KINETIC THEORY IN THE EXPANDING UNIVERSE

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Contents

1

Tools: relativity

In this section we introduce some results from the general theory of relativity. More details can be found, for example, in Weinberg (1972).

The fundamental quantity in the theory is the metric tensor $g_{\mu\nu}$. In fixing it we adopt the usual point of view in cosmology that local metric irregularities due to stars and galaxies, and the like, are ignorable in the large. This is canonized into what is known as the "cosmological principle"; i.e., that the universe, taken on average, is and always has been homogeneous and isotropic. That is to say, at any epoch the universe appears the same in all spatial directions when observed from any spatial point. The first person to use this principle to derive, in a mathematically satisfactory way, the form of the metric tensor was H. P. Robertson (Robertson, 1929). He showed, in units in which $\hbar = c = 1$, that

$$ds^2 \equiv -g_{\mu\nu}\,dx^\mu\,dx^\nu = dt^2 - e^{-f}h_{ij}\,dx^i\,dx^j. \qquad (1.1)$$

Here the h_{ij} are functions of the spatial variables x^1, x^2, x^3 and f is an arbitrary, real, function of the time t. Robertson gives as an example the case in which

$$f = t \times \text{constant},$$

which is what we now call the de Sitter inflationary universe (de Sitter, 1917). This, as we shall shortly show, is one of the two cases in which the curvature of the universe remains constant in time. The other case is the so-called Einstein cosmology (Einstein, 1917) in which

$$f = 0.$$

Introducing polar coordinates, as Robertson did, we can rewrite (1.1) in the now familiar form

$$ds^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2\right). \qquad (1.2)$$

The scales of R and r can be chosen so that $k = 0, \pm 1$. Some general relativity formulae will be useful. In terms of the metric tensor, $g^{\mu\nu}$, the covariant derivatives of an arbitrary vector field A_ν or $A^\mu = g^{\mu\nu}A_\nu$, where $g^{\mu\nu}$ is given by the relation

$$g^{\mu\nu}g_{\lambda\nu} = \delta^\mu_\lambda, \tag{1.3}$$

are defined by the equations

$$A^\nu{}_{;\mu} = A^\nu{}_{,\mu} + \Gamma^\nu_{\mu\lambda}A^\lambda, \tag{1.4}$$

and

$$A_{\nu;\mu} = A_{\nu,\mu} - \Gamma^\lambda_{\mu\nu}A_\lambda, \tag{1.5}$$

where, as is customary, the comma denotes the ordinary derivative and $\Gamma^\mu_{\nu\sigma}$ is the Christoffel symbol defined in terms of the metric by

$$\Gamma^\mu_{\nu\sigma} = \tfrac{1}{2}(g_{\lambda\sigma,\nu} + g_{\nu\lambda,\sigma} - g_{\nu\sigma,\lambda})g^{\lambda\mu}. \tag{1.6}$$

For the metric of (1.2) the $g_{\mu\nu}$ are given by

$$g_{00} = -1, \tag{1.7}$$

$$g_{i0} = 0,$$

and

$$g_{ij} = R^2(t)\hat{g}_{ij}, \tag{1.8}$$

where, in polar coordinates,

$$\hat{g}_{rr} = \frac{1}{1 - kr^2}, \tag{1.9}$$

$$\hat{g}_{\theta\theta} = r^2,$$

$$\hat{g}_{\phi\phi} = r^2 \sin^2\theta,$$

$$\hat{g}_{ij} = 0, \quad i \neq j.$$

We are implicitly using here what are known as "comoving" coordinates. This means that a galaxy, say, is assigned values of r, θ, and ϕ which remain the same as the universe expands. The points on the mesh which define the coordinate grid, expand with the grid. If we call

$$g = -\det g_{\mu\nu} \tag{1.10}$$

then, for our metric

$$g = \frac{R^6 r^4 \sin^2\theta}{1 - kr^2}, \tag{1.11}$$

and the invariant volume element is given by

$$dV = g^{\frac{1}{2}} \, dx_1 \, dx_2 \, dx_3 \, dx_0. \qquad (1.12)$$

Because of the simplicity of the metric the only nonvanishing Christoffel components are

