screws. Each merate gives the number of screws of a particular type. The complete holor designates the inventory of a particular tool room (as regards screws in stock). Transformation of coordinates is meaningless here. Another example has merates that specify the loop currents in a given  $l$ -loop electric network. Again, transformation does not enter. And there are examples in which transformation is possible but not of much interest. Thus, we have a host of practical applications of holors where the holor is definitely not a tensor. And we have, of course, a host of other applications where the holor is a tensor.

\* The name *tensor* is often said to be the invention of Woldemar Voigt,<sup>26</sup> who used it to represent a bivector in crystal physics. The word, however, is much older, having been applied in connection with quaternions by Hamilton. See *The Mathematical papers of Sir William Rowan Hamilton* (Cambridge University Press), p. 237.

### 0.08. Holor notation

Heaviside<sup>13</sup> suggested boldfaced type to distinguish vectors from scalars, and this convention has been widely accepted. Some means of distinguishing other holors is definitely needed. In particular, a distinctive symbol should be established for matrices, though no such standard seems to have been fixed. Schouten<sup>27</sup> invented a complicated notation for holors in 1914. Struik<sup>28</sup> developed another ingenious system in 1922.

A quite different nomenclature for holors developed from invariance theory. A quadratic form is usually written as

$$A = \sum_1^n a_{ij} x^i x^j,$$

where  $x^1, x^2, \dots, x^n$  are real numbers,  $x^i$  is a vector in  $n$ -space, and  $a_{ij}$  is a matrix. Such expressions were used by Riemann in his *Habilitationschrift* (1854) and were incorporated into Ricci's work. Thus, tensor analysis was printed in ordinary type, and the kind of quantity was distinguished by the number and position of the indices. The scheme was further developed by Levi-Civita, Schouten and Struik,<sup>29</sup> Veblen, Weyl, Cartan, Eisenhart, T. Y. Thomas, and many others, so that it may now be considered as a universal notation for all holors.\*

Einstein introduced the summation convention (1916), which has become an important characteristic of modern index notation. Whereas Ricci always wrote a summation with the customary  $\Sigma$ , as

$$\Sigma_{rs} a_{rs} dx_r dx_s.$$

Einstein omitted the summation sign and wrote

$$a_{rs} dx^r dx^s.$$

He says,<sup>24</sup> "A glance at the equations of this paragraph shows that there is always a summation with respect to the indices which occur twice under a sign of summation. . . and only with respect to indices which occur twice. It is therefore possible, without loss of clearness, to omit the sign of summation. In its place we introduce the convention: If an index occurs twice in one term of an expression, it is always to be summed unless the contrary is expressly stated." This apparently trivial change had an astonishing effect on the simplicity and power of index notation.<sup>†</sup>

\* Note that both Schouten and Struik abandoned their "direct" representations after 1922 and employed the standard tensor notation.

† The quotation ignores the important distinction between subscripts and superscripts. A more modern treatment is given in Chapter 2.

**0.09. Derivatives**

Derivatives of holors are frequently encountered. For example, a contravariant vector transforms as

$$v^{i'} = \frac{\partial x^{i'}}{\partial x^i} v^i$$

and a covariant vector as

$$u_{i'} = \frac{\partial x^i}{\partial x^{i'}} u_i.$$

For a metric space, Riemann (1854) wrote

$$(ds)^2 = g_{ij} dx^i dx^j.$$

He also used geodesics defined by the second-order differential equation,

$$\frac{d^2 x^i}{ds^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

Christoffel<sup>30</sup> (1869) introduced the Christoffel symbols which were defined in terms of the metric coefficients  $g_{ij}$ .

In 1917, Levi-Civita<sup>31</sup> extended the idea of parallelism to a Riemannian  $n$ -space. The covariant derivative appeared, and the limitations of Riemann and Christoffel were removed to give a more general treatment. The linear connection  $\Gamma_{jk}^i$  is defined as a holor whose transformation equation is<sup>30</sup>

$$\Gamma_{jk}^i = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{k'}}{\partial x^k} \Gamma_{j'k'}^{i'} + \frac{\partial^2 x^{i'}}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x^{i'}}.$$

The linear connection may be considered as the sum of a symmetric and an antisymmetric part:

$$\Gamma_{jk}^i = S_{jk}^i + \Omega_{jk}^i.$$

The covariant derivative employs a symbolism used by Schouten,<sup>32</sup>

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} + \Gamma_{jk}^i v^k.$$

Also noteworthy are the Riemann-Christoffel tensor,

$$R_{ijl}^k = \frac{\partial \Gamma_{jl}^k}{\partial x^i} - \frac{\partial \Gamma_{il}^k}{\partial x^j} + \Gamma_{jl}^m \Gamma_{im}^k - \Gamma_{il}^m \Gamma_{jm}^k,$$

and the Ricci tensors,

$$R_{ij} = R'_{ijl} \quad \text{and} \quad R_{ij} = R'_{lij}.$$

These curvature tensors are very important in Einstein's general relativity.

### **0.10. Outline of the book**

Our book begins with three chapters on holor notation and holor algebra. At this stage, we are interested in index notation but not at all in transformation properties. Thus, we consider addition of holors as well as uncontracted and contracted multiplication.

Part II introduces coordinate transformations and defines tensors, akinetors, and geometric objects.

Part III introduces holor calculus in a very general form in which the linear connection is entirely arbitrary except for its transformation equation. Note that the keystone of tensor calculus is a linear connection holor that is itself not a tensor.

Part IV depends on the introduction of a metric. It thus includes the important subjects of Riemannian spaces and the special case of Euclidean space.