Cambridge Tracts in Mathematics
and Mathematical Physics

GENERAL EDITORS
F. SMITHIES, PH.D. AND J. A. TODD, F.R.S.

No. 56

THE THEORY OF CLUSTER SETS

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CAMBRIDGE
AT THE UNIVERSITY PRESS
1966

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York NY 10011–4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 1966
First paperback edition 2004

A catalogue record for this book is available from the British Library

Library of Congress Catalogue Card Number: 66-18115

ISBN 0 521 04695 5 hardback
ISBN 0 521 60481 8 paperback

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Chapter 9. PRIME ENDS

PREFACE

This book is intended as an introduction to the theory of cluster sets, and as such does not claim to be comprehensive. Thus, we have had to be selective and omit some things we should have liked to include, but the space available in a Cambridge Tract is limited, and the theory continues to grow.

The founder of the theory was Painlevé who, in 1895, first gave a name—that of *domaine d'indétermination*—and the status of a distinct mathematical notion to the set of limit points of a function at a boundary point of its domain of definition. This set is now called the *cluster set* of the function at the point in question. Although it arose in connexion with analytic functions the notion is applicable to more general mappings and situations and gives rise to a theory which is essentially topological. Applied, as it has been for the most part, to problems in the theory of functions, it forms a chapter in the expanding field of Topological Function Theory. While this is our main theme and point of view, we have tried not to lose sight of the more general background.

When this book was planned there was no other on the subject available, and the need for an organized account seemed obvious. However, K. Noshiro's book *Cluster Sets* in the 'Ergebnisse' series appeared at about the time when the first draft of our book was finished. Far from regretting this, we regarded it as a fortunate coincidence since the objectives of the two series in which they appear are rather different and in consequence the two books would, to a considerable extent, complement one another. Since then, other heavy commitments have delayed completion and in the interval the theory has developed considerably. We have, therefore, had to introduce new material and reorganize parts of what we had written, but without attempting to cover the whole of the new territory recently opened up.

Although a general familiarity with classical function theory is assumed, we have included for completeness and ease of reference in Chapters 2 and 3 most of the distinctive function-theoretic material required in later chapters. The only other body of distinctive material required is some elementary set-theoretic topology which will be familiar to every analyst.

Since the interest of a theory rests ultimately on its applications, a word should perhaps be said about our final chapter. In this we give a fairly extended treatment of the prime ends of a plane domain by the combined cluster-set and function-theoretic methods which have recently yielded solutions of some long-standing problems in this subject. As well as providing an illustration of the power of the methods now available, this chapter, we hope, may be useful in itself.

Our bibliography is confined principally to those writings to which we actually refer and is far from complete. But references omitted by us will generally be found in Noshiro's bibliography, where the coverage up to the date of its publication is much wider; we have, however, added references to the most recent literature.

As regards the arrangement of the book Chapter 1 is introductory, Chapter 2 contains a brief expository account of the classical theory of radial limit values, with some additional material on Blaschke products. In Chapter 3 we give results from the theory of conformal mapping which will be needed in later chapters. The experienced reader may proceed directly to Chapter 4 where the intrinsic properties of cluster sets of functions defined in the unit disc, but not necessarily analytic, are introduced. In Chapter 5 we study the boundary behaviour of an important class of functions, Seidel's Class U, which are analytic in the disc, together with the Gross–Iversen Theorem on exceptional values and its generalizations. In Chapter 6 we study the global cluster set of a function meromorphic in the disc and its relations with the global exceptional and asymptotic sets. This we call boundary theory in the large. Chapter 7 is on boundary theory in the small; in it we apply the same techniques to the local cluster set, that is to say, the cluster set at a point of the unit circumference, and its relations with the local exceptional and asymptotic sets. The main results of these two chapters are the appropriate analogues of Iversen's Theorem 1.6

for the disc in the large and in the small respectively. In Chapter 8 we have collected a number of recent results on the classification of the singularities of functions meromorphic in the disc and their distribution on the circumference, together with uniqueness and existence theorems under given boundary conditions. Chapter 9, which concludes the book, is on boundary correspondence and prime ends.

We are grateful to various friends, who have seen parts of the book at different stages, for much helpful criticism and in particular to F. W. Gehring and G. Piranian. We are also much indebted for the suggestions and comments of the referee and our editors. We cannot hope to have avoided all the pitfalls but we should certainly have fallen into more of them had it not been for their help.

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