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A variant of K-theory: $K_\pm$

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1 Introduction

Topological K-theory [2] has many variants which have been developed and exploited for geometric purposes. There are real or quaternionic versions, 'Real' K-theory in the sense of [1], equivariant K-theory [14] and combinations of all these.

In recent years K-theory has found unexpected application in the physics of string theories [6] [12] [13] [16] and all variants of K-theory that had previously been developed appear to be needed. There are even variants, needed for the physics, which had previously escaped attention, and it is one such variant that is the subject of this paper.

This variant, denoted by $K_\pm(X)$, was introduced by Witten [16] in relation to 'orientifolds'. The geometric situation concerns a manifold X with an involution τ having a fixed sub-manifold Y. On X one wants to study a pair of complex vector bundles (E^+, E^-) with the property that τ interchanges them. If we think of the virtual vector bundle $E^+ - E^-$, then τ takes this into its negative, and $K_\pm(X)$ is meant to be the appropriate K-theory of this situation.

In physics, X is a 10-dimensional Lorentzian manifold and maps $\Sigma \to X$ of a surface Σ describe the world-sheet of strings. The symmetry requirements for the appropriate Feynman integral impose conditions that the putative K-theory $K_\pm(X)$ has to satisfy.

The second author proposed a precise topological definition of $K_\pm(X)$ which appears to meet the physics requirements, but it was not entirely clear how to link the physics with the geometry.

In this paper we elaborate on this definition and also a second (but equivalent) definition of $K_\pm(X)$. Hopefully this will bring the geometry and physics closer together, and in particular link it up with the analysis of Dirac operators.

Although $K_{\pm}(X)$ is defined in the context of spaces with involution it is rather different from Real K-theory or equivariant K-theory (for $G = Z_2$), although it has superficial resemblances to both. The differences will become clear as we proceed, but at this stage it may be helpful to consider the analogy with cohomology. Equivariant cohomology can be defined (for any compact Lie group G), and this has relations with equivariant K-theory. But there is also 'cohomology with local coefficients', where the fundamental group $\pi_1(X)$ acts on the abelian coefficient group. In particular for integer coefficients Z the only such action is via a homomorphism $\pi_1(X) \to Z_2$, i.e. by an element of $H^1(X; Z_2)$ or equivalently a double-covering $\tilde{X}$ of X.

This is familiar for an unoriented manifold with $\tilde{X}$ its oriented double-cover. In this situation, if say X is a compact n-dimensional manifold, then we do not have a fundamental class in $H^n(X; Z)$ but in $H^n(X; \tilde{Z})$ where $\tilde{Z}$ is the local coefficient system defined by $\tilde{X}$. This is also called 'twisted cohomology'.

Here $\tilde{X}$ has a fixed-point-free involution τ and, in such a situation, our group $K_{\pm}(\tilde{X})$ is the precise K-theory analogue of twisted cohomology. This will become clear later.

In fact K-theory has more sophisticated twisted versions. In [8] Donovan and Karoubi use Wall's graded Brauer group [15] to construct twistings from elements of $H^1(X; Z_2) \times H^3(X; Z)_{\text{torsion}}$. More general twistings of K-theory arise from automorphisms of its classifying space, as do twistings of equivariant K-theory. Among these are twistings involving a general element of $H^3(X; Z)$ (i.e., one which is not necessarily of finite order). These are also of interest in physics, and have recently been the subject of much attention [3] [5] [9]. Our $K_{\pm}$ is a twisted version of equivariant K-theory,[1] and this paper can be seen as a preliminary step towards these other more elaborate versions.

2 The first definition

Given a space X with involution we have two natural K-theories, namely $K(X)$ and $K_{Z_2}(X)$ – the ordinary and equivariant theories respectively. Moreover we have the obvious homomorphism

$$\phi : K_{Z_2}(X) \to K(X) \tag{2.1}$$

[1] It is the twisting of equivariant K-theory by the non-trivial element of $H^1_{Z_2}(\mathrm{pt}) = Z_2$. From the point of view of the equivariant graded Brauer group, $K_{\pm}(X)$ is the K-theory of the graded cross product algebra $C(X) \otimes M \rtimes Z_2$, where $C(X)$ is the algebra of continuous functions on X, and M is the graded algebra of 2×2-matrices over the complex numbers, graded in such a way that (i, j) entry has degree $i + j$. The action of Z_2 is the combination of the geometric action given on X and conjugation by the permutation matrix on M.