$$\begin{aligned}
\Gamma^0_{i\,j} &= R\dot{R}\hat{g}_{ij} \qquad (1.13)\\
\Gamma^i_{0j} &= \frac{\dot{R}}{R}\,\delta^i_j = \Gamma^i_{j0}\\
\Gamma_{1\,1}^{\,1} &= \frac{kr}{1-kr^2}\\
\Gamma_{2\,2}^{\,1} &= -(1-kr^2)r\\
\Gamma_{3\,3}^{\,1} &= -(1-kr^2)r\sin^2\theta\\
\Gamma_{1\,2}^{\,2} &= \Gamma_{2\,1}^{\,2} = \Gamma_{1\,3}^{\,3} = \Gamma_{3\,1}^{\,3} = 1/r\\
\Gamma_{3\,3}^{\,2} &= -\sin\theta\cos\theta\\
\Gamma_{2\,3}^{\,3} &= \Gamma_{3\,2}^{\,3} = \cot\theta.
\end{aligned}$$

In most of our later work we shall restrict ourselves to the $k = 0$ (spatially flat) case. If we write the spatially flat metric as

$$ds^2 = dt^2 - R^2(t)(dx_1^2 + dx_2^2 + dx_3^2), \qquad (1.14)$$

we have, with this Cartesian choice,

$$\begin{aligned}
g_{00} &= -1 \qquad (1.15)\\
g_{i0} &= 0\\
g_{ij} &= R^2(t)\delta_{ij},
\end{aligned}$$

and

$$\begin{aligned}
\Gamma^0_{ij} &= R\dot{R}\delta_{ij} \qquad (1.16)\\
\Gamma_{0j}^{\;i} &= \dot{R}\delta^i_j/R\\
\Gamma^i_{jk} &= 0.
\end{aligned}$$

Because of the rotational symmetry of the sections of space orthogonal to a given time direction the spatial components of the Riemann–Christoffel tensor $R_{\alpha\beta\gamma\delta}$ take this form in the $k \neq 0$ case:

$$R_{ijkl} = \Lambda(g_{jk}g_{il} - g_{jl}g_{ik}). \qquad (1.17)$$

This satisfies the general symmetry conditions

$$R_{\alpha\beta\nu\delta} = -R_{\alpha\beta\delta\nu} = -R_{\beta\alpha\nu\delta} = R_{\nu\delta\beta\alpha} = R_{\beta\alpha\delta\nu}. \tag{1.18}$$

In (1.17) Λ is a time-dependent spatial scalar. To evaluate it we use the general definition of $R_{\alpha\beta\nu\delta}$

$$R_{\alpha\beta\nu\delta} = \tfrac{1}{2}\left(\frac{\partial^2 g_{\alpha\beta}}{\partial x^\nu \partial x^\delta} - \frac{\partial^2 g_{\beta\nu}}{\partial x^\alpha \partial x^\delta} - \frac{\partial^2 g_{\alpha\delta}}{\partial x^\beta \partial x^\nu} + \frac{\partial^2 g_{\beta\delta}}{\partial x^\alpha \partial x^\nu}\right) + g_{\eta\sigma}(\Gamma^\eta_{\nu\alpha}\Gamma^\sigma_{\beta\delta} - \Gamma^\eta_{\delta\alpha}\Gamma^\sigma_{\beta\nu}). \tag{1.19}$$

Because of the structure of (1.17) any spatial component will do. Thus, for example,

$$R_{1221} = \frac{r^2 R^2}{1 - kr^2}(k + \dot{R}^2), \tag{1.20}$$

from which it follows that

$$\Lambda = \frac{k + \dot{R}^2}{R^2}. \tag{1.21}$$

The Ricci tensor is defined to be

$$\begin{aligned} R_{\mu\kappa} &= g^{\lambda\nu}R_{\lambda\mu\nu\kappa} = R_{\kappa\mu} \\ &= g^{ij}R_{i\mu jk} + g^{00}R_{0\mu 0\kappa}. \end{aligned} \tag{1.22}$$

We begin by evaluating its spatial components. Thus

$$\begin{aligned} R_{lm} &= g^{ij}R_{iljm} + g^{00}R_{0l0m} \\ &= g^{ij}\Lambda\{g_{lj}g_{im} - g_{lm}g_{ij}\} - R_{0l0m} = -2\Lambda g_{lm} - R_{0l0m} \end{aligned} \tag{1.23}$$

