

Cambridge University Press
052160480X - The Lebesgue Integral
J. C. Burkill
Frontmatter
[More information](#)

**Cambridge Tracts in Mathematics
and Mathematical Physics**

GENERAL EDITORS

H. BASS, J. F. C. KINGMAN, F. SMITHIES
J. A. TODD AND C. T. C. WALL

No. 40

THE LEBESGUE INTEGRAL

Cambridge University Press
052160480X - The Lebesgue Integral
J. C. Burkill
Frontmatter
[More information](#)

THE LEBESGUE INTEGRAL

BY
J. C. BURKILL
Fellow of Peterhouse, Cambridge

CAMBRIDGE UNIVERSITY PRESS
CAMBRIDGE
LONDON · NEW YORK · MELBOURNE

Cambridge University Press
052160480X - The Lebesgue Integral
J. C. Burkill
Frontmatter
[More information](#)

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

<http://www.cambridge.org>

© Cambridge University Press 1951

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 1951
Reprinted 1953 1958 1961 1963 1965 1971 1975
First paperback edition 2004

A catalogue record for this book is available from the British Library

ISBN 0 521 04382 4 hardback
ISBN 0 521 60480 X paperback

Cambridge University Press
052160480X - The Lebesgue Integral
J. C. Burkill
Frontmatter
[More information](#)

PREFACE

My aim is to give an account of the theory of integration due to Lebesgue in a form which may appeal to those who have no wish to plumb the depths of the theory of real functions. There is no novelty of treatment in this tract; the presentation is essentially that of Lebesgue himself. The groundwork in analysis and calculus with which the reader is assumed to be acquainted is, roughly, what is in Hardy's *A Course of Pure Mathematics*.

It has long been clear that anyone who uses the integral calculus in the course of his work, whether it be in pure or applied mathematics, should normally interpret integration in the Lebesgue sense. A few simple principles then govern the manipulation of expressions containing integrals.

To appreciate this general remark, the reader is asked to turn to p. 42; calculations such as are contained in Examples 4–8 might confront anyone having to carry through a mathematical argument. Consider in more detail Example 4; the result has the *look* of being right rather than wrong, but the limiting process involved is by no means simple, and the justification of it without an appeal to Lebesgue's principles would be tiresome. Anyone with a grasp of these principles will see that the easily proved fact, that $(1 - t/n)^n$ increases to its limit e^{-t} , ensures the validity of the passage to the limit.

The attitude which the working mathematician may take towards the more general concepts of integration has been expressed by Hardy, Littlewood and Pólya in *Inequalities*. After dealing with inequalities between finite sets of numbers and extending them to infinite series, they turn to inequalities between integrals and begin Chapter VI with these preliminary remarks on Lebesgue integrals:

The integrals considered in this chapter are Lebesgue integrals, except in §§ 6.15–6.22, where we are concerned with Stieltjes integrals. It may be convenient that we should state here how much knowledge of the theory we assume. This is for the most part very little, and all that the reader usually

Cambridge University Press
 052160480X - The Lebesgue Integral
 J. C. Burkill
 Frontmatter
[More information](#)

vi

P R E F A C E

needs to know is that there is *some* definition of an integral which possesses the properties specified below. There are naturally many of our theorems which remain significant and true with the older definitions, but the subject becomes *easier*, as well as more comprehensive, if the definitions presupposed have the proper degree of generality.

Since Lebesgue's original exposition a number of different approaches to the theory have been discovered, some of them having attractions of simplicity or generality. It is possible to arrive quickly at the integral without any stress on the idea of measure. I believe, however, that there is an ultimate gain in knowing the outlines of the theory of measure, and I have developed this first in as intuitive a way as possible.

During several years of lecturing on this topic I must have adopted ideas from so many of the books and papers on it that detailed acknowledgement would now be difficult. My greatest debts are to the classical books of de la Vallée Poussin, Carathéodory and Saks, and the straightforward account (having a similar scope to this) given by Titchmarsh in his *Theory of Functions*. I also wish to record that one of my many debts to G. H. Hardy lay in his encouragement to write this tract.

J. C. B.

September, 1949

Reprinting has allowed me to put some details into § 2.2 which had been left to the reader. The first paragraph of § 2.7 mentioning the role of an axiom of choice in the Lebesgue theory has been recast. I might have helped the reader more by discussing this axiom at its first appearance—on p. 3. in enumerating the sets E_m . To do this now would disturb the type too much, and I can help him most by urging him to read an account of the foundations of the subject such as is given in the books specified on p. 87.

There are other less important alterations.

I thank Mr Ingham and Professor Besicovitch for constructive criticism.

J. C. B.

June, 1958

CONTENTS

<i>Art.</i>	<i>Page</i>
<i>Author's Preface</i>	v
 <i>Chapter I. SETS OF POINTS</i> 	
1·1 The algebra of sets	1
1·2 Infinite sets	3
1·3 Sets of points. Descriptive properties	4
1·4 Covering theorems	6
1·5 Plane sets	7
 <i>Chapter II. MEASURE</i> 	
2·1 Measure	10
2·2 Measure of open sets	10
2·3 Measure of closed sets	11
2·4 Open and closed sets	12
2·5 Outer and inner measure. Measurable sets	13
2·6 The additive property of measure	14
2·7 Non-measurable sets	15
2·8 Further properties of measure	16
2·9 Sequences of sets	18
2·10 Plane measure	21
2·11 Measurability in the sense of Borel	23
2·12 Measurable functions	23
 <i>Chapter III. THE LEBESGUE INTEGRAL</i> 	
3·1 The Lebesgue integral	26
3·2 The Riemann integral	27
3·3 The scope of Lebesgue's definition	28
3·4 The integral as the limit of approximative sums	30
3·5 The integral of an unbounded function	31
3·6 The integral over an infinite range	33

viii	CONTENTS	
<i>Art.</i>		<i>Page</i>
3·7	Simple properties of the integral	34
3·8	Sets of measure zero	37
3·9	Sequences of integrals of positive functions	38
3·10	Sequences of integrals (integration term by term)	40
 <i>Chapter IV. DIFFERENTIATION AND INTEGRATION</i> 		
4·1	Differentiation and integration as inverse processes	44
4·2	The derivatives of a function	44
4·3	Vitali's covering theorem	46
4·4	Differentiability of a monotonic function	48
4·5	The integral of the derivative of an increasing function	49
4·6	Functions of bounded variation	50
4·7	Differentiation of the indefinite integral	52
4·8	Absolutely continuous functions	54
 <i>Chapter V. FURTHER PROPERTIES OF THE INTEGRAL</i> 		
5·1	Integration by parts	58
5·2	Change of variable	58
5·3	Multiple integrals	61
5·4	Fubini's theorem	63
5·5	Differentiation of multiple integrals	65
5·6	The class L^p	65
5·7	The metric space L^p	67
 <i>Chapter VI. THE LEBESGUE-STIELTJES INTEGRAL</i> 		
6·1	Integration with respect to a function	70
6·2	The variation of an increasing function	71
6·3	The Lebesgue-Stieltjes integral	72
6·4	Integration by parts	75
6·5	Change of variable. Second mean-value theorem	77
	Solutions of some examples	80