which 'forgets' about the Z_2-action. We can reformulate this by introducing the space $(X \times Z_2)$ with the involution $(x, 0) \to (\tau(x), 1)$. Since this action is free we have

$$K_{Z_2}(X \times Z_2) \cong K(X)$$

and (2.1) can then be viewed as the natural homomorphism for K_{Z_2} induced by the projection

$$\pi : X \times Z_2 \to X. \tag{2.2}$$

Now, whenever we have such a homomorphism, it is part of a long exact sequence (of period 2) which we can write as an exact triangle

$$\begin{tikzcd} K^*_{Z_2}(X) \arrow[rr, "\phi"] & & K^*(X) \arrow[dl, "\delta"] \\ & K^*_{Z_2}(\pi) \arrow[ul] & \end{tikzcd} \tag{2.3}$$

where $K^* = K^0 \oplus K^1$, δ has degree 1 mod 2 and the relative group $K^*_{Z_2}(\pi)$ is just the relative group for a pair, when we replace π by a Z_2-homotopically equivalent inclusion. In this case a natural way to do this is to replace the X factor on the right of (2.2) by $X \times I$ where $I = [0, 1]$ is the unit interval with τ being reflection about the mid-point $\frac{1}{2}$. Thus, explicitly

$$K^*_{Z_2}(\pi) = K^*_{Z_2}(X \times I, X \times \partial I) \tag{2.4}$$

where ∂I is the (2-point) boundary of I.

We now take the group in (2.4) (with the degree shifted by one) as our definition of $K^*_\pm(X)$. It is then convenient to follow the notation of [1] where $R^{p,q} = R^p \oplus R^q$ with the involution changing the sign of the first factor, and we use K-theory with compact supports (so as to avoid always writing the boundary). Then our definition of $K_\pm$ becomes

$$K^0_\pm(X) = K^1_{Z_2}(X \times R^{1,0}) \cong K^0_{Z_2}(X \times R^{1,1}) \tag{2.5}$$

(and similarly for K^1).

Let us now explain why this definition fits the geometric situation we began with (and which comes from the physics). Given a vector bundle E we can form the pair (E, τ^*E) or the virtual bundle

$$E - \tau^* E.$$

Under the involution, E and τ^*E switch and the virtual bundle goes into its negative. Clearly, if E came from an equivariant bundle, then $E \cong \tau^*E$ and the virtual bundle is zero. Hence the virtual bundle depends only the element defined by E in the cokernel of ϕ, and hence by the image of E in the next term of the exact sequence (2.3), i.e. by

$$\delta(E) \in K^0_{\pm}(X).$$

This explains the link with our starting point and it also shows that one cannot always define $K_{\pm}(X)$ in terms of such virtual bundles on X. In general the exact sequence (2.3) does not break up into short exact sequences and δ is not surjective.

At this point a physicist might wonder whether the definition of $K_{\pm}(X)$ that we have given is the right one. Perhaps there is another group which is represented by virtual bundles. We will give two pieces of evidence in favour of our definition, the first pragmatic and the second more philosophical.

First let us consider the case when the involution τ on X is trivial. Then $K^*_{Z_2}(X) = R(Z_2) \otimes K^*(X)$ and $R(Z_2) = Z \oplus Z$ is the representation ring of Z_2 and is generated by the two representations:

$$\begin{array}{ll} 1 & \text{(trivial representation)} \\ \rho & \text{(sign representation).} \end{array}$$

The homomorphism ϕ is surjective with kernel $(1-\rho)K^*(X)$ so $\delta = 0$ and

$$K^0_{\pm}(X) \cong K^1(X). \tag{2.6}$$

This fits with the requirements of the physics, which involves a switch from type IIA to type IIB string theory. Note also that it gives an extreme example when ∂ is not surjective.

Our second argument is concerned with the general passage from physical (quantum) theories to topology. If we have a theory with some symmetry then we can consider the quotient theory, on factoring out the symmetry. Invariant states of the original theory become states of the quotient theory but there may also be new states that have to be added. For example if we have a group G of geometric symmetries, then closed strings in the quotient theory include strings that begin at a point x and end at $g(x)$ for $g \in G$. All this is similar to what happens in topology with (generalized) cohomology theories, such as K-theory. If we have a morphism of theories, such as ϕ in (2.1) then the third theory we get fits into a long-exact sequence. The part coming from $K(X)$ is only part of the answer – other elements have to be added. In ordinary cohomology where we start with cochain complexes the process of forming a quotient theory involves an ordinary quotient (or short exact sequence) at the level of cochains. But this becomes a long exact sequence at the cohomology level. For K-theory the analogue is to start with bundles over small open sets and at this level we can form the naïve quotients, but the K-groups arise when we impose the matching conditions to get bundles, and then we end up with long exact sequences.

