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E. F. Collingwood and A. J. Lohwater
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BY
E. F. COLLINGWOOD, F.R.S.
AND
A. J. LOHWATER

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CONTENTS

Preface *page ix*

Chapter 1. INTRODUCTION

1	The cluster set. Definition	1
2	Definition of a partial cluster set	2
3	Illustrations	3
4	Theorems of Weierstrass and Painlevé	4
5	Proofs of the theorems of Painlevé and Besicovitch	5
<i>The range and the asymptotic set</i>		
6	Definitions. Theorems of Picard, Cartwright and Iversen	7
7	The notion of capacity	9
8	Counter-example for non-isolated singularities	10
9	Case of thin sets of non-isolated singularities	11
10	Historical note and summary	13

*Chapter 2. FUNCTIONS ANALYTIC IN A
CIRCULAR DISC*

Theorems of Fatou and F. and M. Riesz

1	Introduction	17
2	Fatou's theorem on radial limits	17
3	Lindelöf's theorem and Fatou's theorem for angular limits	19
4	Theorem of F. and M. Riesz	21
5	Existence theorems on radial limits	22

Blaschke products

6	Product representation of a bounded function	<i>page</i> 28
7	Boundary properties of Blaschke products	31
8	Further properties of Blaschke products. A uniqueness theorem	35
9	Functions of bounded characteristic and functions omitting three values	38
10	Proof that Fatou's theorem is best possible	43

*Chapter 3. TOPICS IN THE THEORY OF
 CONFORMAL MAPPING*

1	Introduction	46
2	Boundary correspondence under conformal mapping of a Jordan domain	46
3	The Riesz theorem on conformal mapping of a Jordan domain with a rectifiable boundary.	49
4	Measure of boundary sets under conformal enlargement of domain	54
5	Angular limits of univalent functions	56
6	Correspondence of sets of capacity zero under conformal mapping	64

*Chapter 4. INTRINSIC PROPERTIES OF
 CLUSTER SETS*

1	Global cluster set, range and asymptotic set	66
2	Existence theorems for global cluster sets	68
3	Sets of maximum indetermination. Existence theorem for an arbitrary function	72

CONTENTS		vii
4	Maximality theorem for a continuous function	<i>page</i> 75
5	Maximality theorems for an arbitrary function	78
6	Symmetric maximality theorems on cluster sets of an arbitrary function	81
7	Bagemihl's theorem on ambiguous points of an arbitrary function	83
8	Normal functions. More general classes of functions	86
9	Generalizations	88

Chapter 5. CLUSTER SETS OF FUNCTIONS
ANALYTIC IN THE UNIT DISC

1	The Iversen–Beurling theorem	89
2	Extensions of the Schwarz reflexion principle	94
3	The Gross–Iversen theorem on exceptional values. Generalizations	101
4	Properties of functions of Seidel's class U	107
5	Scope of the foregoing theorems	109
6	One-sided cluster-set theorems	110

Chapter 6. BOUNDARY THEORY IN THE LARGE

1	Method of the inverse function	114
2	The set $\Gamma(f)$ for bounded functions	120
3	Boundary theorems in the large	122
4	Cluster sets on spiral paths	129

Chapter 7. BOUNDARY THEORY IN THE SMALL

1	Introduction	133
2	The main theorem in the small	135

*Chapter 8. FURTHER BOUNDARY PROPERTIES OF
 FUNCTIONS MEROMORPHIC IN THE DISC.
 CLASSIFICATION OF SINGULARITIES*

1	Introduction	<i>page</i> 144
2	Functions with angular limits. Theorems of Privalov and Plessner	145
3	Radial cluster sets and uniqueness theorems	149
4	Classification and distribution of singularities on K	150
5	Meier's analogue of Plessner's Theorem	153
6	A theorem on polynomial approximation	155
7	An existence theorem on curvilinear cluster sets	162
8	Some consequences of the existence theorem	165

Chapter 9. PRIME ENDS

1	Introduction	167
2	Definition of a prime end	168
3	Preliminary lemmas	171
4	Correspondence of frontiers under conformal mapping	172
5	Elementary properties of the space of prime ends	173
6	Alternative metrics and generalizations	175
7	Principal and subsidiary points	176
8	The classification of prime ends	180
9	The distribution of the subsets \mathcal{E}_v in \mathcal{E}	182
10	A domain having a bi-connected set of accessible points	185
11	The set of asymmetrical prime ends	188
	<i>Bibliography</i>	190
	<i>Index of symbols</i>	207
	<i>Index</i>	209

Cambridge University Press
0521604818 - The Theory of Cluster Sets
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Frontmatter
[More information](#)

PREFACE

This book is intended as an introduction to the theory of cluster sets, and as such does not claim to be comprehensive. Thus, we have had to be selective and omit some things we should have liked to include, but the space available in a Cambridge Tract is limited, and the theory continues to grow.

