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THE LEBESGUE INTEGRAL

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PREFACE

My aim is to give an account of the theory of integration due to Lebesgue in a form which may appeal to those who have no wish to plumb the depths of the theory of real functions. There is no novelty of treatment in this tract; the presentation is essentially that of Lebesgue himself. The groundwork in analysis and calculus with which the reader is assumed to be acquainted is, roughly, what is in Hardy's *A Course of Pure Mathematics*.

It has long been clear that anyone who uses the integral calculus in the course of his work, whether it be in pure or applied mathematics, should normally interpret integration in the Lebesgue sense. A few simple principles then govern the manipulation of expressions containing integrals.

To appreciate this general remark, the reader is asked to turn to p. 42; calculations such as are contained in Examples 4–8 might confront anyone having to carry through a mathematical argument. Consider in more detail Example 4; the result has the *look* of being right rather than wrong, but the limiting process involved is by no means simple, and the justification of it without an appeal to Lebesgue's principles would be tiresome. Anyone with a grasp of these principles will see that the easily proved fact, that $(1-t/n)^n$ *increases* to its limit e^{-t}, ensures the validity of the passage to the limit.

The attitude which the working mathematician may take towards the more general concepts of integration has been expressed by Hardy, Littlewood and Pólya in *Inequalities*. After dealing with inequalities between finite sets of numbers and extending them to infinite series, they turn to inequalities between integrals and begin Chapter VI with these preliminary remarks on Lebesgue integrals:

> The integrals considered in this chapter are Lebesgue integrals, except in §§ 6·15–6·22, where we are concerned with Stieltjes integrals. It may be convenient that we should state here how much knowledge of the theory we assume. This is for the most part very little, and all that the reader usually

needs to know is that there is *some* definition of an integral which possesses the properties specified below. There are naturally many of our theorems which remain significant and true with the older definitions, but the subject becomes *easier*, as well as more comprehensive, if the definitions presupposed have the proper degree of generality.

Since Lebesgue's original exposition a number of different approaches to the theory have been discovered, some of them having attractions of simplicity or generality. It is possible to arrive quickly at the integral without any stress on the idea of measure. I believe, however, that there is an ultimate gain in knowing the outlines of the theory of measure, and I have developed this first in as intuitive a way as possible.

During several years of lecturing on this topic I must have adopted ideas from so many of the books and papers on it that detailed acknowledgement would now be difficult. My greatest debts are to the classical books of de la Vallée Poussin, Carathéodory and Saks, and the straightforward account (having a similar scope to this) given by Titchmarsh in his *Theory of Functions*. I also wish to record that one of my many debts to G. H. Hardy lay in his encouragement to write this tract.

J. C. B.

September, 1949

Reprinting has allowed me to put some details into § 2·2 which had been left to the reader. The first paragraph of § 2·7 mentioning the role of an axiom of choice in the Lebesgue theory has been recast. I might have helped the reader more by discussing this axiom at its first appearance—on p. 3, in enumerating the sets E_m. To do this now would disturb the type too much, and I can help him most by urging him to read an account of the foundations of the subject such as is given in the books specified on p. 87.

There are other less important alterations.

I thank Mr Ingham and Professor Besicovitch for constructive criticism.

J. C. B.

June, 1958

CONTENTS