It is also instructive to consider the special case when the involution is free so that we have a double covering $\tilde{X} \to X$ and the exact triangle (2.3), with $\tilde{X}$ for X, becomes the exact triangle

$$\begin{array}{ccc} K^*(X) & \xrightarrow{\phi} & K^*(\tilde{X}) \\ \nwarrow & & \swarrow \delta \\ & K^*_{Z_2}(L) & \end{array} \tag{2.7}$$

Here L is the real line bundle over X associated to the double covering $\tilde{X}$ (or to the corresponding element of $H^1(X, Z_2)$), and we again use compact supports. Thus (for $q = 0, 1 \mod 2$)

$$K^q_{\pm}(\tilde{X}) = K^{q+1}(L). \tag{2.8}$$

If we had repeated this argument using equivariant cohomology instead of equivariant K-theory we would have ended up with the twisted cohomology mentioned earlier, via a twisted suspension isomorphism

$$H^q(X, \tilde{Z}) = H^{q+1}(L). \tag{2.9}$$

This shows that, for free involutions, $K_{\pm}$ **is precisely the analogue of twisted cohomology,** so that, for example, the Chern character of the former takes values in the rational extension of the latter.

3 Relation to Fredholm operators

In this section we shall give another definition of $K_{\pm}$ which ties in naturally with the analysis of Fredholm operators, and we shall show that this definition is equivalent to the one given in Section 2.

We begin by recalling a few basic facts about Fredholm operators. Let H be complex Hilbert space, $\mathcal{B}$ the space of bounded operators with the norm topology and $\mathcal{F} \subset \mathcal{B}$ the open subspace of Fredholm operators, i.e. operators A so that $\ker A$ and $\operatorname{coker} A$ are both finite-dimensional. The index defined by

$$\text{index } A = \dim \ker A - \dim \operatorname{coker} A$$

is then constant on connected components of $\mathcal{F}$. If we introduce the adjoint A^* of A then

$$\operatorname{coker} A = \ker A^*$$

so that

$$\text{index } A = \dim \ker A - \dim \ker A^*.$$

More generally if we have a continuous map

$$f : X \to \mathfrak{F}$$

(i.e. a family of Fredholm operators, parametrized by X), then one can define

$$\text{index } f \in K(X)$$

and one can show [2] that we have an isomorphism

$$\text{index} : [X, \mathfrak{F}] \cong K(X) \tag{3.1}$$

where [,] denotes homotopy classes of maps. Thus $K(X)$ has a natural definition as the 'home' of indices of Fredholm operators (parametrized by X): it gives the complete homotopy invariant.

Different variants of K-theory can be defined by different variants of (3.1). For example real K-theory uses real Hilbert space and equivariant K-theory for G-spaces uses a suitable H-space module of G, namely $L_2(G) \otimes H$. It is natural to look for a similar story for our new groups $K_{\pm}(X)$. A first candidate might be to consider Z_2-equivariant maps

$$f : X \to \mathfrak{F}$$

where we endow $\mathfrak{F}$ with the involution $A \to A^*$ given by taking the adjoint operator. Since this switches the role of kernel and cokernel it acts as -1 on the index, and so is in keeping with our starting point.

As a check we can consider X with a trivial involution, then f becomes a map

$$f : X \to \widehat{\mathfrak{F}}$$

where $\widehat{\mathfrak{F}}$ is the space of self-adjoint Fredholm operators. Now in [4] it is shown that $\widehat{\mathfrak{F}}$ has three components

$$\widehat{\mathfrak{F}}_+, \widehat{\mathfrak{F}}_-, \widehat{\mathfrak{F}}_*$$

where the first consists of A which are essentially positive (only finitely many negative eigenvalues), the second is given by essentially negative operators. These two components are trivial, in the sense that they are contractible, but the third one is interesting and in fact [4]

$$\widehat{\mathfrak{F}}_* \sim \Omega\mathfrak{F} \tag{3.2}$$

where Ω denotes the loop space. Since

$$[X, \Omega\mathfrak{F}] \cong K^1(X)$$

this is in agreement with (2.6) – though to get this we have to discard the two trivial components of $\widehat{\mathcal{F}}$, a technicality to which we now turn.

