CAMBRIDGE TRACTS IN MATHEMATICS

72. *Completeness and basis properties of sets of special functions*

J. R. HIGGINS
Lecturer in Mathematics
The Cambridgeshire College of Arts and Technology

72. Completeness and basis properties of sets of special functions

CAMBRIDGE UNIVERSITY PRESS
CAMBRIDGE
LONDON · NEW YORK · MELBOURNE

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 1977
First paperback edition 2004

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data
Higgins, John Rowland, 1935–
Completeness and basis properties of sets of special functions.
(Cambridge tracts in mathematics; 72)
Bibliography: p.
Includes indexes.
1. Functions, Special. 2. Series, Orthogonal.
3. Lp spaces. I. Title. II. Series.
QA351.H57 515′.5 76-19630

ISBN 0 521 21376 2 hardback
ISBN 0 521 60488 5 paperback

To Nan and P.B.

Contents

Preface *page* ix

1 FOUNDATIONS 1

1.1 Notes on metric spaces 1
1.2 Notes on the L^p spaces 8
1.3 Orthogonal sequences in Hilbert space 11
1.4 Biorthogonal systems in Hilbert space 19
1.5 Postscript to chapter 1 26

2 ORTHOGONAL SEQUENCES 28

2.1 Complete sequences of polynomials 28
2.2 The Vitali completeness criterion 33
2.3 The Dalzell completeness criterion 39
2.4 The functions of Rademacher, Walsh and Haar 45
2.5 CON sequences and the reproducing kernel 53
2.6 The method of isometric transformation 55
2.7 CON sequences of complex functions 64

3 NON-ORTHOGONAL SEQUENCES 71

3.1 The stability of bases 71
3.2 A complex variable method 84
3.3 Non-orthogonal Fourier–Bessel and Legendre functions 88
3.4 Some theorems of Müntz and Szász 95

4 DIFFERENTIAL AND INTEGRAL OPERATORS 98

4.1 Sturm–Liouville systems 100
4.2 Singular boundary value problems 109
4.3 Integral operators 112

Appendix 1 Supplementary theorems *page* 116

Appendix 2 Definitions of special functions 120

Appendix 3 Some complete sequences of special functions 122

Bibliography 126

Subject index 131

Preface

Some years ago I came across the need for precise information concerning the basis properties of sets of special functions, and the methods available for testing for such properties. This material proved to be rather widely scattered, so I began a collection of notes on the subject which have formed the foundations of the present little book.

I hope that the book will prove useful to graduate students of mathematics, particularly those whose research interests are developing in the direction of bases in Hilbert and Banach spaces: it could bridge the gap that exists between the scant treatment this topic usually receives in standard texts on functional analysis on the one hand, and the rather formidable specialist books such as Marti (1969) and Singer (1970) on the other. There is no harm in having some experience on the practical side of the business before aiming to become managing director!

I hope the book will appeal to workers in other scientific fields as well. An appendix has been included which lists many of the standard results, and this may help to make the book useful as a source of reference.

It is assumed that the reader's education will have included the usual first courses in real variable (including Lebesgue integration) and complex variable. Although a knowledge of functional analysis would be an advantage it is not strictly necessary, and all the principles of functional analysis which are used in the text are listed in an appendix, along with certain other facts which do not fit easily into the presentation.

Introductory sections on metric and L^p spaces have been included, in note form since it is assumed that most readers will already be familiar with this material. These sections are intended to serve as a 'run up' to the main part of the book.

The subject matter touches upon many important topics in both pure and applied mathematics; for example, bases in Banach space, orthogonal series, properties of special functions, interpolation and approximation, eigenfunctions and boundary value problems, probability, and information theory. These, together with a variety of methods of proof both ancient and modern, give the subject a certain charm. This is a source of satisfaction to me, and I hope it will prove equally satisfactory to the reader; if so, my work will have been well rewarded.

I wish to express my gratitude for having been allowed some remission of teaching duties at the Cambridgeshire College of Arts and Technology for purposes of writing and research over a three-year period, during which parts of the book were written. I am particularly grateful to those of my colleagues who, as a consequence, had to shoulder an extra burden of work.

I would also like to thank Dr F. Smithies, fellow of St John's College, Cambridge, for reading the manuscript and making many valuable suggestions; as a result of this the book has been improved in every respect. Appreciation is also due to the staff of Cambridge University Press for accepting the book as a 'tract', and for their courteous efficiency.

Cambridge J. R. HIGGINS
March 1976