As

$$R_{0l0m} = \ddot{R}R\hat{g}_{lm}, \tag{1.24}$$

$$R_{lm} = -\hat{g}_{lm}[2k + 2\dot{R}^2 + \ddot{R}R].$$

On the other hand

$$\begin{aligned} R_{0\alpha} &= g^{\lambda\nu}R_{\lambda 0\nu\alpha} \\ &= g^{ij}R_{i0j\alpha} + g^{00}R_{000\alpha}, \end{aligned} \tag{1.25}$$

from which it follows that

$$R_{00} = g^{ij}R_{i0j0} = 3\ddot{R}/R, \tag{1.26}$$

while

$$R_{0i} = 0. \tag{1.27}$$

The invariant curvature ${}^{(4)}R$ is defined to be

$$\begin{aligned} {}^{(4)}R &= g^{\lambda\nu} g^{\mu\kappa} R_{\lambda\mu\nu\kappa} \\ &= g^{\mu\kappa} R_{\mu\kappa} = g^{00} R_{00} + g^{ij} R_{ij} \\ &= -6\left(\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R}\right). \end{aligned} \tag{1.28}$$

The statement that this curvature remains constant in time leads, with $k = 0$, to the equation

$$\frac{\ddot{R}\dot{R}}{R} - 2\frac{\dot{R}^3}{R^2} + \dddot{R} = 0. \tag{1.29}$$

This equation, as advertised, has only the solutions

$$R = R_0 e^{at} \tag{1.30}$$

and

$$R = R_0, \tag{1.31}$$

which is to say only the Einstein and de Sitter spaces evolve with constant curvature.

The covariant divergence of a contravariant tensor is defined to be

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{g^{\frac{1}{2}}} \frac{\partial}{\partial x^{\mu}} (g^{\frac{1}{2}} T^{\mu\nu}) + \Gamma^{\nu}_{\mu\lambda} T^{\mu\lambda}. \tag{1.32}$$

We may verify, using this definition, that

$$(R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu(4)} R)_{;\mu} = 0. \tag{1.33}$$

We have for a covariant tensor

$$T_{\mu\nu;\lambda} = \frac{\partial T_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\rho}_{\mu\lambda} T_{\nu\rho} - \Gamma^{\rho}_{\nu\lambda} T_{\mu\rho} \tag{1.34}$$

or

$$T_{\mu\nu;\mu} = \frac{\partial T_{\mu\nu}}{\partial x^{\mu}} - \Gamma^{\rho}_{\mu\mu} T_{\nu\rho} - \Gamma^{\rho}_{\nu\mu} T_{\mu\rho}. \tag{1.35}$$

A Killing vector is a four-vector that satisfies the condition

$$A_{\mu;\nu} + A_{\nu;\mu} = 0. \tag{1.36}$$

For any component ν – no sum,

$$A_{\nu;\nu} = 0. \tag{1.37}$$

We can also write the Killing condition as

$$A_{\mu;\nu} + A_{\nu;\mu} = A_{\mu,\nu} + A_{\nu,\mu} - 2\Gamma^{\alpha}_{\mu\nu}A_{\alpha} \qquad (1.38)$$
$$= g_{\mu\rho}A^{\rho}_{,\nu} + g_{\nu\rho}A^{\rho}_{,\mu} + g_{\mu\nu,\rho}A^{\rho} = 0.$$

As we shall see, an important remark for the kinetic theory in a Robertson-Walker expanding universe is, that for nonconstant $R(t)$, there are no nonvanishing *timelike* spatially independent Killing vectors.† To this end, suppose that A^{ρ} were such a vector. We could then find a frame of reference in which the only nonvanishing component is A^0. In this frame (1.38) takes the form

$$g_{\mu 0}A^{0}_{,\nu} + g_{\nu 0}A^{0}_{,\mu} + g_{\mu\nu,0}A^{0} = 0. \qquad (1.39)$$

If we take $\mu = \nu = 1$ then

$$g_{11,0}A^{0} = \frac{1}{1 - kr^2}\frac{\mathrm{d}}{\mathrm{d}t}R^2A^0 = 0. \qquad (1.40)$$

from which the conclusion follows.

We next turn to cosmodynamics.

† The more general statement is that for nonstationary Robertson–Walker matrixes there is no spacelike Killing vector. I am grateful to E. Weinberg for a proof of this theorem. The weaker version given in the text is sufficient for our purposes.