Lying behind the isomorphism (3.1) is Kuiper's Theorem [11] on the contractibility of the unitary group of Hilbert spaces. Hence to establish that our putative definition of $K_{\pm}$ coincides with the definition given in Section 2 we should expect to need a generalization of Kuiper's Theorem incorporating the involution $A \to A^*$ on operators. The obvious extension turns out to be false, precisely because $\widehat{\mathcal{F}}$, the fixed-point set of $*$ on $\mathcal{F}$, has the additional contractible components. There are various ways we can get round this but the simplest and most natural is to use 'stabilization'. Since $H \cong H \oplus H$ we can always stabilize by adding an additional factor of H. In fact Kuiper's Theorem has two parts in its proof:

(1) The inclusion $U(H) \to U(H \oplus H)$ defined by $u \to u \oplus 1$ is homotopic to the constant map.
(2) This inclusion is homotopic to the identity map given by the isomorphism $H \cong H \oplus H$.

The proof of (1) is an older argument (sometimes called the 'Eilenberg swindle'), based on a correct use of the fallacious formula

$$\begin{aligned} 1 &= 1 + (-1+1) + (-1+1) \dots \\ &= (1+-1) + (1+-1) + \dots \\ &= 0. \end{aligned}$$

The trickier part, and Kuiper's contribution, is the proof of (2).

For many purposes, as in K-theory, the stronger version is a luxury and one can get by with the weaker version (1), which applies rather more generally. In particular (1) is consistent with taking adjoints (i.e. inverses in $U(H)$), which is the case we need.

With this background explanation we now introduce formally our second definition, and to distinguish it temporarily from $K_{\pm}$ as defined in Section 2, we put

$$\mathcal{K}_{\pm}(X) = [X, \mathcal{F}]^s_* \tag{3.3}$$

where $*$ means we use Z_2-maps compatible with $*$ and s means that we use **stable** homotopy equivalence. More precisely the Z_2-maps

$$f : X \to \mathcal{F}(H) \quad g : X \to \mathcal{F}(H)$$

are called stably homotopic if the 'stabilized' maps

$$f^s : X \to \mathcal{F}(H \oplus H) \quad g^s : X \to \mathcal{F}(H \oplus H)$$

given by $f^s = f \oplus J,\ g^s = g \oplus J$ are homotopic, where J is a fixed (essentially unique) automorphism of H with

$$J = J^*,\ \ J^2 = 1, \quad +1 \text{ and } -1 \text{ both of infinite multiplicity.} \tag{3.4}$$

Note that under such stabilization the two contractible components $\widehat{\mathcal{F}}_+$ and $\widehat{\mathcal{F}}_-$ of $\widehat{\mathcal{F}}(H)$ both end up in the interesting component $\widehat{\mathcal{F}}_*$ of $\widehat{\mathcal{F}}(H \oplus H)$.

The first thing we need to observe about $\mathcal{K}_\pm(X)$ is that it is an abelian group. The addition can be defined in the usual way by using direct sums of Hilbert spaces. Moreover we can define the negative degree groups $\mathcal{K}_\pm^{-n}(X)$ (for $n \geq 1$) by suspension (with trivial involution on the extra coordinates), so that

$$\mathcal{K}_\pm^{-n}(X) = \mathcal{K}_\pm(X \times S^n, X \times \infty).$$

However, at this stage we do not have the periodicity theorem for $\mathcal{K}_\pm(X)$. This will follow in due course after we establish the equivalence with $K_\pm(X)$. As we shall see our construction of (4.2) is itself closely tied to the periodicity theorem.

Our aim in the subsequent sections will be to show that there is a natural isomorphism

$$\mathcal{K}_\pm(X) \cong K_\pm(X). \tag{3.5}$$

This isomorphism will connect us up naturally with Dirac operators and so should tie in nicely with the physics.

4 Construction of the map

Our first task is to define a natural map

$$K_\pm(X) \to \mathcal{K}_\pm(X). \tag{4.1}$$

We recall from (2.5) that

$$\begin{aligned} K_\pm(X) &= K_{Z_2}(X \times R^{1,1}) \\ &= K_{Z_2}(X \times S^2,\ X \times \infty) \end{aligned}$$

where S^2 is the 2-sphere obtained by compactifying $R^{1,1}$, and ∞ is the added point. Note that Z_2 now acts on S^2 by a **reflection,** so that it reverses its orientation.