The founder of the theory was Painlevé who, in 1895, first gave a name—that of *domaine d'indétermination*—and the status of a distinct mathematical notion to the set of limit points of a function at a boundary point of its domain of definition. This set is now called the *cluster set* of the function at the point in question. Although it arose in connexion with analytic functions the notion is applicable to more general mappings and situations and gives rise to a theory which is essentially topological. Applied, as it has been for the most part, to problems in the theory of functions, it forms a chapter in the expanding field of Topological Function Theory. While this is our main theme and point of view, we have tried not to lose sight of the more general background.

When this book was planned there was no other on the subject available, and the need for an organized account seemed obvious. However, K. Noshiro's book *Cluster Sets* in the 'Ergebnisse' series appeared at about the time when the first draft of our book was finished. Far from regretting this, we regarded it as a fortunate coincidence since the objectives of the two series in which they appear are rather different and in consequence the two books would, to a considerable extent, complement one another. Since then, other heavy commitments have delayed completion and in the interval the theory has developed considerably. We have, therefore, had to introduce new material and reorganize parts of what we had written, but without attempting to cover the whole of the new territory recently opened up.

Although a general familiarity with classical function theory is assumed, we have included for completeness and ease of

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0521604818 - The Theory of Cluster Sets
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Frontmatter
[More information](#)

x

PREFACE

reference in Chapters 2 and 3 most of the distinctive function-theoretic material required in later chapters. The only other body of distinctive material required is some elementary set-theoretic topology which will be familiar to every analyst.

Since the interest of a theory rests ultimately on its applications, a word should perhaps be said about our final chapter. In this we give a fairly extended treatment of the prime ends of a plane domain by the combined cluster-set and function-theoretic methods which have recently yielded solutions of some long-standing problems in this subject. As well as providing an illustration of the power of the methods now available, this chapter, we hope, may be useful in itself.

Our bibliography is confined principally to those writings to which we actually refer and is far from complete. But references omitted by us will generally be found in Noshiro's bibliography, where the coverage up to the date of its publication is much wider; we have, however, added references to the most recent literature.

As regards the arrangement of the book Chapter 1 is introductory, Chapter 2 contains a brief expository account of the classical theory of radial limit values, with some additional material on Blaschke products. In Chapter 3 we give results from the theory of conformal mapping which will be needed in later chapters. The experienced reader may proceed directly to Chapter 4 where the intrinsic properties of cluster sets of functions defined in the unit disc, but not necessarily analytic, are introduced. In Chapter 5 we study the boundary behaviour of an important class of functions, Seidel's Class U , which are analytic in the disc, together with the Gross–Iversen Theorem on exceptional values and its generalizations. In Chapter 6 we study the global cluster set of a function meromorphic in the disc and its relations with the global exceptional and asymptotic sets. This we call boundary theory in the large. Chapter 7 is on boundary theory in the small; in it we apply the same techniques to the local cluster set, that is to say, the cluster set at a point of the unit circumference, and its relations with the local exceptional and asymptotic sets. The main results of these two chapters are the appropriate analogues of Iversen's Theorem 1.6

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Frontmatter
[More information](#)

P R E F A C E

xi

for the disc in the large and in the small respectively. In Chapter 8 we have collected a number of recent results on the classification of the singularities of functions meromorphic in the disc and their distribution on the circumference, together with uniqueness and existence theorems under given boundary conditions. Chapter 9, which concludes the book, is on boundary correspondence and prime ends.

We are grateful to various friends, who have seen parts of the book at different stages, for much helpful criticism and in particular to F. W. Gehring and G. Piranian. We are also much indebted for the suggestions and comments of the referee and our editors. We cannot hope to have avoided all the pitfalls but we should certainly have fallen into more of them had it not been for their help.

E. F. C.
A. J. L.