Thus to define a map (4.1) it is sufficient to define a map

$$K_{Z_2}(X \times S^2) \to \mathcal{K}_\pm(X). \tag{4.2}$$

This is where the Dirac operator enters. Recall first that, if we ignore involutions, there is a basic map

$$K(X \times S^2) \to [X, \mathcal{F}] \cong K(X) \tag{4.3}$$

which is the key to the Bott periodicity theorem. It is given as follows. Let D be the Dirac operator on S^2 from positive to negative spinors and let V be a complex vector bundle on $X \times S^2$, then we can extend, or couple, D to V to get a family D_V of elliptic operators along the S^2-fibres. Converting this, in the usual way, to a family of (bounded) Fredholm operators we get the map (4.3).

We now apply the same construction but keeping track of the involutions. The new essential feature is that Z_2 reverses the orientation of S^2 and hence takes the Dirac operator D into its adjoint D^*. This is precisely what we need to end up in $\mathcal{K}_\pm(X)$ so defining (4.2).

Remark 4.1. Strictly speaking the family D_V of Fredholm operators does not act in a fixed Hilbert space, but in a bundle of Hilbert spaces. The problem can be dealt with by adding a trivial operator acting on a complementary bundle W (so that $W + V$ is trivial).

5 Equivalence of definition

Let us sum up what we have so far. We have defined a natural homomorphism

$$K_\pm(X) \to \mathcal{K}_\pm(X)$$

and we know that this is an isomorphism for spaces X with trivial involution – both groups coinciding with $K^1(X)$. Moreover, if for general X, we ignore the involutions, or equivalently replace X by $X \times \{0, 1\}$, we also get an isomorphism, both groups now coinciding with $K^0(X)$.

General theory then implies that we have an isomorphism for all X. We shall review this general argument.

Let A, B be representable theories, defined on the category of Z_2-spaces, so that

$$A(X) = [X, \mathcal{A}]$$
$$B(X) = [X, \mathcal{B}]$$

where [,] denotes homotopy classes of Z_2-maps into the classifying spaces of $\mathcal{A}$, $\mathcal{B}$ of the theories. A natural map $A(X) \to B(X)$ then corresponds to a Z_2-map $\mathcal{A} \to \mathcal{B}$. Showing that A and B are isomorphic theories is equivalent to showing that $\mathcal{A}$ and $\mathcal{B}$ are Z_2-homotopy equivalent.

If we forget about the involutions then isomorphism of theories is the same as ordinary homotopy equivalence. Restricting to spaces X with trivial involution corresponds to restricting to the fixed-point sets of the involution on $\mathcal{A}$ and $\mathcal{B}$.

Now there is a general theorem in homotopy theory [10] which asserts (for reasonable spaces including Banach manifolds such as $\mathcal{F}$) that, if a Z_2-map $\mathcal{A} \to \mathcal{B}$ is both a homotopy equivalence ignoring the involution and for the fixed-point sets, then it is a Z_2-homotopy equivalence. Translated back into the theories A, B it says that the map $A(X) \to B(X)$ is an isomorphism provided it holds for spaces X with the trivial Z_2-action, and for Z_2-spaces X of the form $Y \times \{0, 1\}$.

This is essentially the situation we have here with

$$A = K_{\pm} \qquad B = \mathcal{K}_{\pm}.$$

Both are representable. The representability of the first

$$K_{\pm}(X) \cong K_{Z_2}(X \times R^{1,1})$$

arises from the general representability of K_{Z_2}, the classifying space being essentially the double loop space of $\mathcal{F}(H \otimes C^2)$ with an appropriate Z_2-action. The second is representable because

$$\mathcal{K}_{\pm}(X) = [X, \mathcal{F}]_*^s = [X, \mathcal{F}_s]_* \tag{5.1}$$

where $\mathcal{F}^s$ is obtained by stabilizing $\mathcal{F}$. More precisely

$$\mathcal{F}^s = \lim_{n \to \infty} \mathcal{F}_n$$

where $\mathcal{F}_n = \mathcal{F}(H \otimes C^n)$ and the limit is taken with respect to the natural inclusions, using J of (3.4) as a base point. The assertion in (5.1) is easily checked and it simply gives two ways of looking at the stabilization process.

We have thus established the equivalence of our two definitions $K_{\pm}$ and $\mathcal{K}_{\pm}$.

6 Free involutions

We shall now look in more detail at the case of free involutions and, following the notation of Section 1, we shall denote the free Z_2-space by $\tilde{X}$ and its quotient by X.

The reason for introducing the stabilization process in Section 3 concerned fixed points. We shall now show that, for free involutions, we can dispense with stabilization. Let

$$\mathcal{F} \to \mathcal{F}^s$$

be the natural inclusion of $\mathcal{F}$ in the direct limit space. This inclusion is a Z_2-map and a homotopy equivalence, though **not** a Z_2-homotopy equivalence (because of the fixed points). Now given the double covering $\tilde{X} \to X$ we can form the associated fibre bundles $\mathcal{F}_X$ and $\mathcal{F}^s_X$ over X with fibres $\mathcal{F}$ and $\mathcal{F}^s$. Thus

$$\mathcal{F}_X = \tilde{X} \times_{Z_2} \mathcal{F} \qquad \mathcal{F}^s_X = \tilde{X} \times_{Z_2} \mathcal{F}^s$$

and we have an inclusion

$$\mathcal{F}_X \to \mathcal{F}^s_X$$

which is fibre preserving. This map is a homotopy equivalence on the fibres and hence, by a general theorem [7] (valid in particular for Banach manifolds) a fibre homotopy equivalence. It follows that the homotopy classes of sections of these two fibrations are isomorphic. But these are the same as

$$\left[\tilde{X}, \mathcal{F}\right]_* \text{ and } \left[\tilde{X}, \mathcal{F}\right]^s_* = \mathcal{K}_\pm(\tilde{X}).$$

This show that, for a free involution, we can use $\mathcal{F}$ instead of $\mathcal{F}^s$. Moreover it gives the following simple description of $K_\pm(\tilde{X})$

$$K_\pm(\tilde{X}) = \text{ Homotopy classes of sections of } \mathcal{F}_X. \tag{6.1}$$

This is the K-theory analogue of twisted cohomology described in Section 1. A corresponding approach to the higher twist of K-theory given by an element of $H^3(X; Z)$ will be developed in [3].

7 The real case

Everything we have done so far extends, with appropriate modifications, to real K-theory. The important difference is that the periodicity is now 8 rather than 2 and that, correspondingly, we have to distinguish carefully between **self-adjoint** and **skew-adjoint** Fredholm operators. Over the complex numbers multiplication by i converts one into the other, but over the real numbers there are substantial differences.

We denote by $\mathcal{F}^1(R)$ the interesting component of the space of real self-adjoint Fredholm operators $\widehat{\mathcal{F}}(R)$ on a real Hilbert space (discarding two contractible components as before). We also denote by $\mathcal{F}^{-1}(R)$ the space of skew-adjoint Fredholm operators. Then in [4] it is proved that

$$[X, \mathcal{F}^1(R)] \cong KO^1(X) \tag{7.1}$$

$$[X, \mathcal{F}^{-1}(R)] \cong KO^{-1}(X) \cong KO^7(X) \tag{7.2}$$

showing that these are essentially different groups.

Using (7.1), stabilizing, and arguing precisely as before, we define

$$KO_{\pm}(X) = KO^1_{Z_2}(X \times R^{1,0}) \cong KO_{Z_2}(X \times R^{1,7})$$
$$\mathcal{KO}_{\pm}(X) = [X, \mathcal{F}(R)]^s_*$$

where (in (2.5)) the mod 2 periodicity of K has been replaced by the mod 8 periodicity of KO. But we cannot now just use the Dirac operator on S^2 because this is not real. Instead we have to use the Dirac operator on S^8, which then gives us our map

$$KO_{\pm}(X) \to [X, \mathcal{F}(R)]^s_*. \tag{7.3}$$

The same proof as before establishes the isomorphism of (7.3), so that

$$KO_{\pm}(X) \cong \mathcal{KO}_{\pm}(X)$$

and more generally for q modulo 8

$$KO^q_{\pm}(X) \cong \mathcal{KO}^q_{\pm}(X). \tag{7.4}$$

In [4] there is a more systematic analysis of Fredholm operators in relation to Clifford algebras and using this it is possible to give more explicit descriptions of $KO^q_{\pm}(X)$, for all q, in terms of Z_2-mappings into appropriate spaces of Fredholm operators. This would fit in with the different behaviour of the Dirac operator in different dimensions (modulo 8).

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